Phys 4050 Assignment 6

1. Diamangetic susceptibility of atomic hydrogen. The wave function of the hydrogen atom in its ground state (1s) is

$$\psi = (\pi a_0^3)^{-1/2} \exp(-r/a_0)$$

where a_o is the Bohr radius. The charge density is $-e\,|\psi|^2$ according to the statistical interpretation of the wave function. Show that for this state $\langle r^2 \rangle = 3\,a_o^2$ and calculate the molar diamagnetic susceptibility of atomic hydrogen.

2. Triplet excited states. Some organic molecules have a triplet (S=1) excited state at an energy $k_B\,\Delta$ above a single (S=0) ground state.

   a) Find an expression for the magnetic moment $\langle\mu\rangle$ in a field B.

   b) Show that the susceptibility for $T >> \Delta$ is approximately independent of Δ.

3. Heat capacity from internal degrees of freedom. Consider a two level system with an energy splitting $k_B\,\Delta$ between upper and lower states; the splitting may arise from a magnetic field or in other ways.

   a) Show that the heat capacity per system is

$$C = k_B \frac{(\Delta/T)^2 e^{\Delta/T}}{(1+e^{\Delta/T})^2}$$

   b) Show that for $T >> \Delta$ one obtains $C = k_B\,(\Delta/2T)^2$

4. Two level system. Consider two levels having energies Δ and $-\Delta$.

   a) Show that the energy is given by

$$U = -\Delta \tanh\left(\frac{\Delta}{k_B T}\right)$$

   b) Show that the heat capacity is given by

$$C = k_B \left(\frac{\Delta}{k_B T}\right)^2 \mathrm{sech}^2\left(\frac{\Delta}{k_B T}\right)$$

   c) If the system has a random composition such that all values of Δ are equally likely up to some limit Δ_o show the heat capacity is linearly proportional to the temperature provided $k_B\,T << \Delta_o$.

5. Paramagnetism of $S = 1$ system.

a) Find the magnetization as a function of magnetic field and temperature for a system of spins with $S = 1$, moment μ and concentration n.

b) Show that in the limit $\mu B \ll kT$ the result is $M = (2n\mu^2 / 3kT)\, B$.

6. Rotating coordinate system. We define the vector

$$\vec{F}(t) = F_x(t)\,\hat{x} + F_y(t)\,\hat{y} + F_z(t)\,\hat{z}$$

Let the coordinate system of the unit vectors $\hat{x}, \hat{y}$ & $\hat{z}$ rotate with an instantaneous angular velocity $\vec{\Omega}$ so that

$$\frac{d\hat{x}}{dt} = \Omega_y \hat{z} - \Omega_z \hat{y} \quad \text{etc.}$$

a) Show that $$\frac{d\vec{F}}{dt} = \left(\frac{d\vec{F}}{dt}\right)_R + \vec{\Omega} \times \vec{F}$$

where $\left(\frac{d\vec{F}}{dt}\right)_R$ is the time derivative of $\vec{F}$ as viewed in the rotating frame R.

b) Show that
$$\left(\frac{d\vec{M}}{dt}\right)_R = \gamma \vec{M} \times (\vec{B}_a + \vec{\Omega}/\gamma)$$

This is the equation of motion of $\vec{M}$ in a rotating coordinate system. The transformation to a rotating system is extraordinarily useful.

c) Let $\vec{\Omega} = -\gamma B_a \hat{z}$ thus in the rotating frame there is no static magnetic field. Still in the rotating frame, we now apply a dc pulse $B_1 \hat{x}$ for a time t. If the magnetization is initially along $\hat{z}$, find an expression for the pulse length t such that the magnetization will be directed along $-\hat{z}$ at the end of the pulse. Neglect relaxation effects.

d) Describe this pulse as viewed from the laboratory frame of reference.