Assignment 1

1a) electric field inside a conductor $\vec{E} = 0$.

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Gauss law $\Rightarrow$ $\rho = \frac{\nabla . \vec{E}}{4\pi} = 0$.

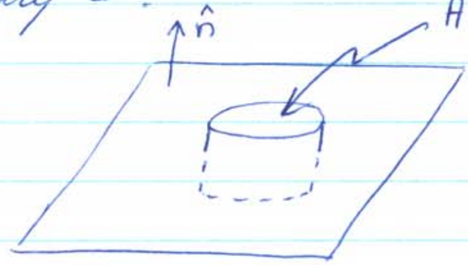
b) $\Phi(b) - \Phi(a) = -\underbrace{\int_a^b \vec{E} . d\vec{l}}_{= 0}$

$\Rightarrow \Phi(b) = \Phi(a)$ for any 2 points a + b inside conductor.

c). Just outside a conductor, there can be no tangential fields since charge would move until an equal opposing field was established.

$\therefore$ $\vec{E}$ is $\perp$ to conductor surface.

Next consider a conductor with surface charge density σ.



Consider a pillbox of negligible height and area A

Gauss law $\displaystyle\int_{\substack{\text{surface} \\ \text{of pillbox}}} \vec{E} . d\vec{s} = 4\pi \int_{\text{pillbox}} \rho \, dV.$

$$\int_{Top} \vec{E}\cdot d\vec{s} + \underbrace{\int_{sides} \vec{E}\cdot d\vec{s}}_{\substack{=0 \text{ since} \\ \vec{E} \perp \text{conductor} \\ \text{surface.}}} + \underbrace{\int_{Bottom} \vec{E}\cdot d\vec{s}}_{\vec{E}=0 \text{ in cond.}} = 4\pi \sigma A.$$

$$\int_{Top} E\, da = 4\pi \sigma A.$$

$$E A = 4\pi \sigma A$$

$$E = 4\pi \sigma.$$

One can argue that $\int_{Top} E\, da \neq E A$. However we can take $\lim_{A \to 0}$ on both sides of the equation.

$\therefore\ \vec{E} = 4\pi\sigma\, \hat{n}$ holds in general.

2a) Because of spherical symmetry $\vec{E} = E(r)\,\hat{r}$.

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Gauss Law $\int_S \vec{E} \cdot d\vec{s} = 4\pi \int_V \rho\, dV$.

Let S be sphere of radius r.

$$\therefore E(r)\, 4\pi r^2 = 4\pi \int \rho\, dV.$$

$r < a$. $$E(r)\, 4\pi r^2 = 4\pi q.$$

$$\vec{E}(r) = \frac{q}{r^2}\,\hat{r}$$

$a < r < b$ $\vec{E} = 0$ inside conductor.

$r > b$ $$E(r)\, 4\pi r^2 = 4\pi \int \rho\, dV.$$

$$= 4\pi q.$$

$$\vec{E}(r) = \frac{q}{r^2}\,\hat{r}.$$

b) Component of electric field $\perp$ to cond. surface of radius $r = a$ is

$$E_\perp(r = a) = \frac{q}{a^2}\,\hat{r} \cdot (-\hat{r})$$

normal unit vector to surface

$$= -\frac{q}{a^2}.$$

∴ charge density on $r=a$ cond. surface

$$\sigma(r=a) = \frac{E_\perp(r=a)}{4\pi}$$

$$= \frac{-q}{4\pi a^2}.$$

Component of electric field $\perp$ to cond. surface of radius $r=b$ is:

$$E_\perp(r=b) = \frac{q}{b^2}\,\hat{r}\cdot\hat{r}$$

normal unit vector to surface

$$= \frac{q}{b^2}.$$

∴ charge density $\sigma(r=b) = \dfrac{q}{4\pi b^2}$.

c) Let $\Phi(r=\infty) = 0$.

$r \geq b$ $\quad \Phi(r) - \Phi(\infty) = -\int_\infty^r \vec{E}\cdot d\vec{l}$ (Choose $d\vec{l} = d\vec{r}$)

$$\Phi(r) = -\int_\infty^r \frac{q}{r^2}\,dr$$

$$= -q\left[\frac{-1}{r}\right]_\infty^r$$

$$= \frac{q}{r}.$$

$a \leq r \leq b$ $\quad \Phi = \dfrac{q}{b}$ = const. inside conductor.

