Assignment 5

1a) $\nabla \cdot \left(\frac{\vec{J}}{r}\right) = \frac{1}{r} \nabla \cdot \vec{J} + \vec{J} \cdot \nabla \left(\frac{1}{r}\right)$ (1)

$\nabla' \cdot \left(\frac{\vec{J}}{r}\right) = \frac{1}{r} \nabla' \cdot \vec{J} + \vec{J} \cdot \nabla' \left(\frac{1}{r}\right)$ (2)

using vector identity from back of notes

But $\nabla' \left(\frac{1}{r}\right) = \frac{-1}{r^2} \nabla' r$

$= \frac{-1}{r^2} \nabla' |\vec{r} - \vec{r}'|$

$= \frac{-1}{r^2} \left(-\nabla |\vec{r} - \vec{r}'|\right)$ To prove $\nabla r = -\nabla' r$ let $\vec{r} = (x, y, z)$, $\vec{r}' = (x', y', z')$

$= \frac{1}{r^2} \nabla r$

$= -\nabla \left(\frac{1}{r}\right)$

$\therefore (2) \Rightarrow \nabla' \cdot \left(\frac{\vec{J}}{r}\right) = \frac{1}{r} \nabla' \cdot \vec{J} - \vec{J} \cdot \nabla \left(\frac{1}{r}\right)$

$\vec{J} \cdot \nabla \left(\frac{1}{r}\right) = \frac{1}{r} (\nabla' \cdot \vec{J}) - \nabla' \cdot \left(\frac{\vec{J}}{r}\right)$

Substitute this into (1) & get:

$\nabla \cdot \left(\frac{\vec{J}}{r}\right) = \frac{1}{r} (\nabla \cdot \vec{J}) + \frac{1}{r} (\nabla' \cdot \vec{J}) - \nabla' \cdot \left(\frac{\vec{J}}{r}\right)$

b) $\nabla \cdot \vec{J}(\vec{r}', t_r) = \frac{\partial \vec{J}}{\partial t_r} \cdot \nabla t_r$ where $t_r = t - \frac{r}{c}$

$$= -\frac{1}{c} \frac{\partial \vec{J}}{\partial t_r} \cdot \nabla r$$

c) $\nabla' \cdot \vec{J}(\vec{r}', t_r) = \partial' \cdot \vec{J}(x', y', z', t_r)$

$$= \frac{\partial J_{x'}(\vec{r}', t_r)}{\partial x'} + \frac{\partial J_{y'}(\vec{r}', t_r)}{\partial y'} + \frac{\partial J_{z'}(\vec{r}', t_r)}{\partial z'}$$

$$+ \frac{\partial \vec{J}}{\partial t_r} \cdot \nabla' t_r$$

$$= -\frac{\partial \rho}{\partial t}(\vec{r}', t_r) + \frac{\partial \vec{J}}{\partial t_r} \cdot \nabla' t_r$$

using the Continuity Equation

$$= -\frac{\partial \rho}{\partial t}(\vec{r}', t_r) - \frac{1}{c} \frac{\partial \vec{J}}{\partial t_r} \cdot (\nabla' r)$$

d) $\nabla \cdot \vec{A} = \frac{1}{c} \int \nabla \cdot \left(\frac{\vec{J}(\vec{r}', t_r)}{r} \right) d^3 r'$

$$= \frac{1}{c} \int \left[\frac{\nabla \cdot \vec{J}}{r} + \frac{\nabla' \cdot \vec{J}}{r} - \nabla' \cdot \left(\frac{\vec{J}}{r} \right) \right] d^3 r' \quad \text{using (a)}$$

$$= \frac{1}{c} \int \left[\frac{1}{r} \left(\frac{-1}{c} \right) \frac{\partial \vec{J}}{\partial t_r} \cdot \nabla r + \frac{1}{r} \left(-\frac{\partial \rho}{\partial t} - \frac{1}{c} \frac{\partial \vec{J}}{\partial t_r} \cdot (\nabla' r) \right) \right] d^3 r'$$

$$- \frac{1}{c} \int \nabla' \cdot \left(\frac{\vec{J}}{r} \right) d^3 r'$$

$$\nabla \cdot \vec{A} = \frac{1}{c} \int_V \frac{-1}{r} \frac{\partial \rho}{\partial t} \, d^3 r' - \frac{1}{c} \int_S \frac{\vec{J}}{r} \cdot d^2 \vec{r}'$$

