Physics 3010 Test 10

Name: ____________________________

1. (4 marks)

a) What is the definition of the moment of inertia tensor?

$$I_{ij} = \sum_{\alpha} m_\alpha \left\{ \delta_{ij} \sum_k x_{\alpha k}^2 - x_{\alpha i} x_{\alpha j} \right\}$$

b) Explain why or why not the following matrix describes a moment of inertia tensor?

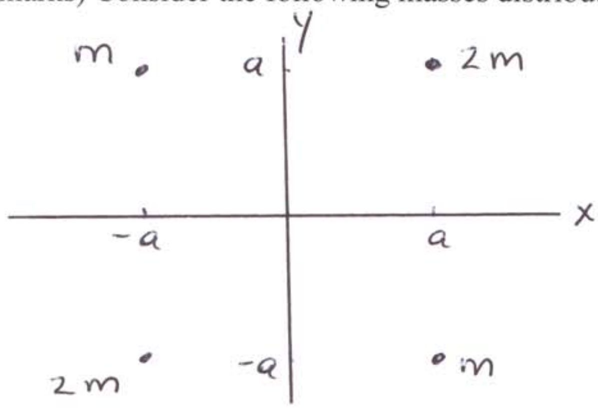
$$I = I_0 \begin{pmatrix} 1 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\underset{\sim}{I}$ is not symmetric $\therefore$ can't be a moment of inertia tensor. i.e. $I_{ij} \neq I_{ji}$

2. (2 marks) What is nutation?

A spinning top precesses about a vertical axis. The axis can wobble and is called nutation.

3. (4 marks) Consider the following masses distributed as follows.



a) Find the moment of inertia tensor.

$$I_{11} = \sum_{\alpha} m_\alpha (r_\alpha^2 - x_\alpha^2)$$

$$= \left(2m(2a^2 - a^2) + m(2a^2 - a^2)\right) \times 2$$

$$= 6ma^2$$

Similarly $I_{22} = 6ma^2$

$$I_{33} = \sum_{\alpha} m_\alpha (r_\alpha^2 - z_\alpha^2)$$

$$= \sum_{\alpha} m_\alpha r_\alpha^2$$

$$= 12ma^2$$

$$I_{12} = -\sum_{\alpha} m_\alpha x_\alpha y_\alpha$$

$$= \left(-2ma^2 - m(-a^2)\right) \times 2$$

$$= -2ma^2$$

$$I_{13} = -\sum_{\alpha} m_\alpha x_\alpha z_\alpha \implies \underset{\sim}{I} = ma^2 \begin{pmatrix} 6 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

$$= 0$$

b) Find the principal axes.

For eigenvalues solve $\det \begin{pmatrix} 6-\lambda & -2 & 0 \\ -2 & 6-\lambda & 0 \\ 0 & 0 & 12-\lambda \end{pmatrix} = 0$

$$\left[(6-\lambda)^2 - 4\right](12-\lambda) = 0.$$

$$\left[\lambda^2 - 12\lambda + 36 - 4\right](12-\lambda) = 0.$$

$$(\lambda - 8)(\lambda - 4)(12-\lambda) = 0.$$

$$\therefore \lambda = 4, 8, 12.$$

$\lambda = 4$ Principal axis solve $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow 2x_1 - 2x_2 = 0 \Rightarrow x_1 = x_2$$
$$10x_3 = 0 \quad x_3 = 0$$

$\therefore$ principal axis is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. corresponding to $\lambda = 4ma^2$

Similarly principal axis for $\lambda = 8ma^2$ is $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

" " $\lambda = 12ma^2$ is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Total = 10 marks