Assignment 7

1a) $$x_\mu x_\mu = x'_\mu x'_\mu$$

$$= \lambda_{\mu\sigma} x_\sigma \lambda_{\mu\rho} x_\rho$$

$$= \lambda_{\mu\sigma} \lambda_{\mu\rho} x_\sigma x_\rho$$

$$\therefore \delta_{\sigma\rho} = \lambda_{\mu\sigma} \lambda_{\mu\rho}$$

$$= \lambda^T_{\sigma\mu} \lambda_{\mu\rho}$$

$$\therefore 1 = \lambda^T \lambda \quad \text{or} \quad 1 = \lambda \lambda^T$$

b) $$\lambda \lambda^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma^2(1-\beta^2) & \gamma^2(-i\beta+i\beta) \\ 0 & 0 & \gamma^2(-i\beta+i\beta) & \gamma^2(-\beta^2+1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \lambda \lambda^T = 1$$

2a) A 4-vector M_μ transforms according to $M'_\mu = \lambda_{\mu\nu} M_\nu$.

$\therefore \quad M'_4 = \gamma\,(M_4 - i\beta M_3)$

If $M = (\vec{x}, 0)$, $M' = (\vec{x}', 0)$ $\Rightarrow$ $0 = \gamma(0 - i\beta x_3)$
$= -i\beta\gamma x_3.$

This is nonsense. $\therefore$ $(\vec{x}, 0)$ is not a 4-vector.

b)
$$x_1 = x_1'$$
$$x_2 = x_2'$$
$$x_3 = \gamma\,(x_3' - i\beta x_4')$$
$$x_4 = \gamma\,(x_4' + i\beta x_3')$$

$$\square \equiv \left(\nabla, \frac{\partial}{\partial x_4}\right)$$

Obviously
$$\frac{\partial}{\partial x_1'} = \frac{\partial}{\partial x_1}$$
$$\frac{\partial}{\partial x_2'} = \frac{\partial}{\partial x_2}$$

$$\frac{\partial}{\partial x_3'} = \frac{\partial x_3}{\partial x_3'}\frac{\partial}{\partial x_3} + \frac{\partial x_4}{\partial x_3'}\frac{\partial}{\partial x_4}$$
$$= \gamma\frac{\partial}{\partial x_3} + i\beta\gamma\frac{\partial}{\partial x_4}.$$

$$= \lambda_{33}\frac{\partial}{\partial x_3} + \lambda_{34}\frac{\partial}{\partial x_4}$$

$$\frac{\partial}{\partial x_4'} = \frac{\partial x_3}{\partial x_4'}\frac{\partial}{\partial x_3} + \frac{\partial x_4}{\partial x_4'}\frac{\partial}{\partial x_4}$$

$$= -i\beta\gamma\frac{\partial}{\partial x_3} + \gamma\frac{\partial}{\partial x_4}$$

$$= \lambda_{43}\frac{\partial}{\partial x_3} + \lambda_{44}\frac{\partial}{\partial x_4}.$$

$\therefore\ \Box' = \lambda\,\Box$

Hence $\Box$ is a 4-vector.

c) t is not a Lorentz invariant since a moving observer measures a time

$$t' = \gamma\left(t - \frac{\beta}{c}x_3\right) \neq t.$$

d) A tensor satisfies $T'_{\mu\nu} = \lambda_{\mu\alpha}\lambda_{\nu\beta}T_{\alpha\beta}$

$$\lambda_{\mu\alpha}\lambda_{\nu\beta}\delta_{\alpha\beta} = \lambda_{\mu\alpha}\lambda_{\nu\alpha}$$

$$= \delta_{\mu\nu} \quad \text{from \#1}$$

$$\equiv \delta'_{\mu\nu}$$

$\therefore\ \delta_{\mu\nu}$ is a tensor.

3) $$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} J_\mu$$

Let $\mu = 1 \Rightarrow$ $$\frac{\partial F_{1\nu}}{\partial x_\nu} = \frac{4\pi}{c} J_1$$

$$\frac{\partial F_{11}}{\partial x_1} + \frac{\partial F_{12}}{\partial x_2} + \frac{\partial F_{13}}{\partial x_3} + \frac{\partial F_{14}}{\partial x_4} = \frac{4\pi}{c} J_1$$

$$0 + \frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} + \frac{\partial(-iE_1)}{\partial(ict)} = \frac{4\pi}{c} J_1$$

$$(\nabla \times \vec{B})_1 - \frac{1}{c}\frac{\partial E_1}{\partial t} = \frac{4\pi}{c} J_1$$

Combining this result with eqns. resulting when $\mu = 2, 3$ gives:

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}.$$

Let $\mu = 4 \Rightarrow$ $$\frac{\partial F_{4\nu}}{\partial x_\nu} = \frac{4\pi}{c} J_4$$

$$\frac{\partial F_{41}}{\partial x_1} + \frac{\partial F_{42}}{\partial x_2} + \frac{\partial F_{43}}{\partial x_3} + \frac{\partial F_{44}}{\partial x_4} = \frac{4\pi}{c} J_4.$$

$$i\frac{\partial E_1}{\partial x_1} + i\frac{\partial E_2}{\partial x_2} + i\frac{\partial E_3}{\partial x_3} = \frac{4\pi}{c} ic\rho.$$

$$\therefore \nabla \cdot \vec{E} = 4\pi\rho.$$

4a) We need to show $T'_{\mu\nu} = \lambda_{\mu\alpha} \lambda_{\nu\beta} T_{\alpha\beta}$.

