Assignment 3

101. $\psi = 5\cos(10y + 6t)$

Amplitude $= 5$

Wavevector $|\vec{k}| = 10 \text{ cm}^{-1}$

Wavelength $\lambda = \frac{2\pi}{10} = \frac{\pi}{5}$ cm.

Angular Frequency $\omega = 6$ rad/sec

Frequency $\nu = \frac{6}{2\pi} = \frac{3}{\pi} \text{ sec}^{-1}$

Period $T = \frac{\pi}{3}$ sec

Speed of Wave $v = \frac{\omega}{k} = \frac{6}{10} = .6$ cm/sec

Propagation direction $\hat{k} = -\hat{y}$

102a. Maxwells Eqns. in Linear Nonconducting Media
(No charges or currents)

$$\nabla \cdot \vec{E} = 0 \quad (1) \qquad \nabla \cdot \vec{B} = 0 \quad (3)$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad (2) \qquad \nabla \times \vec{B} = \frac{\epsilon\mu}{c}\frac{\partial \vec{E}}{\partial t} \quad (4)$$

$$\nabla \times (\nabla \times \vec{B}) = \frac{\epsilon\mu}{c}\frac{\partial}{\partial t}(\nabla \times \vec{E})$$

$$\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \frac{\epsilon\mu}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right) \quad \text{using (2)}$$

$$-\nabla^2 \vec{B} = -\frac{\epsilon\mu}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{using (3)}$$

$$\nabla^2 \vec{B} - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0. \qquad (5)$$

b. Let $\vec{B} = \vec{B}_0 e^{i(\vec{k}.\vec{r} - \omega t)}$

$$\nabla^2 \vec{B} = -k^2 \vec{B}$$

$$\frac{\partial^2 \vec{B}}{\partial t^2} = -\omega^2 \vec{B}.$$

$$\therefore (5) \Rightarrow \quad -k^2 \vec{B} + \frac{\epsilon\mu}{c^2}\omega^2 \vec{B} = 0.$$

$$k^2 = \frac{\epsilon\mu}{c^2}\omega^2$$

$$k = \sqrt{\epsilon\mu}\,\frac{\omega}{c}$$

$\therefore$ plane wave solution for $\vec{B}$ satisfies (5) provided above eqn. holds.

c. Let $\vec{B} = \vec{B}_0 e^{i(\vec{k}.\vec{r} - \omega t)}$

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}.\vec{r} - \omega t)}$$

$$\nabla \times \vec{B} = \frac{\mu\epsilon}{c}\frac{\partial \vec{E}}{\partial t} \Rightarrow \quad i\vec{k} \times \vec{B}_0 = \frac{\mu\epsilon}{c}(-i\omega)\vec{E}_0$$

$$\vec{k} \times \vec{B}_0 = -\mu\epsilon\frac{\omega}{c}\vec{E}_0.$$

or $\hat{k} \times \vec{B}_o = -\sqrt{\mu\epsilon}\ \vec{E}_o$ using result from b

$\therefore\ \hat{k}$, $\vec{B}_o$ & $\vec{E}_o$ are mutually perpendicular.

Also $\Rightarrow |\vec{B}_o| = \sqrt{\mu\epsilon}\ |\vec{E}_o|$.

10 3. Field Energy Density

$$u = \frac{1}{8\pi}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right) \quad \text{for real fields.}$$

$$= \frac{1}{8\pi}\left((Re\,\vec{E})\cdot(Re\,\vec{D}) + (Re\,\vec{B})\cdot(Re\,\vec{H})\right)$$

Averaging over time we get:

$$\langle u \rangle = \frac{1}{8\pi}\langle (Re\,\vec{E})\cdot(Re\,\vec{D}) + (Re\,\vec{B})\cdot(Re\,\vec{H}) \rangle$$

For a linear media we shall show the above expression is equivalent to

$$\langle u \rangle = \frac{1}{16\pi}\left(\vec{E}\cdot\vec{D}^* + \vec{B}\cdot\vec{H}^*\right) \quad \text{where fields are}$$

complex.

