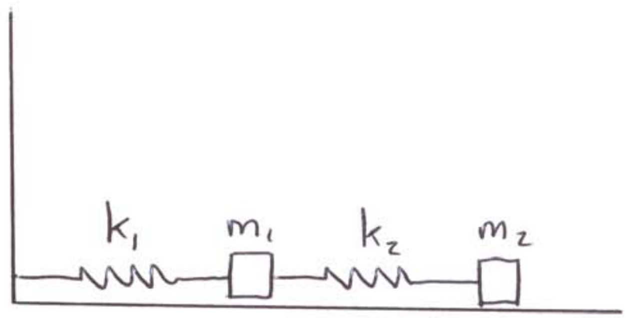
Physics 3010 Test 11

Name: ______________________________

1. Consider two masses connected via springs as shown below.



$k_2 = 2k_1$

$m_2 = 2m_1$

a) (2 marks) Write down the Lagrangian. Be sure to clearly define your general coordinates.

Let x_1 be extension of spring 1 from $=$m.

" x_2 " " 2 " "

$$U = \frac{k_1}{2} x_1^2 + \frac{k_2}{2} x_2^2$$

$$T = \frac{m_1}{2} \dot{x}_1^2 + m_2 (\dot{x}_1 + \dot{x}_2)^2$$

$$\therefore L = T - U$$

b) (6 marks) Solve the Lagrangian equations to find the eigenfrequencies for the motion.

$$\frac{\partial L}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = 0$$

$$-k_1 x_1 - m_1 \ddot{x}_1 - m_2(\ddot{x}_1 + \ddot{x}_2) = 0$$

Let $x_1 = A e^{i\omega t}$ $\quad x_2 = B e^{i\omega t}$

$$-k_1 A + m_1 \omega^2 A + m_2 \omega^2 (A+B) = 0$$

$$(-k_1 + m_1\omega^2 + m_2\omega^2)A + m_2\omega^2 B = 0 \quad (1)$$

$$\frac{\partial L}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = 0$$

$$-k_2 x_2 - m_2(\ddot{x}_1 + \ddot{x}_2) = 0$$

$$-k_2 B + \omega^2 m_2 (A+B) = 0$$

$$m_2\omega^2 A + (-k_2 + m_2\omega^2) B = 0 \quad (2)$$

$$(1) \,\&\, (2) \Rightarrow \begin{pmatrix} -k + 3m\omega^2 & 2m\omega^2 \\ 2m\omega^2 & -2k + 2m\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \det\begin{pmatrix} -k + 3m\omega^2 & 2m\omega^2 \\ 2m\omega^2 & -2k + 2m\omega^2 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} -1 + 3\omega^2/\omega_0^2 & 2\omega^2/\omega_0^2 \\ \omega^2/\omega_0^2 & -1 + \omega^2/\omega_0^2 \end{pmatrix} = 0 \quad \text{where } \omega_0 \equiv \sqrt{\frac{k}{m}}$$

$$\left(-1 + \frac{3\omega^2}{\omega_0^2}\right)\left(-1 + \frac{\omega^2}{\omega_0^2}\right) - 2\frac{\omega^4}{\omega_0^4} = 0$$

$$\frac{\omega^4}{\omega_0^4} - 4\frac{\omega^2}{\omega_0^2} + 1 = 0 \Longrightarrow \omega^2 = \omega_0^2\left\{\frac{4 \pm \sqrt{16 - 4}}{2}\right\}$$

$$\omega = \omega_0\sqrt{2 \pm \sqrt{3}}$$

c) (2 marks) The following two questions require very short answers with almost no algebra! You can even do them if you didn't get parts a and b.

i. What is the eigenfrequency if the second mass is zero?

$$\omega = \omega_0 \equiv \sqrt{\frac{k}{m}}$$ i.e. just one mass on one spring

ii. What is the eigenfrequency if the second string constant k_2 is infinite?

i.e. m_1 & m_2 firmly tied together

$$\omega = \sqrt{\frac{k}{m_1 + 2m_1}} = \frac{\omega_0}{\sqrt{3}}$$

Total = 10 marks