# Assignment 2

1. Lorentz Force $\vec{F} = q\left(\frac{\vec{v}}{c} \times \vec{B} + \vec{E}\right)$.

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$\therefore$ magnetic field alone $\Rightarrow \vec{F}_M = q\,\frac{\vec{v}}{c} \times \vec{B}$.

Work done by $\vec{F}_M$ when charge moves displacement $\vec{v}\,dt$ is
$$dW = \vec{F}_M \cdot \vec{v}\,dt$$
$$= 0.$$

$\therefore$ magnetic fields do no work since $\vec{F}_M \perp \vec{v}$.

2. We must solve $\nabla\cdot(\vec{A} + \nabla\chi) = 0$. for $\chi$.

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$$\nabla\cdot\vec{A} + \nabla^2\chi = 0.$$

$$\nabla^2\chi = -\nabla\cdot\vec{A} \qquad (1)$$

Recall Poisson's eqn. $\nabla^2\Phi = -4\pi\rho$

$$\Rightarrow \Phi(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d^3r'$$

Similarly we can solve (1) if we replace $\Phi$ by $\chi$ and $\rho$ by $\frac{\nabla\cdot\vec{A}}{4\pi}$.

$$\therefore \chi(\vec{r}) = \frac{1}{4\pi}\int \frac{\nabla'\cdot\vec{A}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d^3r'$$

where $\nabla' = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right)$.

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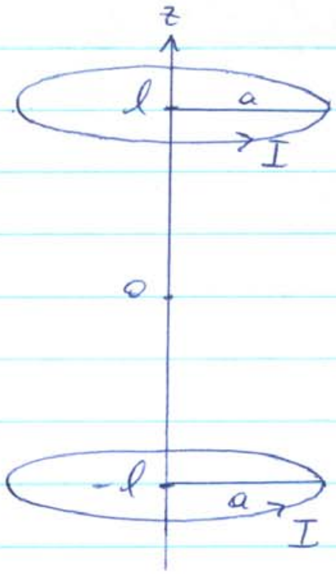



a) Field along $z$ axis due to coil at origin is

$$\vec{B}(0,0,z) = \frac{2\pi I a^2}{c(a^2+z^2)^{3/2}}\,\hat{z}.$$

Field due to coil 1 at $z=-l$ is:

$$\vec{B}_1(0,0,z) = \frac{2\pi I a^2}{c\left[a^2+(z+l)^2\right]^{3/2}}\,\hat{z}.$$

Field due to coil 2 at $z=+l$ is:

$$\vec{B}_2(0,0,z) = \frac{2\pi I a^2}{c\left[a^2+(z-l)^2\right]^{3/2}}\,\hat{z}$$

$\therefore$ field of both coils on $z$ axis is

$$\vec{B}(0,0,z) = \vec{B}_1(0,0,z) + \vec{B}_2(0,0,z)$$

$$= \frac{2\pi I a^2}{c}\left\{\left[a^2+(z+l)^2\right]^{-3/2} + \left[a^2+(z-l)^2\right]^{-3/2}\right\}\hat{z}$$

b) $\vec{B}(0,0,z) = \hat{z}\; B(z)$

$$B(z) = B(0) + z \left.\frac{\partial B}{\partial z}\right|_0 + \frac{z^2}{2} \left.\frac{\partial^2 B}{\partial z^2}\right|_0 + \frac{z^3}{3} \left.\frac{\partial^3 B}{\partial z^3}\right|_0 + \ldots$$