Solution

Let $\begin{cases} \vec{E} = \vec{E}_o e^{-i\omega t} \\ \vec{B} = \vec{B}_o e^{-i\omega t} \end{cases}$ where $\begin{cases} \vec{E}_o = \vec{E}_1 + i\vec{E}_2 \\ \vec{B}_o = \vec{B}_1 + i\vec{B}_2 . \end{cases}$

Also $\vec{D} = \epsilon \vec{E}$ & $\vec{B} = \mu \vec{H}$.

$\therefore \vec{E} \cdot \vec{D}^* + \vec{B} \cdot \vec{H}^*$

$= (\vec{E}_1 + i \vec{E}_2) e^{-i\omega t} \cdot \epsilon (\vec{E}_1 - i \vec{E}_2) e^{i\omega t}$

$+ (\vec{B}_1 + i \vec{B}_2) e^{-i\omega t} \cdot \frac{1}{\mu} (\vec{B}_1 - i \vec{B}_2) e^{i\omega t}$

$= \epsilon (\vec{E}_1 + i \vec{E}_2) \cdot (\vec{E}_1 - i \vec{E}_2) + \frac{1}{\mu} (\vec{B}_1 + i \vec{B}_2) \cdot (\vec{B}_1 - i \vec{B}_2)$

$= \epsilon (E_1^2 + E_2^2) + \frac{1}{\mu} (B_1^2 + B_2^2) \qquad (1)$

$Re(\vec{E}) = Re \left\{ (\vec{E}_1 + i \vec{E}_2) e^{-i\omega t} \right\}$

$= Re \left\{ (\vec{E}_1 + i \vec{E}_2)(\cos \omega t - i \sin \omega t) \right\}$

$= \vec{E}_1 \cos \omega t + \vec{E}_2 \sin \omega t$

Similarly $Re\, \vec{B} = \vec{B}_1 \cos \omega t + \vec{B}_2 \sin \omega t$

$(Re\, \vec{E}) \cdot (Re\, \vec{D}) + (Re\, \vec{B}) \cdot (Re\, \vec{H})$

$= (\vec{E}_1 \cos \omega t + \vec{E}_2 \sin \omega t) \cdot \epsilon (\vec{E}_1 \cos \omega t + \vec{E}_2 \sin \omega t)$

$+ (\vec{B}_1 \cos \omega t + \vec{B}_2 \sin \omega t) \cdot \frac{1}{\mu} (\vec{B}_1 \cos \omega t + \vec{B}_2 \sin \omega t)$

$\therefore 2 \langle (Re\, \vec{E}) \cdot (Re\, \vec{D}) + (Re\, \vec{B}) \cdot (Re\, \vec{H}) \rangle$

$= 2 \left\{ \epsilon \left(E_1^2 \langle \cos^2 \omega t \rangle + E_2^2 \langle \sin^2 \omega t \rangle + 2 \vec{E}_1 \cdot \vec{E}_2 \langle \cos \omega t \cdot \sin \omega t \rangle \right) \right.$

$\left. + \frac{1}{\mu} \left(B_1^2 \langle \cos^2 \omega t \rangle + B_2^2 \langle \sin^2 \omega t \rangle + 2 \vec{B}_1 \cdot \vec{B}_2 \langle \cos \omega t \sin \omega t \rangle \right) \right\}$

$$= 2\left\{ \epsilon\left(E_1^2 \frac{1}{2} + E_2^2 \frac{1}{2} + 0 \right) \right.$$

$$\left. + \frac{1}{\mu}\left(B_1^2 \frac{1}{2} + B_2^2 \frac{1}{2} + 0 \right) \right\}$$

$$= \epsilon(E_1^2 + E_2^2) + \frac{1}{\mu}(B_1^2 + B_2^2) \quad (2)$$

$\therefore (1) + (2) \Rightarrow \langle u \rangle = \frac{1}{16\pi}\left(\vec{E} \cdot \vec{D}^* + \vec{B} \cdot \vec{H}^* \right)$ for complex fields in linear media.

