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## Assignment 6

1) $$\psi_{1s} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

$$\langle r^2 \rangle = \int_0^\infty 4\pi r^2 \, dr \; \psi_{1s}^* \, r^2 \, \psi_{1s}$$

$$= \frac{4\pi}{\pi a_0^3} \int_0^\infty r^4 e^{-2r/a_0} \, dr$$

| | | |
|---|---|---|
| $r^4$ | $+$ | $e^{-2r/a_0}$ |
| $4r^3$ | $-$ | $-\frac{a_0}{2} e^{-2r/a_0}$ |
| $12r^2$ | $+$ | $\left(\frac{-a_0}{2}\right)^2 e^{-2r/a_0}$ |
| $24r$ | $-$ | $\vdots$ |
| $24$ | $+$ | |
| $0$ | $-$ | |

$$= \frac{4\pi}{\pi a_0^3} \, 24 \left(\frac{a_0}{2}\right)^5$$

$$\therefore \langle r^2 \rangle = 3 a_0^2$$

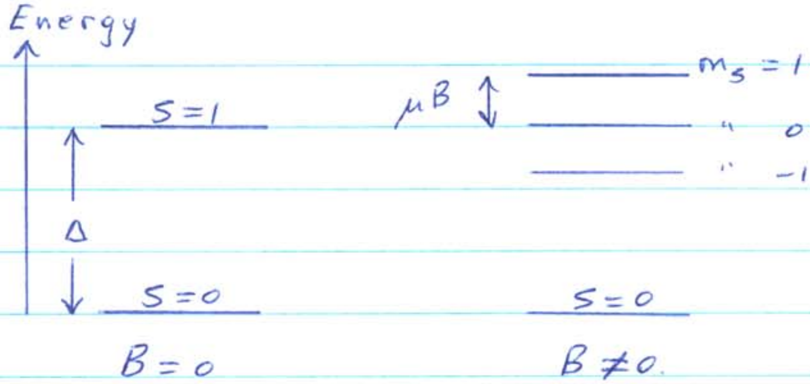
Diamagnetic Susceptibility per mole is

$$\chi = - \frac{N Z e^2}{6 m c^2} \langle r^2 \rangle$$

$$\chi_H = - \frac{6 \times 10^{23} \text{ atoms/mole} \times 1 \times (4.8 \times 10^{-10} \text{ esu})^2 \times 3 \times (.5 \times 10^{-8} \text{ cm})^2}{6 \times 9.11 \times 10^{-28} \text{ gm} \times (3 \times 10^{10} \text{ cm/sec})^2}$$

$$= -2.37 \times 10^{-6} \text{ cm}^3/\text{mole}.$$

3a)




$$\langle \mu \rangle = \mu \left( N_{m_s=1} - N_{m_s=-1} \right)$$

mag. moment per unit vol.

$$= \mu N \frac{e^{-(\Delta+\mu B)/kT} - e^{-(\Delta-\mu B)/kT}}{1 + e^{-\Delta/kT}\left(1 + e^{-\mu B/kT} + e^{\mu B/kT}\right)}$$

# molecules per unit vol.

$$= \mu N \frac{e^{-\Delta/kT}\left(e^{-\mu B/kT} - e^{\mu B/kT}\right)}{1 + e^{-\Delta/kT}\left(1 + e^{-\mu B/kT} + e^{\mu B/kT}\right)}$$

$$\simeq \mu N \frac{e^{-\Delta/kT}\left(1 - \frac{\mu B}{kT} - \left(1 + \frac{\mu B}{kT}\right)\right)}{1 + 3e^{-\Delta/kT}}$$