But due to reflection symmetry about the $xy$ plane (containing $z=0$) we require

$$B(z) = B(-z) \Rightarrow \left.\frac{\partial B}{\partial z}\right|_0 = \left.\frac{\partial^3 B}{\partial z^3}\right|_0 = \ldots = 0.$$

c) $$\frac{\partial B}{\partial z} = \frac{2\pi I a^2}{c}\left\{ -\frac{3}{2}\, 2(z+\ell)\left[a^2+(z+\ell)^2\right]^{-5/2} -\frac{3}{2}\, 2.(z-\ell)\left[a^2+(z-\ell)^2\right]^{-5/2}\right\}$$

$$= -\frac{6\pi I a^2}{c}\left\{ (z+\ell)\left[a^2+(z+\ell)^2\right]^{-5/2} + (z-\ell)\left[a^2+(z-\ell)^2\right]^{-5/2}\right\}$$

$$\frac{\partial^2 B}{\partial z^2} = -\frac{6\pi I a^2}{c}\left\{ \left[a^2+(z+\ell)^2\right]^{-5/2} - \frac{5}{2}\, 2.(z+\ell)^2\left[a^2+(z+\ell)^2\right]^{-7/2} + \left[a^2+(z-\ell)^2\right]^{-5/2} - \frac{5}{2}\, 2(z-\ell)^2\left[a^2+(z-\ell)^2\right]^{-7/2}\right\}$$

$$\left.\frac{\partial^2 B}{\partial z^2}\right|_{z=0} = -\frac{6\pi I a^2}{c}\left\{ 2.\left[a^2+\ell^2\right]^{-5/2} - 10\,\ell^2\left[a^2+\ell^2\right]^{-7/2}\right\}$$

$$0 = \left.\frac{\partial^2 B}{\partial z^2}\right|_{z=0} \Rightarrow 0 = 2\left[a^2+\ell^2\right]^{-5/2} - 10\ell^2\left[a^2+\ell^2\right]^{-7/2}$$

$$= a^2+\ell^2 - 5\ell^2.$$

$$0 = a^2 - 4l^2.$$

$$\therefore a = 2l.$$

Hence if $a = 2l$, $B(z) = B(0) + O(z^4)$.

d i) charge of electron $= 4.8 \times 10^{-10}$ esu
$= 1.6 \times 10^{-19}$ Coul.

$\therefore 1.6 \times 10^{-19}$ Coul $= 4.8 \times 10^{-10}$ esu.

$$1 \text{ Coul} = 3 \times 10^9 \text{ esu}.$$

$$1 \text{ amp} = 3 \times 10^9 \text{ esu/sec}.$$

$$\therefore I(\text{amps}) \cdot 3 \times 10^9 = I(\text{esu/sec}).$$

$$I(\text{amp}) \cdot \frac{3 \times 10^9}{3 \times 10^{10}} = \frac{I(\text{esu/sec})}{3 \times 10^{10}}.$$

$$\therefore \frac{I(\text{amps})}{10} = \frac{I(\text{esu/sec})}{c}$$

ii) From part a we find:

$$\vec{B}(0,0,0) = \hat{z}\, \frac{2\pi I a^2}{c} \left\{ 2 \left[ a^2 + l^2 \right]^{-3/2} \right\}$$

Helmholtz Condition is $a = 2l$.

$$\vec{B}(0,0,0) = \hat{z}\, \frac{2\pi I a^2}{c}\, 2 \left( a^2 + \frac{a^2}{4} \right)^{-3/2}$$

$$= \hat{z}\, \frac{4\pi I a^2}{c}\, \frac{1}{a^3} \left( \frac{5}{4} \right)^{-3/2}$$

$$\vec{B}(0,0,0) = \hat{z}\ \frac{4\pi I}{ac}\left(\frac{4}{5}\right)^{3/2}$$

$$= \hat{z}\ \frac{32\pi}{5^{3/2}}\ \frac{I}{ac}.$$

For Helmholtz coil with N turns

$$\boxed{\vec{B}(0,0,0) = \hat{z}\ \frac{32\pi}{5^{3/2}}\ \frac{NI}{ac}.} \qquad (1)$$

We are given I in amps. But I in above formula is in esu/sec. But $\frac{I(esu/sec)}{c} = \frac{I(amps)}{10}$.