5 4. From notes $\vec{E} = \vec{E}_0 e^{i(kz - \omega t)}$, $k = \alpha + i\beta$ where:

$$\alpha = \frac{\omega}{c}\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1 + \left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2} + 1 \right]^{1/2}$$

$$\beta = \frac{\omega}{c}\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1 + \left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2} - 1 \right]^{1/2}$$

We consider case where $\frac{4\pi\sigma}{\omega\epsilon} \ll 1$.

$$\Rightarrow \sqrt{1 + \left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2} \simeq 1 + \frac{1}{2}\left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2$$

$$\therefore \alpha \simeq \frac{\omega}{c}\sqrt{\frac{\mu\epsilon}{2}}\left[1 + \frac{1}{2}\left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2 + 1 \right]^{1/2}$$

$$= \frac{\omega}{c}\sqrt{\mu\epsilon}\left[1 + \frac{1}{4}\left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2 \right]^{1/2}$$

$$\alpha \simeq \frac{\omega}{c}\sqrt{\mu\epsilon}\left[1 + \frac{1}{8}\left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2\right]$$

$$= \frac{\omega}{c}\sqrt{\mu\epsilon}\left[1 + \frac{1}{2}\left(\frac{2\pi\sigma}{\omega\epsilon}\right)^2\right]$$

$$\beta \simeq \frac{\omega}{c}\sqrt{\frac{\mu\epsilon}{2}}\left[1 + \frac{1}{2}\left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2 - 1\right]^{1/2}$$

$$= \frac{\omega}{c}\sqrt{\frac{\mu\epsilon}{2}}\ \frac{1}{\sqrt{2}}\ \frac{4\pi\sigma}{\omega\epsilon}$$

$$= \frac{2\pi\sigma}{c}\sqrt{\frac{\mu}{\epsilon}}$$

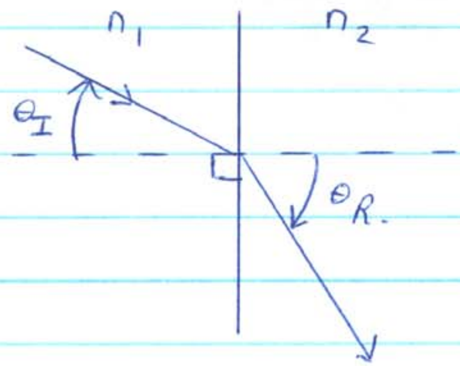
5 5.

Radiation	Frequency $\frac{\omega}{2\pi}$	δ
UV	10^{16} Hz.	6.74×10^{-8} cm.
visible	10^{14}	6.74×10^{-7}
infrared	10^{13}	2.13×10^{-6}
microwave	10^{10}	6.74×10^{-5}
TV, FM	10^{6}	6.74×10^{-3}
radiofrequency	10^{4}	6.74×10^{-2}

$$\delta = \frac{c}{\sqrt{2\pi\sigma\mu\omega}} \qquad \text{or} \qquad \delta = \frac{3 \times 10^{10}}{2\pi\sqrt{5 \times 10^{17}\,\nu}} \text{ cm.}$$

$$= \frac{c}{2\pi\sqrt{\sigma\mu\nu}} \qquad\qquad = 6.74\,\nu^{-1/2} \text{ cm.}$$

5 6.



Snell's Law: $n_1 \sin\theta_I = n_2 \sin\theta_R$.

When $\theta_R = 90°$, $\theta_I = \theta_c \Rightarrow n_1 \sin\theta_c = n_2 \sin 90°$

$$\sin\theta_c = \frac{n_2}{n_1}$$

If $n_1 = n_{glass} = 1.5$ & $n_2 = n_{air} = 1$.

$$\sin\theta_c = \frac{1}{1.5}$$

$$\theta_c = 41.8°.$$

7.