S ← surface at infinity where $\frac{\vec{J}}{r} = 0$.

$$= -\frac{1}{c} \frac{\partial}{\partial t} \int_V \frac{\rho(\vec{r}', t_r) \, d^3 r'}{r}$$

$$\therefore \nabla \cdot \vec{A} = -\frac{1}{c} \frac{\partial \Phi}{\partial t}$$

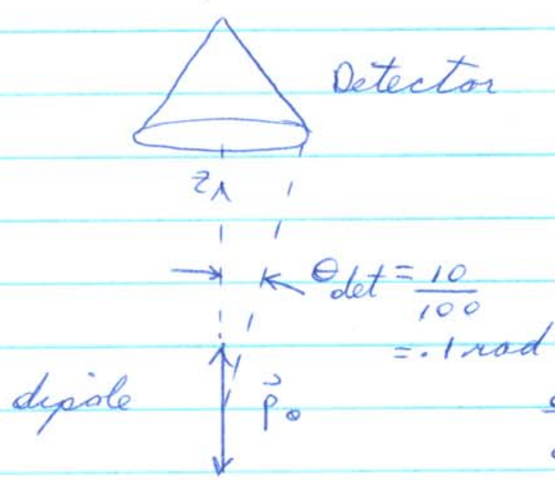
2a) $r \gg \lambda \gg s$

r = distance from oscillating dipole to observer

λ = wavelength of radiation
~ 5000 Å for visible light

s = size of dipole
$\sim$ Å for molecules

b).



Power radiated by dipole into solid angle $d\Omega$ is

$$\frac{dP}{d\Omega} = r^2 \langle \vec{S} \rangle \cdot \hat{r}$$

$$= K \sin^2 \theta \quad K = \text{constant}$$

Area of detector $= \pi r^2$
$= \pi (10\,cm)^2$
$= 100\pi\ cm^2$

Surface area of 1 meter sphere $= 4\pi (100\,cm)^2$
$= 40{,}000\,\pi\,cm^2.$

$\therefore$ fraction of solid angle seen by detector is
$\frac{100\pi}{40{,}000\pi} = \frac{1}{400} = 2.5 \times 10^{-3}.$

Power detected $P_{det} = \int_0^{\theta_{det}} K \sin^2\theta \; 2\pi \sin\theta \, d\theta.$

$$= 2\pi K \int_0^{\theta_{det}} -(1-\cos^2\theta)\, d\cos\theta$$

$$= 2\pi K \left(\frac{\cos^3\theta}{3} - \cos\theta \right)_0^{\theta_{det}}$$

$$= 2\pi K \left(\frac{\cos^3\theta_{det} - 1}{3} - \cos\theta_{det} + 1 \right)$$