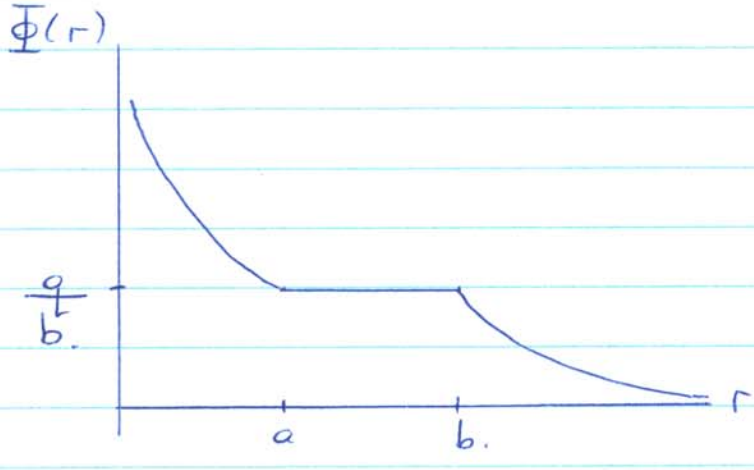
$r < a$ $\quad \Phi(r) - \Phi(a) = -\int_a^r \vec{E} \cdot d\vec{l}$ (Choose $d\vec{l} = d\vec{r}$)

$$\Phi(r) - \frac{q}{b} = -\int_a^r \frac{q}{r^2}\, dr$$

$$= -q\left[\frac{-1}{r}\right]_a^r$$

$$= \frac{q}{r} - \frac{q}{a}$$

$$\Phi(r) = q\left[\frac{1}{r} + \frac{1}{b} - \frac{1}{a}\right]$$



3a)
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$$\rho = \begin{cases} \rho_0 & r < a \\ 0 & r > a \end{cases}$$

Gauss law $\displaystyle\int_S \vec{E} \cdot d\vec{s} = 4\pi \int_V \rho \, dV$

Due to spherical symmetry $\vec{E} = E(r)\,\hat{r}$.

Let S be sphere of radius r.

$r < a$. $\quad E(r)\, 4\pi r^2 = 4\pi \rho_0 \frac{4}{3}\pi r^3.$

$$\vec{E}(r) = \frac{4\pi}{3} \rho_0 \vec{r}$$

$r > a$ $\quad \vec{E}(r) = \frac{4\pi}{3} \frac{a^3}{r^2} \rho_0 \; \hat{r}.$

b) $\displaystyle \Phi(r) - \Phi(0) = -\int_0^r \vec{E} \cdot d\vec{l}$ $\quad$ (Choose $d\vec{l} = d\vec{r}$).

$r \leq a$ $\quad \displaystyle \Phi(r) = -\int_0^r \frac{4\pi}{3} \rho_0 \, r \, dr.$

$$= -\frac{4\pi}{3} \rho_0 \left. \frac{r^2}{2} \right|_0^r$$

$$= -\frac{2\pi}{3} \rho_0 r^2.$$

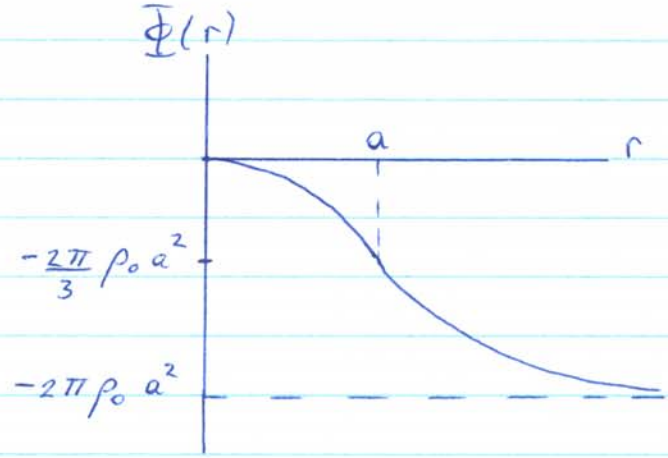
$r \geq a$ $\quad \displaystyle \Phi(r) - \Phi(a) = -\int_a^r \vec{E} \cdot d\vec{l}.$

$$= -\frac{4\pi}{3} a^3 \rho_0 \int_a^r \frac{dr}{r^2}.$$

$$\Phi(r) - \left(-\frac{2\pi}{3}\rho_0 a^2\right) = -\frac{4\pi}{3} a^3\rho_0 \left[\frac{-1}{r}\right]_a^r$$

$$= \frac{4\pi}{3} a^3\rho_0 \frac{1}{r} - \frac{4\pi}{3} a^2\rho_0 .$$

$$\Phi(r) = \frac{4\pi}{3} a^3\rho_0\left(\frac{1}{r} - \frac{3}{2a}\right).$$



c) $a = 2\text{ cm}$, $\rho_0 = \frac{3}{2\pi}$ esu/cm^3.