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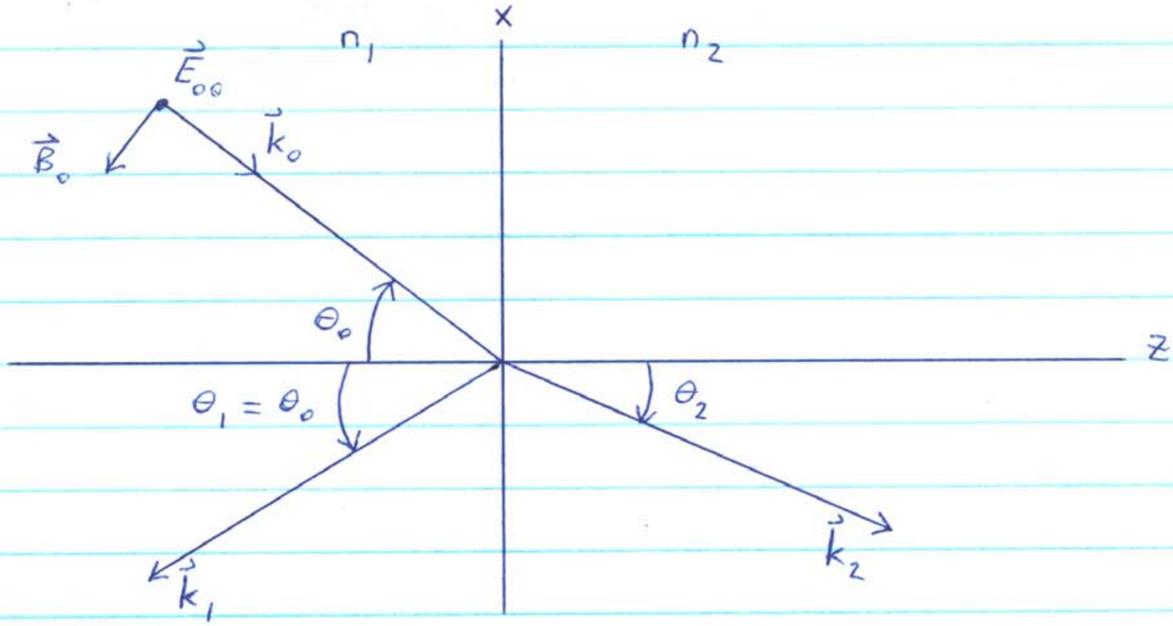


$\hat{k}_0 = (-\sin\theta_0, 0, \cos\theta_0)$

$\hat{k}_1 = (-\sin\theta_0, 0, -\cos\theta_0)$

$\hat{k}_2 = (-\sin\theta_2, 0, \cos\theta_2)$

Incident wave is s polarized. i.e. $\vec{E}_{00} = (0, E_{00}, 0)$.

First we write down equations for each wave specifying that the electric field is perpendicular to the propagation vector.

$\hat{k}_0 \cdot \vec{E}_{00} = 0$ (Obviously satisfied)

$\hat{k}_1 \cdot \vec{E}_{10} = 0$ or $-E_{10x}\sin\theta_0 - E_{10z}\cos\theta_0 = 0$ (0a)

$\hat{k}_2 \cdot \vec{E}_{20} = 0$ or $-E_{20x}\sin\theta_2 + E_{20z}\cos\theta_2 = 0.$ (0b)

Next boundary conditions are applied. (as in notes)

1) Perpendicular component of $\vec{D} = \epsilon \vec{E}$ is continuous.

$$\epsilon_1 \left[\vec{E}_{00} + \vec{E}_{10} \right]_z = \epsilon_2 \left[\vec{E}_{20} \right]_z$$

or $n_1^2 \; E_{10z} = n_2^2 \; E_{20z}$ $\qquad (1)$

2) Tangential component of $\vec{E}$ is continuous.

$$\left[\vec{E}_{00} + \vec{E}_{10} \right]_{x,y} = \left[\vec{E}_{20} \right]_{x,y}$$

or $\quad E_{10x} = E_{20x} \qquad (2a)$

$\quad E_{00} + E_{10y} = E_{20y} \qquad (2b)$

3) Perpendicular component of $\vec{B}$ is continuous
Tangential " " $\vec{H} = \vec{B}$ " ", $(\mu = 1)$

$$n_1 \left(\hat{k}_0 \times \vec{E}_{00} \right) + n_1 \left(\hat{k}_1 \times \vec{E}_{10} \right) = n_2 \left(\hat{k}_2 \times \vec{E}_{20} \right)$$

$$\hat{k}_0 \times \vec{E}_{00} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\theta_0 & 0 & \cos\theta_0 \\ 0 & E_{00} & 0 \end{vmatrix}$$

$$= \left(-E_{00} \cos\theta_0 ,\; 0 ,\; -E_{00} \sin\theta_0 \right)$$

$$\hat{k}_1 \times \vec{E}_{10} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta_0 & 0 & -\cos\theta_0 \\ E_{10x} & E_{10y} & E_{10z} \end{vmatrix}$$