where we used $\mu B \ll kT$

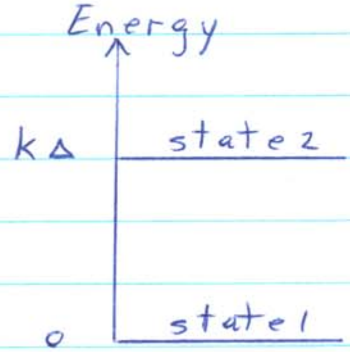
$$\langle \mu \rangle = -2\frac{\mu^2 B N}{kT} \frac{e^{-\Delta/kT}}{1 + 3e^{-\Delta/kT}}$$

b) If $\Delta \ll kT$ $\quad e^{-\Delta/kT} \simeq 1$

$$\Rightarrow \langle \mu \rangle = -\frac{\mu^2 N B}{2kT}$$

$$\therefore \chi = \frac{\langle \mu \rangle}{B} = -\frac{\mu^2 N}{2kT}$$

4a)




Energy / cm$^3$ $\quad U = \underset{\substack{\uparrow \\ \text{atom} \\ \text{density}}}{N} \quad \dfrac{0 \cdot e^{-0/kT} + k\Delta\, e^{-k\Delta/kT}}{e^{-0/kT} + e^{-k\Delta/kT}}$

$$= N\, k\Delta \, \frac{e^{-\Delta/T}}{1 + e^{-\Delta/T}}$$

Specific Heat per atom

$$C = \frac{1}{N} \frac{\partial U}{\partial T}$$

$$= k\Delta \left\{ \frac{\Delta/T^2\, e^{-\Delta/T}}{1 + e^{-\Delta/T}} + \frac{e^{-\Delta/T} \; (-1)\, e^{-\Delta/T} \; (\Delta/T^2)}{(1 + e^{-\Delta/T})^2} \right\}$$

$$= \frac{k\Delta^2}{T^2} \; \frac{e^{-\Delta/T}(1 + e^{-\Delta/T}) - e^{-2\Delta/T}}{(1 + e^{-\Delta/T})^2}$$

$$C = \frac{k\Delta^2}{T^2} \; \frac{e^{-\Delta/T}}{(1 + e^{-\Delta/T})^2}$$

b) $\quad T >> \Delta \implies \quad C = \dfrac{k\Delta^2}{T^2} \; \dfrac{1 - \Delta/T}{(1 + \; 1 - \Delta/T)^2}$

$$\therefore C = \frac{k\Delta^2}{4T^2} + O\left(\frac{\Delta}{T}\right)^3$$

7a) Energy of N atoms is:

$$U = N \frac{\Delta e^{-\Delta/kT} - \Delta e^{+\Delta/kT}}{e^{-\Delta/kT} + e^{\Delta/kT}}$$

$$= -N\Delta \tanh(\Delta/kT)$$

Specific Heat per atom is:

$$C = \frac{1}{N}\frac{dU}{dT}$$

$$= k\left(\frac{\Delta}{kT}\right)^2 \operatorname{sech}^2(\Delta/kT)$$

b) Averaging C over range of $\Delta$, we get:

$$\bar{C} = \frac{1}{\Delta_0}\int_0^{\Delta_0} C \, d\Delta.$$

$$= \frac{1}{\Delta_0}\int_0^{\Delta_0} k\left(\frac{\Delta}{kT}\right)^2 \operatorname{sech}^2(\Delta/kT)\, d\Delta.$$

$$= \frac{k^2 T}{\Delta_0}\int_0^{x_0} x^2 \operatorname{sech}^2 x \, dx \qquad x \equiv \frac{\Delta}{kT}$$

In limit $\Delta_0 >> kT \Rightarrow x_0 = \infty$.