$$= 2\pi K \left(\frac{\cos^3 0.1}{3} - \cos 0.1 + \frac{2}{3} \right)$$

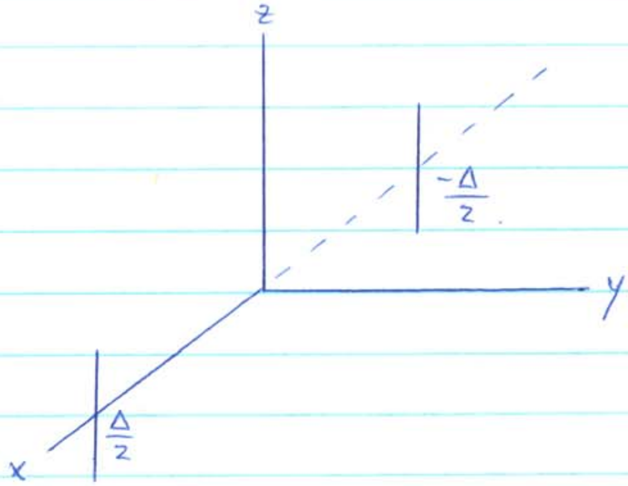
$$= 5 \times 10^{-5} \pi K.$$

Total power emitted by dipole $P_{TOT} = \frac{8\pi}{3} K.$

$\therefore$ fraction of power detected $= \frac{5 \times 10^{-5}}{8/3} = 1.9 \times 10^{-5}.$

$\therefore$ fraction of power detected is less than fraction of solid angle seen by detector since dipole radiates preferentially in plane $\perp$ to $\vec{p}$.

3.



Current in each antenna $\vec{I} = \hat{z}\, I_o \cos\omega t \sin k\left(\frac{d}{2} - |z|\right)$

$$\vec{A}(\vec{r},t) = \underbrace{\vec{A}_1(\vec{r},t)}_{\text{due to antenna 1}} + \underbrace{\vec{A}_2(\vec{r},t)}_{\text{due to antenna 2}}$$

$$\vec{r}\,'_{\pm} = \left(\pm\frac{\Delta}{2},\, 0,\, z'\right)$$

$$\vec{r} = r(\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)$$

$$\imath_{\pm} = |\vec{r} - \vec{r}\,'_{\pm}|$$

$$\imath_{\pm}^2 = \left(r\sin\theta\cos\phi \mp \frac{\Delta}{2}\right)^2 + r^2\sin^2\theta\sin^2\phi + (r\cos\theta - z')^2$$

$$= r^2 \mp \Delta r\sin\theta\cos\phi + \left(\frac{\Delta}{2}\right)^2 - 2rz'\cos\theta + z'^2$$

$$= r^2\left[1 \mp \frac{\Delta}{r}\sin\theta\cos\phi - \frac{2z'}{r}\cos\theta + \frac{\left(\frac{\Delta}{2}\right)^2 + z'^2}{r^2}\right]$$

Assuming $r \gg d, \Delta$ we get:

$$r_{\pm} \simeq r\left[1 \mp \frac{\Delta}{2r}\sin\theta\cos\phi - \frac{z'}{r}\cos\theta\right]$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int \frac{\vec{I}(\vec{r}', t - r_+/c)}{r_+}\,dz' + \frac{1}{c}\int \frac{\vec{I}(\vec{r}', t - r_-/c)}{r_-}\,dz'$$

$$\simeq \hat{z}\frac{I_0}{rc}\left\{\int_{-d/2}^{d/2} e^{i\omega(t - r_+/c)} \sin k\left(\frac{d}{2} - |z'|\right) dz'\right.$$

$$\left. + \int_{-d/2}^{d/2} e^{i\omega(t - r_-/c)} \sin k\left(\frac{d}{2} - |z'|\right) dz'\right\}$$

Now $e^{i\omega(t - r_{\pm}/c)} = e^{i\omega(t - r/c)} \cdot e^{i\frac{\omega}{c}z'\cos\theta} \cdot e^{\pm i\frac{\omega\Delta}{2c}\sin\theta\cos\phi}$

Define $\gamma(\theta,\phi) \equiv \frac{\omega\Delta}{2c}\sin\theta\cos\phi$.

$$\therefore \vec{A}(\vec{r},t) = \hat{z}\,\frac{I_0}{rc}\, e^{i\omega(t - r/c)} \int_{-d/2}^{d/2} e^{i\frac{\omega}{c}z'\cos\theta} \sin k\left(\frac{d}{2} - |z'|\right) dz'$$

$$\cdot\left(e^{i\gamma} + e^{-i\gamma}\right)$$

The integral was evaluated in the notes.