$$= \left(E_{10y}\cos\theta_0,\ E_{10z}\sin\theta_0 + E_{10x}\cos\theta_0,\ -E_{10y}\sin\theta_0 \right)$$

$$\hat{k}_2 \times \vec{E}_{20} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta_2 & 0 & \cos\theta_2 \\ E_{20x} & E_{20y} & E_{20z} \end{vmatrix}$$

$$= \left(-E_{20y}\cos\theta_2,\ E_{20z}\sin\theta_2 + E_{20x}\cos\theta_2,\ -E_{20y}\sin\theta_2 \right)$$

$$-n_1 E_{00}\cos\theta_0 + n_1 E_{10y}\cos\theta_0 = -n_2 E_{20y}\cos\theta_2 \qquad (3a)$$

$$n_1 E_{10z}\sin\theta_0 + n_1 E_{10x}\cos\theta_0 = n_2 E_{20z}\sin\theta_2 + n_2 E_{20x}\cos\theta_2 \quad (3b)$$

$$-n_1 E_{00y}\sin\theta_0 - n_1 E_{10y}\sin\theta_0 = -n_2 E_{20y}\sin\theta_2 \qquad (3c)$$

Equations 0a, 0b, 1, 2a, 3b involve E_{10x}, E_{10z}, E_{20x} & E_{20z}.

Using (1) & (2a) in (0a) we get:

$$-E_{20x}\sin\theta_0 - \frac{n_2^2}{n_1^2} E_{20z}\cos\theta_0 = 0.$$

Multiplying (0b) by $\frac{\sin\theta_0}{\sin\theta_2}$ we get: (For oblique incidence $\theta_2 \neq 0$).

$$-E_{20x}\sin\theta_0 + E_{20x}\cos\theta_2 \frac{\sin\theta_0}{\sin\theta_2} = 0.$$

Subtracting the last two equations gives:

$$E_{20z}\left[-\frac{n_2^2}{n_1^2}\cos\theta_0 - \cos\theta_2\frac{\sin\theta_0}{\sin\theta_2}\right] = 0.$$

For this to be true $\forall\ \theta_0$ we must have $E_{20z} = 0$.

$\Rightarrow E_{10z} = E_{20z} = E_{10x} = E_{20x} = 0$. $\therefore$ all 3 waves are s polarized!

One also sees that (3b) is then trivially satisfied.

Equations (2b), (3a) & (3c) involve E_{10y} & E_{20y}.

Using Snell's law we see that (2b) & (3c) are identical.

$$E_{00} + E_{10y} = E_{20y} \qquad (2b)$$

$$-n_1 E_{00}\cos\theta_0 + n_1 E_{10y}\cos\theta_0 = -n_2 E_{20y}\cos\theta_2 \qquad (3a)$$

or $-E_{00} + E_{10y} = -\beta\alpha E_{20y}$ (3a') where

$$\beta \equiv \frac{n_2}{n_1} \quad \& \quad \alpha \equiv \frac{\cos\theta_2}{\cos\theta_0}.$$

$(2b) - (3a') \Rightarrow \quad 2E_{00} = E_{20y}(1+\beta\alpha)$

$$\boxed{E_{20y} = \frac{2}{1+\beta\alpha}E_{00}}$$

Substituting E_{20y} into (2b) gives:

$$E_{00} + E_{10y} = \frac{2}{1+\beta\alpha} E_{00}$$

$$E_{10y} = \frac{2 - 1 - \beta\alpha}{1+\beta\alpha} E_{00}$$

$$\boxed{E_{10y} = \frac{1-\beta\alpha}{1+\beta\alpha} E_{00}}$$

Reflection Coefficient $R = \dfrac{E_{10}^2}{E_{00}^2}$

$$R = \left(\frac{1-\beta\alpha}{1+\beta\alpha}\right)^2$$

Transmission Coefficient $T = \dfrac{E_{20}^2 \; n_2 \cos\theta_2}{E_{00}^2 \; n_1 \cos\theta_0}$

$$T = \frac{4\alpha\beta}{(1+\beta\alpha)^2}$$

Check: $R + T = \dfrac{(1-\beta\alpha)^2 + 4\alpha\beta}{(1+\beta\alpha)^2}$

$$= \frac{1 - 2\beta\alpha + \beta^2\alpha^2 + 4\alpha\beta}{(1+\beta\alpha)^2}$$

$$= 1$$

One can also check that R & T agree with expressions derived for case of light normally incident on a boundary.

