Assignment 3

1) $M \frac{d^2 u_{lm}}{dt^2} = C[u_{l+1,m} + u_{l-1,m} - 2u_{lm}] + C[u_{l\,m+1} + u_{l\,m-1} - 2u_{lm}]$

$$u_{lm} = u(0) \exp i(l k_x a + m k_y a - \omega t)$$

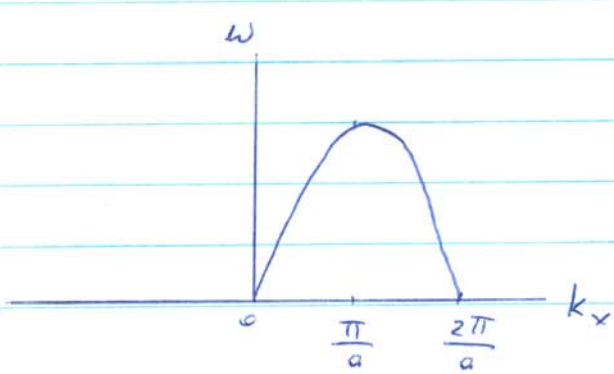
$$-M\omega^2 u_{lm} = C\left[e^{ik_x a} + e^{-ik_x a} - 2\right] u_{lm}$$
$$+ C\left[e^{ik_y a} + e^{-ik_y a} - 2\right] u_{lm}$$

$$\omega^2 M = -C\left[2\cos k_x a - 2 + 2\cos k_y a - 2\right]$$
$$= -2C\left[\cos k_x a + \cos k_y a - 2\right]$$
$$\omega^2 M = 2C\left[2 - \cos k_x a - \cos k_y a\right]$$

Right side is periodic. i.e. $\begin{pmatrix} k_x \\ k_y \end{pmatrix} \in \left[0, \frac{2\pi}{a}\right]$ contain all possible values.

b) $k_y = 0$, $k_x = k$ $\Rightarrow$ $\omega^2 = \frac{2C}{M}\left(1 - \cos ka\right)$

$k_x = k_y \Rightarrow \omega^2 = \frac{2C}{M}\left(2 - 2\cos k_x a\right).$



2a). K.E. of lattice is $\sum_s \frac{m}{2}\left(\frac{du_s}{dt}\right)^2$

P.E. of lattice is $\underset{\substack{\uparrow \\ \text{avoid double counting}}}{\frac{1}{2}}\sum_s C(u_s - u_{s+1})^2$

Total Energy $E = \frac{m}{2}\sum_s\left(\frac{du_s}{dt}\right)^2 + \frac{C}{2}\sum_s (u_s - u_{s+1})^2$

b) Let $u_s = u\cos(\omega t - ska)$

$$E = \frac{m}{2}\sum_s \omega^2 u^2 \sin^2(\omega t - ska)$$

$$+ \frac{C}{2}\sum_s \underbrace{\left[u\cos(\omega t - ska) - u\cos(\omega t - (s+1)ka)\right]^2}_{X}$$

$$X = u^2\Big[\cos(\omega t - ska) - \cos(\omega t - ska)\cos ka$$

$$+ \sin(\omega t - ska)\sin ka\Big]^2$$

$$= u^2\Big[\cos^2(\omega t - ska)(1-\cos ka)^2 + \sin^2(\omega t - ska)\sin^2 ka$$

$$+ 2\cos(\omega t - ska)(1-\cos ka)\sin(\omega t - ska)\sin ka\Big]$$

$$\underset{\substack{\uparrow \text{time} \\ \text{averaged}}}{\langle X \rangle} = u^2\left[\frac{(1-\cos ka)^2}{2} + \frac{\sin^2 ka}{2}\right]$$

$$\therefore \langle E \rangle = \frac{m}{2} \sum_s \omega^2 \frac{u^2}{2} + \frac{C}{2} \sum_s \frac{u^2}{2} \left[(1-\cos ka)^2 + \sin^2 ka \right]$$

$$\langle E \rangle \text{ per atom} = \frac{m\omega^2 u^2}{4} + \frac{C}{4} u^2 (2 - 2\cos ka)$$

From previous problem $2C(1-\cos ka) = m\omega^2$.

$$\therefore \langle E \rangle \text{ per atom} = \frac{m\omega^2 u^2}{2}$$

7a) Charge density of electrons in metal $\rho = \frac{-e}{\frac{4}{3}\pi R^3}$

electric field due to sphere of radius r is

$$E \cdot 4\pi r^2 = 4\pi \rho \frac{4}{3} \pi r^3$$

$$\vec{E} = -\frac{e\vec{r}}{R^3}.$$

Hence ion displaced distance r feels restoring force

$$\vec{F} = -\frac{e^2 \vec{r}}{R^3}.$$

Equation of Motion is $m\frac{d^2\vec{r}}{dt^2} = -\frac{e^2\vec{r}}{R^3}$

let $\vec{r} = \vec{r}_0 e^{i\omega t} \Longrightarrow \omega = \left(\frac{e^2}{mR^3}\right)^{1/2}$

b) For Na $m \simeq 23\, m_{prot.}$ $\Longrightarrow \omega \sim 1.5 \times 10^{13}$ Hz.
$R \sim 3 \mathring{A}$

c) speed of sound $v = \frac{\omega}{k} \sim \frac{a}{\pi}\omega \sim 1500$ m/sec.
(using $a \sim 3 \mathring{A}$)

