Physics 3010 Test 4

Name: ____________________________

1. (3 marks) What conserved quantities result from a Lagrangian that is invariant with respect to:

   a) Time $\Rightarrow$ Energy Conservation

   b) Translation Momentum Conservation

   c) Rotation Angular Momentum Conservation

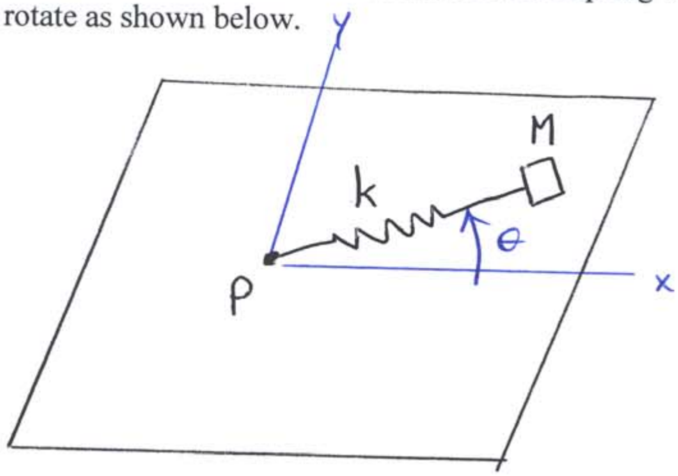
2. (2 marks) Write down the Virial Theorem defining all variables.

$$\underbrace{\langle T \rangle}_{\text{average Kinetic Energy}} = -\frac{1}{2} \left\langle \underbrace{\sum_{\alpha}}_{\text{sum over particles.}} \vec{F}_{\alpha} \cdot \vec{r}_{\alpha} \right\rangle$$

$\vec{F}_{\alpha}$ = force on particle α

$\vec{r}_{\alpha}$ = position of particle α.

3. (5 marks) A mass M on a flat table is connected via a spring to a central point that can freely rotate as shown below.



a) Write down the Lagrangian

Let l be distance of M from pivot.
Let θ be as shown above.

$$T = \frac{M}{2}\left(\dot{l}^2 + l^2\dot{\theta}^2\right)$$

$$U = \frac{k}{2}(l - l_0)^2 \qquad l_0 = \text{equilibrium spring length}$$

$$\therefore L = \frac{M}{2}\left(\dot{l}^2 + l^2\dot{\theta}^2\right) - \frac{k}{2}(l - l_0)^2$$

b) Find the canonical momenta.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = M l^2 \dot{\theta}$$

$$p_l = \frac{\partial L}{\partial \dot{l}} = M\dot{l}$$

c) Write down the Hamiltonian

$$H = -L + p_\theta \dot{\theta} + p_\ell \dot{\ell}$$

$$= -\frac{M}{2}\left(\dot{\ell}^2 + \ell^2 \dot{\theta}^2\right) + \frac{k}{2}(\ell - \ell_0)^2 + p_\theta \dot{\theta} + p_\ell \dot{\ell}$$

$$= \frac{p_\ell^2}{2M} + \frac{p_\theta^2}{2M\ell^2} + \frac{k}{2}(\ell - \ell_0)^2$$

d) Write down Hamilton's equations of motion and obtain a first order differential equation describing the length of the spring.

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \Rightarrow M\ell^2 \dot{\theta} = \text{constant } L$$

$$\dot{\theta} = \frac{L}{M\ell^2}$$

$$\dot{p}_\ell = -\frac{\partial H}{\partial \ell}$$

$$\dot{p}_\ell = \frac{p_\theta^2}{M\ell^3} - k(\ell - \ell_0)$$

$$M\ddot{\ell} - \frac{L^2}{M\ell^3} + k(\ell - \ell_0) = 0.$$

Integrating gives $\Rightarrow$ $$\underbrace{\frac{M\dot{\ell}^2}{2} + \frac{L^2}{2M\ell^2}}_{\text{K.E.}} + \underbrace{\frac{k}{2}(\ell - \ell_0)^2}_{\text{P.E.}} = \underbrace{E}_{\text{energy (constant of integration)}}$$

Total = 10 marks