$$\theta_0 = \theta_2 = 0 \implies \alpha = \frac{\cos\theta_2}{\cos\theta_0} = 0$$

$$\beta = \frac{n_2}{n_1}$$

$$\therefore R = \left(\frac{1 - \frac{n_2}{n_1}}{1 + \frac{n_2}{n_1}} \right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

$$T = \frac{4 \frac{n_2}{n_1}}{\left(1 + \frac{n_2}{n_1}\right)^2} = \frac{4 n_1 n_2}{(n_1 + n_2)^2}$$

These results agree with those found in notes.

Next let's plot R & T versus the angle of incidence. First we show there is no Brewster's angle.

Proof: Suppose there is some angle of incidence $\theta_I \neq 90°$ where $R = 0$.

$$\therefore 1 - \beta\alpha = 0.$$

$$\alpha = \frac{1}{\beta}$$

$$\frac{\cos\theta_2}{\cos\theta_0} = \frac{n_1}{n_2}.$$

$$\cos^2\theta_2 = \frac{n_1^2}{n_2^2}\cos^2\theta_1$$

$$1-\sin^2\theta_2 = \frac{n_1^2}{n_2^2}(1-\sin^2\theta_1)$$

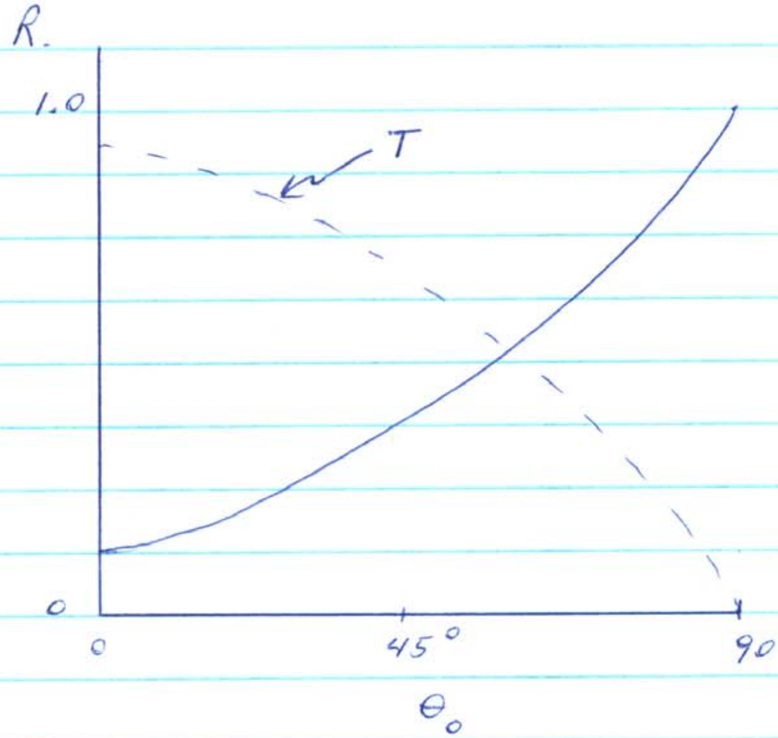
$$n_2^2 - n_1^2\sin^2\theta_1 = n_1^2 - n_1^2\sin^2\theta_1 \quad \text{using Snell's Law}$$