$$\int_{-d/2}^{d/2} e^{i\frac{\omega}{c}z'\cos\theta} \sin k\left(\frac{d}{2} - |z'|\right) dz' = \frac{2}{k}F(\theta)$$

where $F(\theta) \equiv \dfrac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin^2\theta}$

$$\therefore \vec{A}(\vec{r},t) = \hat{z}\,\frac{4I_0}{rkc}\,e^{i\omega(t-r/c)}\,F(\theta)\,\cos\gamma(\theta,\phi).$$

$$\text{or } \boxed{\vec{A}(\vec{r},t) = \frac{4I_0}{rkc}\,e^{i\omega(t-r/c)}\,F(\theta)\cos\gamma(\theta,\phi)\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)}$$

b) Electric field $\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \nabla\Phi$ where $\Phi = 0$ since $\rho = 0$.

$$\boxed{\therefore \vec{E}(\vec{r},t) = -\frac{4iI_0\omega}{rkc^2}\,e^{i\omega(t-r/c)}\,F(\theta)\cos\gamma\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)}$$

Magnetic field $\vec{B} = \nabla\times\vec{A}$.

We are interested in energy that is radiated away. Hence we keep only terms in B of $O\left(\frac{1}{r}\right)$. Higher order terms $\left(\frac{1}{r^2}\right)$ do not contribute to radiation since $\lim_{r\to\infty} r^2|\vec{E}\times\vec{B}| = 0$. if $B \sim \frac{1}{r^2}$.

$$\therefore \vec{B}(\vec{r},t) = \hat{\phi}\,\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)$$

$$rA_\theta = -\frac{4I_0}{kc} e^{i\omega(t-r/c)} F(\theta)\cos\gamma \sin\theta$$

$$\frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} = i\frac{4I_0\omega}{rkc^2} e^{i\omega(t-r/c)} F(\theta)\cos\gamma\sin\theta.$$

$$\therefore \boxed{\vec{B}(\vec{r},t) = 4i\frac{I_0\omega}{rkc^2} e^{i\omega(t-r/c)} F(\theta)\cos\gamma\sin\theta\,\hat{\phi}}$$

c) Time Averaged Poynting Vector

$$\langle\vec{S}\rangle = \frac{c}{8\pi} \operatorname{Re}(\vec{E}\times\vec{B}^*)$$

Radial flow of energy is:

$$\langle\vec{S}\rangle_r = \frac{c}{8\pi}\operatorname{Re}\left(E_\theta\,\hat{\theta}\times B_\phi^*\,\hat{\phi}\right)$$

$$= \hat{r}\frac{c}{8\pi}\,16\left(\frac{I_0\omega}{rkc^2}\right)^2 F^2(\theta)\cos^2\gamma\sin^2\theta.$$

$$\langle\vec{S}\rangle_r = \frac{\hat{r}}{r^2}\frac{2c}{\pi}\left(\frac{I_0}{c}\right)^2 F^2(\theta)\cos^2\gamma\sin^2\theta$$

Power radiated dP into solid angle $d\Omega$ is

$$\frac{dP}{d\Omega} = r^2\langle\vec{S}\rangle\cdot\hat{r}$$

$$\boxed{\frac{dP}{d\Omega} = \frac{2I_0^2}{\pi c}F^2(\theta)\cos^2\gamma\sin^2\theta.}$$

or $\frac{dP}{d\Omega} = \frac{2 I_0^2}{\pi c} \left(\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin\theta} \right)^2 \cos^2\left(\frac{k\Delta}{2}\sin\theta\cos\phi\right)$

d) Half wave antenna $kd = \pi$.