$$\lambda_{\mu\alpha} \lambda_{\nu\beta} T_{\alpha\beta} = \lambda_{\mu\alpha} \lambda_{\nu\beta} \frac{1}{4\pi} \left[F_{\alpha\sigma} F_{\sigma\beta} + \frac{1}{4} \delta_{\alpha\beta} F_{\lambda\rho} F_{\lambda\rho} \right]$$

$$= \frac{1}{4\pi} \Big\{ \lambda_{\mu\alpha} \lambda_{\sigma'\sigma} F_{\alpha\sigma} \lambda_{\sigma'\sigma} \lambda_{\nu\beta} F_{\sigma\beta}$$

$$+ \frac{1}{4} \lambda_{\mu\alpha} \lambda_{\nu\beta} \delta_{\alpha\beta} \lambda_{\lambda'\lambda} \lambda_{\rho'\rho} F_{\lambda\rho} \lambda_{\lambda'\lambda} \lambda_{\rho'\rho} F_{\lambda\rho} \Big\}$$

where we used $\lambda_{\sigma'\sigma} \lambda_{\sigma'\sigma} = 1$ (See #1)

$$= \frac{1}{4\pi} \left\{ F'_{\mu\sigma'} F_{\sigma'\nu} + \frac{1}{4} \delta'_{\mu\nu} F_{\lambda'\rho'} F_{\lambda'\rho'} \right\}$$

since $F_{\mu\nu}$, $\delta_{\mu\nu}$ are tensors

$\therefore \lambda_{\mu\alpha} \lambda_{\nu\beta} T_{\alpha\beta} = T'_{\mu\nu} \Rightarrow T_{\mu\nu}$ is a tensor.

b) $$T_{11} = \frac{1}{4\pi} \left[F_{1\sigma} F_{\sigma 1} + \frac{1}{4} \delta_{11} F_{\lambda\rho} F_{\lambda\rho} \right]$$

$$= \frac{1}{4\pi} \Big[0 + B_3(-B_3) + (-B_2) B_2 + (-iE_1) iE_1$$

$$+ \frac{1}{4} \left(B_3^2 + B_2^2 + B_1^2 - E_1^2 - E_2^2 - E_3^2 \right) 2 \Big]$$

$$= \frac{1}{4\pi} \left[-B_3^2 - B_2^2 + E_1^2 + \frac{1}{2} (\vec{B}^2 - \vec{E}^2) \right]$$

$$\therefore T_{11} = \frac{1}{4\pi} \left[B_1^2 + E_1^2 - \frac{1}{2} (\vec{B}^2 + \vec{E}^2) \right]$$

$$T_{41} = \frac{1}{4\pi}\left[F_{4\sigma} F_{\sigma 1} + \frac{1}{4} \delta_{41} F_{\lambda\rho} F_{\lambda\rho} \right]$$

$$= \frac{1}{4\pi}\left[iE_1 \, 0 + iE_2 (-B_3) + iE_3 B_2 + 0 \, iE_1 \right]$$

$$= \frac{-i}{4\pi}\left(E_2 B_3 - E_3 B_2 \right)$$

$$\therefore T_{41} = \frac{-i}{4\pi} (\vec{E} \times \vec{B})_1$$

$$T_{44} = \frac{1}{4\pi}\left[F_{4\sigma} F_{\sigma 4} + \frac{1}{4} \delta_{44} F_{\lambda\rho} F_{\lambda\rho} \right]$$

$$= \frac{1}{4\pi}\left[iE_1(-iE_1) + iE_2(-iE_2) + iE_3(-iE_3) + 0 + \frac{1}{2}(\vec{B}^2 - \vec{E}^2) \right]$$

$$= \frac{1}{4\pi}\left[\vec{E}^2 + \frac{1}{2}(\vec{B}^2 - \vec{E}^2) \right]$$

$$\therefore T_{44} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$$

5a) $K' = (\rho_0 \vec{E}', 0)$

b) $K' = \lambda K$

or $K = \lambda^T K'$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \rho_0 E_1' \\ \rho_0 E_2' \\ \rho_0 E_3' \\ 0 \end{pmatrix}$$

$$K = \begin{pmatrix} \rho_0 E_1' \\ \rho_0 E_2' \\ \gamma \rho_0 E_3' \\ i\beta\gamma\rho_0 E_3' \end{pmatrix}$$

c) $E_1' = \gamma(E_1 - \beta B_2)$ $\quad B_1' = \gamma(B_1 + \beta E_2)$

$E_2' = \gamma(E_2 + \beta B_1)$ $\quad B_2' = \gamma(B_2 - \beta E_1)$

$E_3' = E_3$ $\quad B_3' = B_3$

d) J is a 4-vector.

$\therefore J = \lambda^T J'$ where $J' = (0, ic\rho_0)$

$J_4 = \lambda^T_{4\nu} J'_\nu$

$ic\rho = \lambda^T_{44}\, ic\rho_0$

$\therefore \rho = \gamma \rho_0$

e) $K_1 = \rho_0 E_1'$

$$= \frac{\rho}{\gamma} \gamma (E_1 - \beta B_2)$$

$$= \rho (E_1 - \beta B_2)$$

Now $\vec{\beta} \equiv (0, 0, \frac{u}{c})$

$\therefore K_1 = \rho [E_1 + (\vec{\beta} \times \vec{B})_1]$

Similarly $K_2 = \rho [E_2 + (\vec{\beta} \times \vec{B})_2]$

$K_3 = \gamma \rho_0 E_3'$

$$= \gamma \frac{\rho}{\gamma} E_3$$

$$= \rho [E_3 + (\vec{\beta} \times \vec{B})_3]$$

$\therefore$ Lorentz force density $\vec{K} = \rho (\vec{E} + \vec{\beta} \times \vec{B})$.