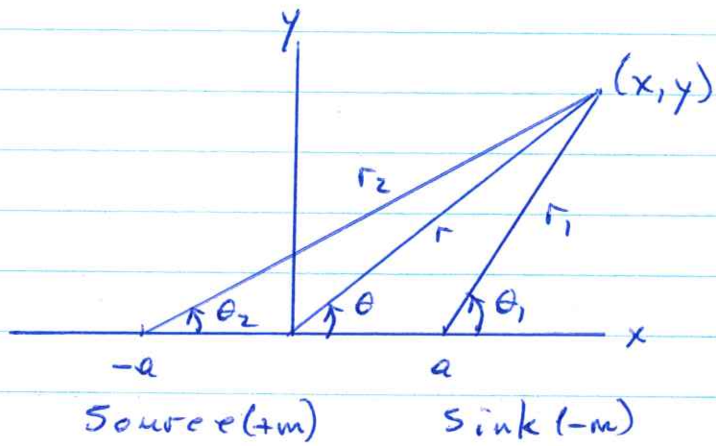
Assignment 5

1. Doublet = Source at $-a$ + Sink at $+a$.



Total Stream Fct. $\psi = \frac{m}{2\pi}\theta_2 - \frac{m}{2\pi}\theta_1$

Now $\tan\theta_1 = \frac{y}{x-a}$ & $\tan\theta_2 = \frac{y}{x+a}$

$= \frac{r\sin\theta}{r\cos\theta - a}$ $\quad = \frac{r\sin\theta}{r\cos\theta + a}$

$$\tan\theta_2 - \tan\theta_1 = r\sin\theta\left(\frac{1}{r\cos\theta + a} - \frac{1}{r\cos\theta - a}\right)$$

$$\frac{\sin\theta_2}{\cos\theta_2} - \frac{\sin\theta_1}{\cos\theta_1} = \frac{-2ar\sin\theta}{r^2\cos^2\theta + a^2}$$

$$\frac{\sin(\theta_2 - \theta_1)}{\cos\theta_1\cos\theta_2} = \frac{-2ar\sin\theta}{r^2\cos^2\theta + a^2}$$

$$\sin(\theta_2 - \theta_1) = \frac{-2ar\sin\theta}{r^2\cos^2\theta + a^2}\cos\theta_1\cos\theta_2.$$

We now take dipole limit.

$$\lim_{\substack{a \to 0 \\ m \to \infty \\ am \to \pi K}} m \sin(\theta_2 - \theta_1) = \lim_{\substack{a \to 0 \\ m \to \infty \\ am \to \pi K}} \frac{-2amr\sin\theta}{r^2\cos^2\theta + a^2} \cos\theta_1 \cos\theta_2$$

$$= \lim_{\substack{a \to 0 \\ am \to \pi K}} \frac{-2\pi K r \sin\theta}{r^2 \cos^2\theta} \cos^2\theta$$

$$= -\frac{2\pi K \sin\theta}{r}$$

For $\theta_2 - \theta_1$ small $\Rightarrow$ $m(\theta_2 - \theta_1) = \dfrac{-2\pi K \sin\theta}{r}$

$\therefore$ stream fct. $\psi = -\dfrac{K \sin\theta}{r}$

2a) From notes: $\psi = Ur\sin\theta + \frac{m\theta}{2\pi}$

Velocity: $v_r = U\cos\theta + \frac{m}{2\pi r}$

$v_\theta = -U\sin\theta$

Stagnation point by symmetry occurs on $-x$ axis.

$\therefore$ set $v_r(\theta = \pi) = 0$

$$-U + \frac{m}{2\pi r} = 0$$

$$r = \frac{m}{2\pi U} \equiv +b.$$

$\therefore$ stagnation point occurs at $x = -b = -\frac{m}{2\pi U}$.

b) $\psi(x=-b) = \psi(\theta=\pi, r=b)$

$$= Ub\sin\pi + \frac{m\pi}{2\pi}$$

$$= \frac{m}{2}$$

c) $\psi = \frac{m}{2} \Rightarrow \frac{m}{2} = Ur\sin\theta + \frac{m\theta}{2\pi}$

$$\frac{m}{2} = Uy + \frac{m\theta}{2\pi}$$

$$Uy = \frac{m}{2\pi}(\pi - \theta)$$

$$y = b(\pi - \theta)$$

d) Bernoulli Equ. along streamline (z = const.) gives:

$$P_0 + \frac{1}{2}\rho U^2 = P + \frac{1}{2}\rho v^2$$

Now $v^2 = v_r^2 + v_\theta^2$

$$= \left(U\cos\theta + \frac{m}{2\pi r}\right)^2 + (-U\sin\theta)^2$$

$$= U^2\left(1 + \frac{2b}{r}\cos\theta + \frac{b^2}{r^2}\right)$$

$$\therefore P = P_0 - \frac{\rho U^2}{2}\left(\frac{2b}{r}\cos\theta + \frac{b^2}{r^2}\right)$$