$\therefore \bar{C} = \text{constant} \times T$

8a) $S=1$

| | $m$ | $E$ |
|---|---|---|
| | $-1$ | $E = \mu B$ |
| | $0$ | $0$ |
| | $+1$ | $-\mu B.$ |

Magnetization $M = n \langle \mu \rangle$

$$= n \frac{\left[ -\mu e^{-\mu B/kT} + 0 + \mu e^{\mu B/kT} \right]}{e^{-\mu B/kT} + 1 + e^{\mu B/kT}}$$

$$= n\mu \frac{2 \sinh \mu B/kT}{1 + 2 \cosh \mu B/kT}$$

b) If $\mu B/kT \ll 1$ then $\sinh \frac{\mu B}{kT} = \frac{e^{\mu B/kT} + e^{-\mu B/kT}}{2}$

$$\simeq \frac{\mu B}{kT}$$

$$\cosh \frac{\mu B}{kT} \simeq 1$$

$\therefore$ magnetization $M = \dfrac{n\mu \; 2 \frac{\mu B}{kT}}{1 + 2}$

$$M = \frac{2}{3} \frac{n \mu^2 B}{kT}$$

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2 a) $\vec{F} = F_x(t)\,\hat{x} + F_y(t)\,\hat{y} + F_z(t)\,\hat{z}$

$$\frac{d\vec{F}}{dt} = \frac{dF_x}{dt}\hat{x} + \frac{dF_y}{dt}\hat{y} + \frac{dF_z}{dt}\hat{z}$$

$$+ F_x(\Lambda_y \hat{z} - \Lambda_z \hat{y}) + F_y(-\Lambda_x \hat{z} + \Lambda_z \hat{x}) + F_z(\Lambda_x \hat{y} - \Lambda_y \hat{x})$$

$$= \left(\frac{d\vec{F}}{dt}\right)_{rot.\ frame} + \hat{x}(F_y\Lambda_z - F_z\Lambda_y) + \hat{y}(-F_x\Lambda_z + F_z\Lambda_x) + \hat{z}(F_x\Lambda_y - F_y\Lambda_x)$$

$$\therefore \frac{d\vec{F}}{dt} = \left(\frac{d\vec{F}}{dt}\right)_{rot.\ frame} + \vec{\Lambda}\times\vec{F}$$

b) (7) $\dfrac{d\vec{M}}{dt} = \gamma\vec{M}\times\vec{B}_a$

Part a $\Rightarrow$ $\left(\dfrac{d\vec{M}}{dt}\right)_{rot} + \vec{\Lambda}\times\vec{M} = \gamma\vec{M}\times\vec{B}_a$.

$$\left(\frac{d\vec{M}}{dt}\right)_{rot} = \gamma\vec{M}\times\left(\vec{B}_a + \frac{\vec{\Lambda}}{\gamma}\right)$$

c) $\vec{\Lambda} = -\gamma B_a\hat{z} \Longrightarrow \left(\dfrac{d\vec{M}}{dt}\right)_{rot} = 0.$

If a dc pulse $B_1\hat{x}$ is applied in rotating frame, then:

$$\left(\frac{d\vec{M}}{dt}\right)_{rot} = \gamma\vec{M}\times(B_1\hat{x})$$

$$= \gamma B_1(0, M_z, -M_y)$$

$\therefore \; M_{x\,rot} = \text{const.}$

$$\left.\begin{aligned} \left(\frac{dM_y}{dt}\right)_{rot} &= \gamma B_1 M_z \\ \left(\frac{dM_z}{dt}\right)_{rot} &= -\gamma B_1 M_y \end{aligned}\right\} \Longrightarrow \frac{d^2}{dt^2}\begin{pmatrix} M_y \\ M_z \end{pmatrix} = -(\gamma B_1)^2 \begin{pmatrix} M_y \\ M_z \end{pmatrix}$$

Taking $M_z(t=0) = M$, $M_y(t=0) = 0$, we get:

$$M_y = M \sin \gamma B_1 t$$

$$M_z = M \cos \gamma B_1 t$$

Magnetization will be directed into $(-\hat{z})$ direction if pulse lasts $\gamma B_1 t = \pi$.

$$t = \frac{\pi}{\gamma B_1} \text{ sec.}$$

d) If field in rotating frame is constant, then in lab frame, field precesses at frequency $\Omega$.