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1a) Dispersion Relation for monatomic linear lattice of N atoms is:

$$\omega^2 = \frac{\omega_{max}^2}{2}(1-\cos ka)$$

modes with wavevector in range k to k+dk is $\frac{L}{2\pi}dk$

$$\therefore\ D(\omega)\,d\omega = \frac{L}{2\pi}\,dk.$$

$$D(\omega) = \frac{L}{2\pi}\frac{dk}{d\omega} \qquad \text{where } L = Na$$

Now $\omega^2 = \frac{\omega_{max}^2}{2}(1-\cos ka)$

$$2\omega\,d\omega = \frac{\omega_{max}^2}{2}\sin ka\ a\,dk.$$

$$\frac{dk}{d\omega} = \frac{4\omega}{a\omega_{max}^2\sin ka}$$

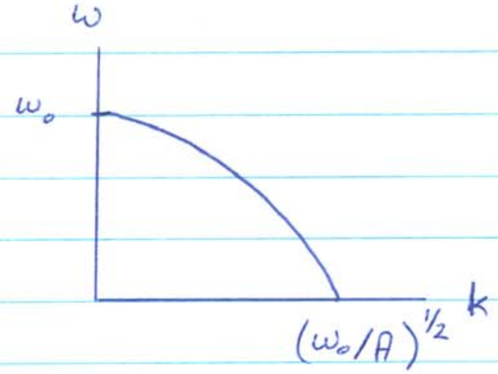
Using boring algebra one can show this becomes:

$$\frac{dk}{d\omega} = \frac{2}{a(\omega_{max}^2-\omega^2)^{1/2}}$$

$$\therefore\ D(\omega) = \frac{L}{2\pi}\frac{2}{a(\omega_{max}^2-\omega^2)^{1/2}}$$

$$= \frac{N}{\pi(\omega_{max}^2-\omega^2)^{1/2}} \qquad \text{for each polarization type}$$

b) $\omega = \omega_0 - Ak^2$



If $\omega > \omega_0$, wave can't propagate $\Rightarrow D(\omega) = 0$.

For $\omega < \omega_0$, # modes per polarization with wavevector between $\vec{k}$ and $\vec{k} + d^3k$ is $\left(\frac{L}{2\pi}\right)^3 d^3k$.

$$\therefore D(\omega)\, d\omega = \int_{\substack{\text{shell of} \\ \text{thickness } dk}} \left(\frac{L}{2\pi}\right)^3 d^3k.$$

$$= \left(\frac{L}{2\pi}\right)^3 4\pi k^2\, dk.$$

$$D(\omega) = \left(\frac{L}{2\pi}\right)^3 4\pi k^2 \frac{dk}{d\omega}$$

Now $\omega = \omega_0 - Ak^2 \Longrightarrow \dfrac{dk}{d\omega} = \dfrac{1}{2Ak}$.

$$\therefore D(\omega) = \left(\frac{L}{2\pi}\right)^3 4\pi k^2 \frac{1}{2Ak}.$$

$$= \left(\frac{L}{2\pi}\right)^3 \frac{2\pi}{A^{3/2}} (\omega_0 - \omega)^{1/2}$$ is density of modes per polarization.

4a) Crystal = rigidly coupled layers
i.e. a 2 dim. lattice

$$\text{Energy } U = \int_0^{\omega_D} \underbrace{D(\omega)\,d\omega}_{\substack{\#\text{ states in energy}\\ \text{range } d\omega}} \quad \hbar\omega \quad \underbrace{\langle n \rangle}_{\text{phonon occupation \#}}$$

Density of States For 2 dim. Lattice

modes with wavevector in range $\vec{k}$ to $\vec{k} + d^2k$ is $\left(\frac{L}{2\pi}\right)^2 d^2k$.

$$D(\omega)\,d\omega = \int \left(\frac{L}{2\pi}\right)^2 d^2k.$$

circumference
of circle of radius k.

$$= \left(\frac{L}{2\pi}\right)^2 2\pi k\, dk$$

$$D(\omega) = \left(\frac{L}{2\pi}\right)^2 2\pi k \frac{dk}{d\omega} \qquad \text{where} \qquad \omega = vk. \quad k = \frac{\omega}{v}$$

$$= \left(\frac{L}{2\pi}\right)^2 \frac{2\pi k}{v}$$

$$U = \int_0^{\omega_D} \left(\frac{L}{2\pi}\right)^2 \frac{2\pi k}{v} \hbar\omega \frac{1}{e^{\hbar\omega/kT} - 1}\, d\omega.$$

$$= \left(\frac{L}{2\pi}\right)^2 \frac{2\pi\hbar}{v^2} \int_0^{\omega_D} \frac{\omega^2}{e^{\hbar\omega/kT} - 1}\, d\omega$$

$$= \left(\frac{L}{2\pi}\right)^2 \frac{2\pi\hbar}{v^2} \left(\frac{kT}{\hbar}\right)^3 \int_0^{x_D} \frac{x^2}{e^x - 1}\, dx \qquad x \equiv \frac{\hbar\omega}{kT}$$

At low temperatures $x_D \to \infty$ and integral yields a value independent of T.

$\therefore \; U \propto T^3$

Hence specific heat $C = \dfrac{\partial U}{\partial T} \sim T^2$.