3. In notes, we showed steady flow between 2 plates gives:

$$P = -\rho g y + f_1(x)$$

$$u = \frac{1}{2\mu} \frac{\partial P}{\partial x} y^2 + c_1 y + c_2$$

Couette flow: $u(y=0) \Rightarrow c_2 = 0$

$$u(y=b) = U \Rightarrow \quad U = \frac{1}{2\mu} \frac{\partial P}{\partial x} b^2 + c_1 b$$

$$c_1 = \frac{1}{b}\left[U - \frac{1}{2\mu} \frac{\partial P}{\partial x} b^2 \right]$$

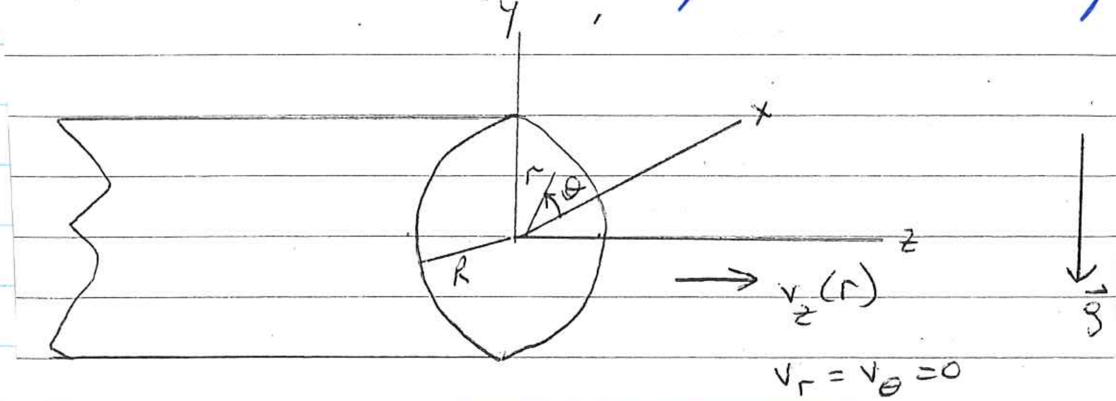
$$\therefore\ u = \frac{1}{2\mu} \frac{\partial P}{\partial x} y^2 + \frac{1}{b}\left[U - \frac{1}{2\mu} \frac{\partial P}{\partial x} b^2 \right] y$$

$$= \frac{y}{b} U + \frac{1}{2\mu} \frac{\partial P}{\partial x}\left[y^2 - \frac{y}{b} b^2 \right]$$

$$\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U} \frac{\partial P}{\partial x}\left[\frac{y}{b} - \frac{y^2}{b^2} \right]$$

$$\frac{u}{U} = \frac{y}{b} + P' \frac{y}{b}\left(1 - \frac{y}{b} \right) \text{ where } P' \equiv \frac{-b^2}{2\mu U} \frac{\partial P}{\partial x}$$

4. Poiseuille Flow: laminar flow in circular pipe



Velocity $\quad v_z = v_z(r)$

$\qquad\qquad v_r = v_\theta = 0$

a) Substituting $\vec{v}$ into Navier Stokes equations for steady flow gives:

$$0 = -\rho g \sin\theta - \frac{\partial P}{\partial r} \qquad (1)$$

$$0 = -\rho g \cos\theta - \frac{1}{r}\frac{\partial P}{\partial \theta} \qquad (2)$$

$$0 = -\frac{\partial P}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) \qquad (3)$$

b) $(1) + (2) \Rightarrow \quad P = -\rho g r \sin\theta + f_1(z)$

$\therefore \frac{\partial P}{\partial z}$ is independent of θ + r.

$$(3) \Rightarrow \quad \frac{\partial P}{\partial z} = \frac{\mu}{r}\frac{d}{dr}\left(r \frac{dv_z}{dr}\right)$$

$$\frac{r^2}{2\mu}\frac{\partial P}{\partial z} = r\frac{dv_z}{dr} + K_1$$

$$\frac{r}{2\mu}\frac{\partial P}{\partial z} = \frac{dv_z}{dr} + \frac{K_1}{r}$$

$$\frac{r^2}{4\mu}\frac{\partial P}{\partial z} = v_z + K_1 \ln r + K_2$$

But v_z is well defined at $r=0 \Longrightarrow K_1 = 0$.

$$\therefore \; v_z = \frac{r^2}{4\mu}\frac{\partial P}{\partial z} - K_2$$

But $v_z(r=R)=0 \Longrightarrow \quad v_z = \frac{1}{4\mu}\frac{\partial P}{\partial z}\left(r^2 - R^2\right)$

c) Volume Flow rate $\quad Q = \int_0^R v_z \, 2\pi r \, dr$

$$= \frac{1}{4\mu}\frac{\partial P}{\partial z} 2\pi \int_0^R (r^2 - R^2)\, r \, dr$$

$$Q = -\frac{\pi}{8\mu}\frac{\partial P}{\partial z} R^4$$