$$\therefore\ \vec{B}(0,0,0) = \hat{z}\ \frac{32\pi}{5^{3/2}}\ \frac{N}{a}\ \frac{I(amps)}{10}.$$

$$= \hat{z}\ \frac{32\pi}{5^{3/2}}\ \frac{50}{30\,cm.}\ \frac{20\ amps}{10.}$$

$$= \hat{z}\ \frac{32\pi}{5^{3/2}}\ \frac{1000}{300}$$

$\therefore\ \vec{B}(0,0,0) = 30\ \hat{z}$ gauss is field at center of coil.

iii) $B_{Earth} \simeq 0.5$ gauss.

$$(1) \Rightarrow \quad \frac{I}{c.} = \frac{5^{3/2}}{32\pi}\ B(0)\ \frac{a}{N.}$$

$$\frac{I(amps)}{10} = \frac{5^{3/2}}{32\pi}\ B(0)(gauss)\ \frac{a(cm)}{(N\ turns)}$$

$$\therefore\ I\,(\text{amps}) = 10\ \frac{5^{3/2}}{32\pi}\ \frac{1}{2}\ \text{gauss}\ \frac{30\text{ cm}}{50\text{ turns}}$$

$$= \frac{1}{3}\ \text{amp}.$$

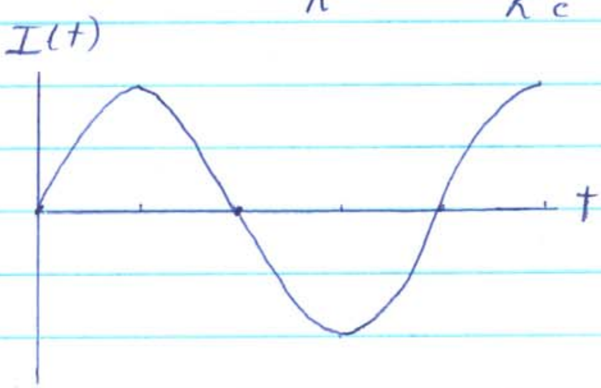
One would need to be careful that direction of field opposes $\vec{B}_{Earth}$ rather than adding to it!

10 4a) Flux through loop $\Phi(t) = BA\cos\omega t$.

b) Voltage $\mathcal{E} = \left| -\frac{1}{c}\frac{\partial \Phi}{\partial t} \right|$

$$= \frac{BA\omega}{c}\sin\omega t.$$

c) Current $I = \frac{\mathcal{E}}{R} = \frac{BA\omega}{Rc}\sin\omega t.$



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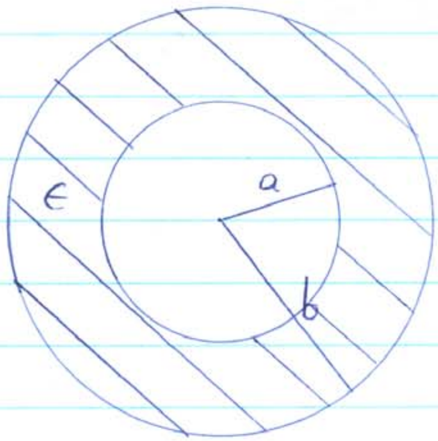


First we find $\vec{D}$ using $\int_S \vec{D} \cdot d\vec{a} = 4\pi \int_V \rho_{free} \, dV$.

By symmetry $\vec{D} = D(r)\,\hat{r}$.

$\therefore \int_S \vec{D} \cdot d\vec{a} = D(r) \int_S da = D(r)\, 4\pi r^2$.

surface of sphere of radius r

$r < a$. $D(r)\, 4\pi r^2 = 0$ (No charge inside conductor)

$$D(r) = 0.$$

$r > a$ $D(r)\, 4\pi r^2 = 4\pi Q$

$$\vec{D}(r) = \frac{Q}{r^2}\,\hat{r}.$$

Next we find electric field.

$r < a$ $\vec{E} = 0$.

$a < r < b$ $\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{Q}{\epsilon r^2}\,\hat{r}$.