i) total charge on sphere $\frac{4\pi}{3} a^3\rho_0$

$$= \frac{4\pi}{3}(2\text{ cm})^3 \frac{3}{2\pi} \text{ esu/cm}^3$$

$$= 16 \text{ esu}.$$

ii) $$\vec{E}(10\text{ cm}) = \frac{16\text{ esu}}{(10\text{ cm})^2}\hat{r}$$

$$= .16\,\hat{r} \text{ esu/cm}^2.$$

$$= .16\,\hat{r} \text{ statvolt/cm}.$$

$$\Phi(10\,cm) = 16\,esu\left(\frac{1}{10\,cm} - \frac{3}{2\times 2\,cm}\right)$$

$$= -10.4\ esu/cm$$

$$= -10.4\ statvolts$$

iii) Work done in moving 5 esu from infinity to 10 cm from center of sphere is

$$W_{10,\infty} \equiv 5\,esu\left[\Phi(10\,cm) - \Phi(\infty)\right]$$

$$= 5\,esu\left[-10.4\ statvolts - (-2\pi)\rho_0 a^2\right]$$

$$= 5\,esu\left[-10.4\ statvolts + 2\pi\frac{3}{2\pi}\frac{esu}{cm^3}\cdot(2\,cm)^2\right]$$

$$= 5\,esu\left[-10.4 + 12\right] statvolts$$

$$\therefore W_{10,\infty} = 8\ ergs.$$

Force between charge & sphere is

$$5\,esu \times E(r=10\,cm)$$

$$= 5\,esu \times .16\ \frac{statvolt}{cm.}$$

$$= .80\ dynes.$$

The force is repulsive since both charges are positive.

4 a)

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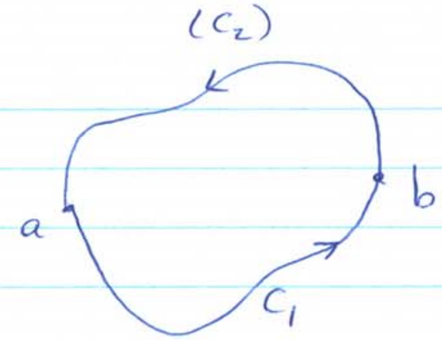


Let a and b be two different points on a closed path.

$$\oint \vec{E}\cdot d\vec{l} = \underset{(C_1)}{\int_a^b} \vec{E}\cdot d\vec{l} + \underset{(C_2)}{\int_b^a} \vec{E}\cdot d\vec{l}$$

$$= -\Phi(b) + \Phi(a) - \Phi(a) + \Phi(b)$$

$$\therefore \oint \vec{E}\cdot d\vec{l} = 0.$$

b) $\oint \vec{E}\cdot d\vec{l} = 0.$ for any closed path

$\int_S (\nabla\times\vec{E})\cdot d\vec{a} = 0$ for any arbitrary surface S

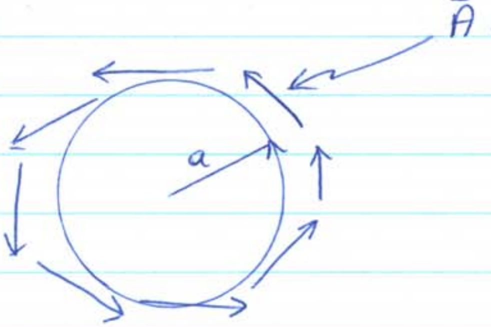
$$\Rightarrow \nabla\times\vec{E} = 0.$$

c) If $\vec{E} = -\nabla\Phi$

$$= -\left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right)$$

then $\nabla\times\vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial\Phi}{\partial x} & -\frac{\partial\Phi}{\partial y} & -\frac{\partial\Phi}{\partial z} \end{vmatrix} = \vec{0}$

d)



$$\oint_{\substack{\text{circle perimeter}\\ \text{of radius } a}} \vec{A}\cdot d\vec{l} \neq 0.$$

5) Obviously $\Phi = 0$ is a solution. Because
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of uniqueness theorem, we know this is the only possible solution.