$$n_2^2 = n_1^2$$

$$n_2 = n_1.$$

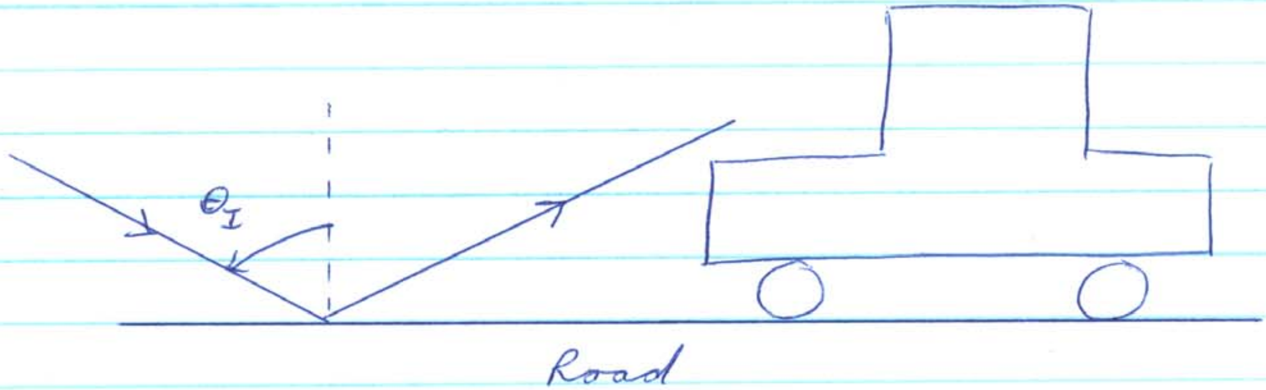
But this is nonsense since the two media are different.

$\therefore$ for s polarized waves there is no incident angle such that $R=0$.



s polarized wave

5 8



Sunlight heats the road producing a layer of warm air near the surface. At the boundary between warm + cool air, light is reflected and transmitted. P polarized light incident at an angle $\theta_I \simeq \theta_{Brewster}$ is not reflected. $\therefore$ the reflected glare will be primarily s polarized. Hence if the driver wears linear polaroid glasses that transmit primarily p polarized light, the glare will be reduced.

10 9a. $\vec{\epsilon}_+ \cdot \vec{\epsilon}_+ = (\hat{x} \cos\omega t + \hat{y} \sin\omega t)\cdot(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$

$$\vec{\epsilon}_+ \cdot \vec{\epsilon}_+ = \cos^2\omega t + \sin^2\omega t$$
$$\langle \vec{\epsilon}_+ \cdot \vec{\epsilon}_+ \rangle = 1$$

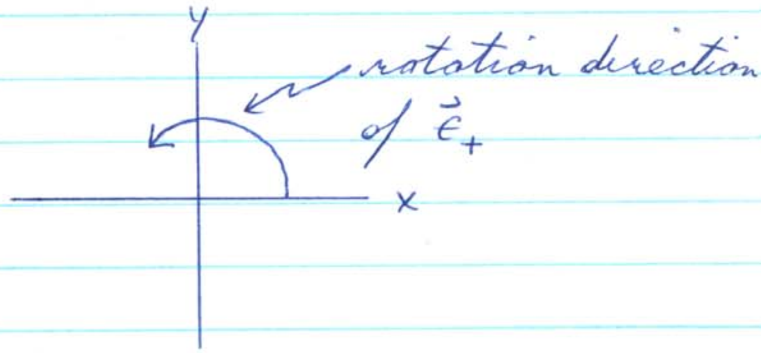
$$\vec{\epsilon}_+ \cdot \vec{\epsilon}_- = (\hat{x}\cos\omega t + \hat{y}\sin\omega t)\cdot(\hat{x}\cos\omega t - \hat{y}\sin\omega t)$$

$$= \cos^2\omega t - \sin^2\omega t$$
$$\langle \vec{\epsilon}_+ \cdot \vec{\epsilon}_- \rangle = \frac{1}{2} - \frac{1}{2}$$
$$= 0$$

Similarly one can show $\langle \vec{\epsilon}_- \cdot \vec{\epsilon}_- \rangle = 1$.

b. $\vec{\epsilon}_+ = \hat{x} \cos \omega t + \hat{y} \sin \omega t$

ωt	$\vec{\epsilon}_+$
0	$\hat{x}$
$\frac{\pi}{2}$	$\hat{y}$
π	$-\hat{x}$
$\frac{3\pi}{2}$	$-\hat{y}$
2π	$\hat{x}$
$\vdots$	$\vdots$



$\therefore \vec{\epsilon}_+$ rotates counterclockwise

$\vec{\epsilon}_- = \hat{x} \cos \omega t - \hat{y} \sin \omega t$ rotates in clockwise fashion.

c. $\vec{E} = \hat{x} \; E_0 \cos \omega t$

$$= \frac{E_0}{2} \left[\vec{\epsilon}_+ + \vec{\epsilon}_- \right]$$

$\therefore$ half of linear polarized wave rotates in clockwise direction and other half in counter-clockwise direction.

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