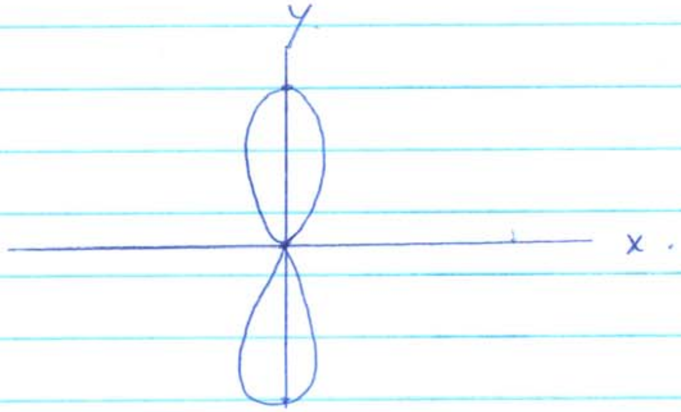
If two antennae are $\frac{\lambda}{2}$ apart then:

$$\frac{k\Delta}{2} = \frac{1}{2}\,\frac{2\pi}{\lambda}\cdot\frac{\lambda}{2} = \frac{\pi}{2}$$

$$\frac{dP}{d\Omega} = \frac{2 I_0^2}{\pi c} \left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right)^2 \cos^2\left(\frac{\pi}{2}\sin\theta\cos\phi\right)$$

In xy plane $\theta = \frac{\pi}{2}$.

$$\therefore \frac{dP}{d\Omega} = \frac{2 I_0^2}{\pi c} \cos^2\left(\frac{\pi}{2}\cos\phi\right)$$



Above is a polar plot of $\frac{dP}{d\Omega}(\theta = \frac{\pi}{2})$ vs. ϕ.

Note energy is radiated preferentially along $\hat{y}$.

4 a) Force on electron $\dfrac{e^2}{a_0^2} = m\,a$.

$\therefore$ acceleration $a = \dfrac{e^2}{m a_0^2}$

$\therefore$ power radiated $P = \dfrac{2}{3}\dfrac{e^2}{c^3}\left(\dfrac{e^2}{m a_0^2}\right)^2$

$$= \frac{2}{3}\frac{e^6}{m^2 c^3 a_0^4}$$

$$P = \frac{2}{3}\frac{(4.8\times10^{-10}\,esu)^6}{(9.11\times10^{-28}\,gm)^2(3\times10^{10}\,cm/sec)^3(.5\times10^{-8}\,cm)^4}$$

$\therefore P = .58$ erg/sec

b) Energy of H atom in ground state is

$$E = \frac{1}{2}\frac{e^2}{a_0}$$

$$= \frac{1}{2}\frac{(4.8\times10^{-10}\,esu)^2}{.5\times10^{-8}\,cm}$$

$$= 2.30\times10^{-11}\ \text{erg}.$$

c) $\therefore$ time for electron to radiate away its energy is

$$\tau \simeq \frac{E}{P} = \frac{3}{4}\frac{m^2c^3a_0^3}{e^4}$$

$$= 4\times10^{-11}\ \text{sec}.$$

5a) Current = electron charge × # revs. an electron makes per second × # electrons N.

$$I = e \times \frac{v}{2\pi r} \times N.$$

$$N = \frac{I \; 2\pi r}{e v} \quad \text{where } v \simeq c$$

$$N = \frac{10^{-6}}{10} \; \frac{2\pi \times 10^{5} \text{cm.}}{4.8 \times 10^{-10} \text{esu}}$$

$$= 1.3 \times 10^{8} \text{ electrons}$$

b) Power radiated by beam $P = N \frac{2}{3} \frac{e^2 a^2}{c^3} \gamma^6$

acceleration $a = \frac{v^2}{R} \simeq \frac{c^2}{R}.$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$v = .9999c \implies \gamma = 70.7$

$\therefore$ power $P = N \frac{2}{3} \frac{e^2 c^4}{c^3 R^2} \gamma^6$

$$= \frac{2}{3} N \frac{e^2 c}{R^2} \gamma^6$$

$$= \frac{2}{3} \times \frac{1.3 \times 10^{8} \, (4.8 \times 10^{-10})^2 \, 3 \times 10^{10} \times (70.7)^6}{(10^5)^2}$$

$P = 7.5$ erg/sec.

$\therefore$ power radiated by beam is 7.5 erg/sec.