6) Consider the following problem.
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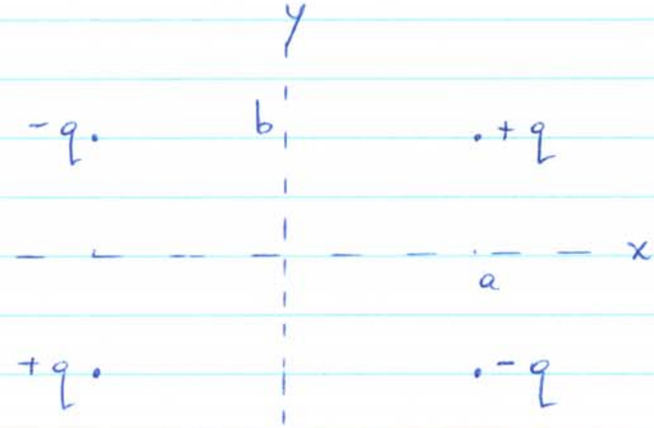

Note: 1) $\nabla^2 \Phi = 4\pi\rho$ in the region $x, y > 0$ is

the same as in problem.

2) $\Phi = 0$ on x & y planes.

$\therefore$ by uniqueness theorem, both problems have the same potential in the quadrant $x, y > 0$.

$$\therefore \Phi(x,y,z) = \frac{+q}{|(x-a,\, y-b,\, z)|} - \frac{q}{|(x-a,\, y+b,\, z)|}$$

$$\frac{-q}{|(x+a,\, y-b,\, z)|} + \frac{q}{|(x+a,\, y+b,\, z)|}$$

$$\Phi(x,y,z) = q\left\{ \left[(x-a)^2+(y-b)^2+z^2\right]^{-1/2} - \left[(x-a)^2+(y+b)^2+z^2\right]^{-1/2} - \left[(x+a)^2+(y-b)^2+z^2\right]^{-1/2} + \left[(x+a)^2+(y+b)^2+z^2\right]^{-1/2} \right\}$$

This only holds in quadrant $x, y > 0$.
In conductor $\Phi = 0$.

7) Consider the following problem.

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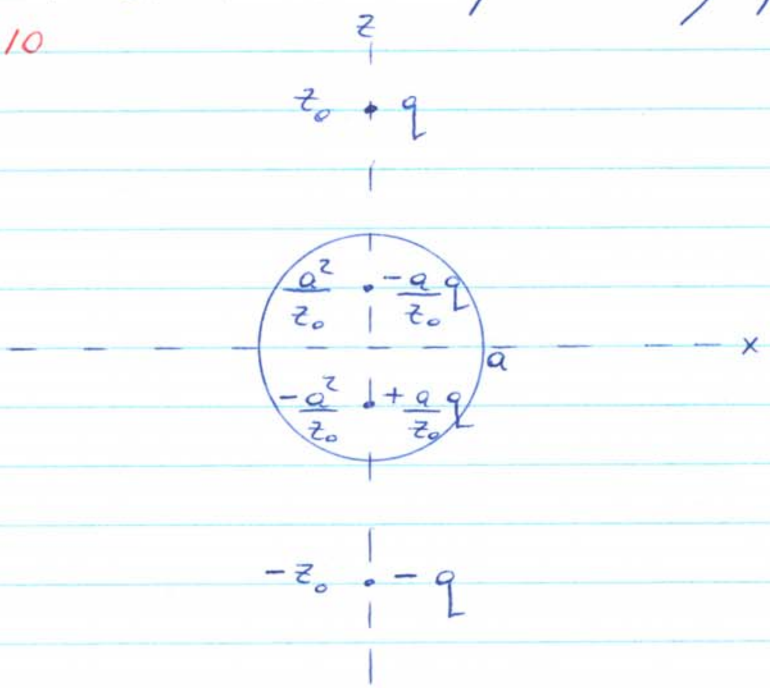

Note: 1) $\nabla^2 \Phi = 4\pi\rho$ in region above plane & hemi-spherical bubble is the same as in problem

2) $\Phi = 0$ on xy plane & on hemisphere.

$\therefore$ by uniqueness theorem, both problems have the same potential in region above plane & bubble.

$$\Phi(\vec{r}) = \frac{q}{|\vec{r} - (0,0,z_0)|} - \frac{\frac{a}{z_0}\, q}{|\vec{r} - (0,0,a^2/z_0)|}$$

$$+ \frac{\frac{a}{z_0}\, q}{|\vec{r} - (0,0,-a^2/z_0)|} - \frac{q}{|\vec{r} - (0,0,-z_0)|}$$