$r > b$ $\vec{E} = \vec{D} = \frac{Q}{r^2}\,\hat{r}$.

Next we find Polarization.

$r < a \quad \vec{P} = 0.$

$$a < r < b \quad \vec{P} = \frac{\vec{D} - \vec{E}}{4\pi} = (\epsilon - 1)\frac{\vec{E}}{4\pi}$$

$$= \frac{\epsilon - 1}{4\pi\epsilon} \frac{Q}{r^2} \hat{r}$$

$r > b \quad \vec{P} = 0.$

a) bound surface charge density at $r = a$. is

$$\sigma_b(r = a) = \vec{P}\Big|_{r=a} \cdot (-\hat{r})$$

$$= -\left(\frac{\epsilon - 1}{4\pi\epsilon}\right)\frac{Q}{a^2}$$

bound surface charge density at $r = b$ is

$$\sigma_b(r = b) = \vec{P}\Big|_{r=b} \cdot \hat{r}$$

$$= \frac{\epsilon - 1}{4\pi\epsilon} \frac{Q}{b^2}$$

One can check that total induced charge is 0.

b) $\Phi(0) - \Phi(\infty) = -\int_{\infty}^{0} \vec{E} \cdot d\vec{l}$.

For convenience pick path $d\vec{l} = dr\,\hat{r}$.

$$\Phi(0) = -\left\{ \int_{\infty}^{b} \frac{Q}{r^2}\,dr + \int_{b}^{a} \frac{Q}{\epsilon r^2}\,dr + \int_{a}^{0} 0\,dr \right\}$$

$$= -\left\{ \left[\frac{-Q}{r}\right]_{\infty}^{b} + \left[\frac{-Q}{\epsilon r}\right]_{b}^{a} \right\}$$

$$= -\left\{ -\frac{Q}{b} - \frac{Q}{\epsilon a} + \frac{Q}{\epsilon b} \right\}$$

$$\Phi(0) = Q\left( \frac{1}{b} + \frac{1}{\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right)\right)$$

Check: if $\epsilon = 1 \Rightarrow \Phi(0) = \frac{Q}{a}$.

6. 5

$$\nabla \cdot \vec{J}_{free} = \nabla \cdot \left\{ \frac{c}{4\pi}\left[ \nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t}\right]\right\}$$

$$= \frac{-1}{4\pi}\frac{\partial}{\partial t}(\nabla \cdot \vec{D})$$

$$= \frac{-1}{4\pi}\frac{\partial}{\partial t}(4\pi \rho_{free})$$

$$\nabla \cdot \vec{J}_{free} = -\frac{\partial \rho_{free}}{\partial t}$$

$\Rightarrow$ continuity eqn. is satisfied.

7. $\nabla \cdot \vec{E} = 4\pi \rho$ $\qquad$ $\int_S \vec{E} \cdot d\vec{a} = 4\pi \int_V \rho \, dV.$

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Flux of electric field coming out of a unit volume is $4\pi$ times charges inside unit vol.

$\nabla \cdot \vec{B} = 0$ $\qquad$ $\int_S \vec{B} \cdot d\vec{a} = 0$

Flux of magnetic field out of any closed surface is 0. i.e. no magnetic monopoles.

$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$ $\qquad$ $\oint \vec{E} \cdot d\vec{l} = -\frac{1}{c} \frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{a}$

A changing magnetic field creates a rotating electric field. Faraday Law

$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$ $\qquad$ $\oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} \int_S \vec{J} \cdot d\vec{a} + \frac{1}{c} \frac{\partial}{\partial t} \int_S \vec{E} \cdot d\vec{a}$

The second $\underbrace{\text{term}}_{}$ $\frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ is called the displacement ~~current~~ term. We see that it creates a rotating magnetic field from a changing electric field. The first term $\frac{4\pi}{c}\vec{J}$ creates a magnetic field due to a current.

This is called Ampere's Law.

Total = 60