8). We must solve the two dimensional Laplace

10 eqn. $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0.$

Let $\Phi(x,y) = X(x)\, Y(y).$

Bound. Conds. are: $\Phi(0,y) = \Phi(a,y) = 0 \quad \forall y.$

$$\Rightarrow X(0) = X(a) = 0.$$

Hence we need periodic solution for $X(x)$. (Case 2)

$$X(x) = A \cos kx + B \sin kx.$$

$X(0) = 0 \Rightarrow A = 0. \Rightarrow X(x) = B \sin kx.$

$X(a) = 0 \Rightarrow \quad B \sin ka = 0.$

$$ka = n\pi.$$

$$k = \frac{n\pi}{a} \qquad n \in N.$$

Next $Y(y) = C e^{ky} + D e^{-ky}$

But $\Phi(x, \infty) = 0 \Rightarrow C = 0. \Rightarrow Y(y) = D e^{-ky}$

$$\therefore \Phi(x,y) = \sum_{n=1}^{\infty} B_n \sin k_n x \; e^{-k_n y} \quad \text{where } k_n = \frac{n\pi}{a}.$$

$$= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \; e^{-\frac{n\pi y}{a}}.$$

Remaining B.C. is $\Phi(x,0) = \Phi_0$.

$$\therefore \Phi_0 = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a}.$$

$$\int_0^a \Phi_0 \sin \frac{m\pi x}{a}\, dx. = \sum_{n=1}^{\infty} B_n \underbrace{\int_0^a \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a}\, dx.}_{=\frac{a}{2}\delta_{nm}}$$

$$\Phi_0 \left[\frac{-\cos \frac{m\pi x}{a}}{\frac{m\pi}{a}} \right]_0^a = B_m \frac{a}{2}$$

$$B_m = \frac{2}{a} \frac{a}{m\pi} \left[-\cos m\pi + 1 \right] \Phi_0$$

$$\therefore B_m = \frac{2}{m\pi} \left[1 - (-1)^m \right] \Phi_0$$

$$\boxed{\therefore \Phi(x,y) = \sum_{n=1}^{\infty} \frac{2\Phi_0}{n\pi} \left(1 - (-1)^n\right) \sin \frac{n\pi x}{a}\, e^{-n\pi y/a}}$$

9). Let $\Phi = R(\rho)\, Q(\phi)$.

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$$0 = \nabla^2 \Phi = Q(\phi) \frac{1}{\rho} \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{R(\rho)}{\rho^2} \frac{d^2 Q(\phi)}{d\phi^2}$$

$$0 = \frac{\rho}{R} \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{Q} \frac{d^2 Q}{d\phi^2}$$

$$\therefore \quad \frac{1}{Q} \frac{d^2 Q}{d\phi^2} = -K.$$

$$\& \quad \frac{\rho}{R} \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = +K.$$

b) $K = 0 \implies \dfrac{d^2 Q}{d\phi^2} = 0 \implies Q = C\phi + D.$

$$\frac{\rho}{R} \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = 0.$$

$$\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = 0.$$

$$\rho \frac{dR}{d\rho} = A.$$

$$\frac{dR}{d\rho} = \frac{A}{\rho}$$

$$R(\rho) = A \ln \rho + B.$$

$K = +k^2 > 0 \qquad \frac{d^2 Q}{d\phi^2} = -k^2 Q.$

$$Q = C \cos k\phi + D \sin k\phi.$$

$$\frac{\rho}{R} \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho} \right) = k^2.$$

let $R = \rho^\lambda. \Rightarrow \rho \frac{d}{d\rho}\left(\rho \lambda \rho^{\lambda - 1} \right) = k^2 \rho^\lambda.$

$$\rho \frac{d}{d\rho}\left(\lambda \rho^\lambda \right) = k^2 \rho^\lambda.$$

$$\lambda^2 = k^2.$$

$$\lambda = \pm k.$$

$$\Rightarrow R(\rho) = A \rho^k + B \rho^{-k}.$$

$K = -k^2 < 0$ similar to above.

c) $Q(\phi) = Q(\phi + 2\pi).$

If $K = -k^2 < 0$ one can quickly show $Q = 0$.

" $K = 0$ one can show $C = 0$.

" $K = +k^2 > 0$ " $C \cos k(2\pi + \phi) + D \sin k(2\pi + \phi)$
$= C \cos k\phi + D \sin k\phi.$

$\Rightarrow k = n$ an integer

$\therefore$ most general possible solution if $\Phi(\rho, \phi) = \Phi(\rho, \phi + 2\pi)$ is:

$$\Phi(\rho, \phi) = A \ln\rho + B + \sum_{n=1}^{\infty} \left(A_n \rho^n + B_n \rho^{-n}\right)$$

$$\cdot\left(C_n \cos n\phi + D_n \sin n\phi\right).$$

where constant D has been incorporated into A + B.

Total = 90.