Assignment 1

1. $$P(4H, 1T) = \frac{5!}{4!\,1!} P_H^4 P_T^1$$

$$= 5 \left(\frac{3}{4}\right)^4 \cdot \frac{1}{4}$$

$$= 40\%.$$

2a) $$P(\text{no molecule in excited state}) = \frac{5!}{5!\,0!} P_{Gnd}^5 P_{Exc}^0$$

$$= \left(\frac{9}{10}\right)^5$$

$$= 59\%$$

b) $$P(1 \text{ excited molecule}) = \frac{5!}{4!\,1!} P_{Gnd}^4 P_{Exc}^1$$

$$= 5 \cdot \left(\frac{9}{10}\right)^4 \cdot \frac{1}{10}$$

$$= 33\%$$

3. $$P(500H, 500T) = \frac{1000!}{500!\,500!} \left(\frac{1}{2}\right)^{500} \left(\frac{1}{2}\right)^{500}$$

$$\ln P = \ln 1000! - 2\ln 500! - 1000 \ln 2$$

$$\simeq 1000 \ln 1000 - 1000 + \frac{1}{2} \ln 2000\pi$$

$$-2\left[500 \ln 500 - 500 + \frac{1}{2} \ln 1000\pi\right]$$

$$- 1000 \ln 2$$

$$\ln P = 1000 \ln 1000 - 1000 \ln 500 - 1000 \ln 2$$

$$+ \frac{1}{2} \ln 2000\pi - \ln 1000\pi$$

$$= \frac{1}{2}\left(\ln 2 + \ln 1000\pi\right) - \ln 1000\pi$$

$$= \frac{1}{2} \ln 2 - \frac{1}{2} \ln 1000\pi$$

$$= \ln \left(\frac{2}{1000\pi}\right)^{1/2}$$

$$\therefore P = \frac{1}{\sqrt{500\pi}}$$

$$= \frac{1}{10\sqrt{5\pi}}$$

$$= 2.5\%$$

4. $$\sigma^2 \equiv \sum_{n=0}^{N} (n-\bar{n})^2 P_N(n)$$

$$= \sum_{n=0}^{N} \left[n^2 - 2n\bar{n} + \bar{n}^2 \right] P_N(n)$$

$$= \sum_{n=0}^{N} n^2 P_N(n) - 2\bar{n} \underbrace{\sum_{n=0}^{N} n P_N(n)}_{=\bar{n}} + \bar{n}^2 \underbrace{\sum_{n=0}^{N} P_N(n)}_{=1}$$

$$= \sum_{n=0}^{N} n^2 P_N(n) - \bar{n}^2$$

Now $$\sum_{n=0}^{N} n^2 P_N(n)$$

$$= \sum_{n=0}^{N} \frac{n^2 \, N!}{n!\,(N-n)!} p^n q^{N-n}$$

$$= \sum_{n=1}^{N} \frac{n \, N!}{(n-1)!\,(N-n)!} p^n q^{N-n}$$

$$= N \sum_{i=0}^{N-1} \frac{(i+1)\,(N-1)!}{i!\,(N-i-1)!} p^{i+1} q^{N-i-1} \quad \text{where } i \equiv n-1$$

$$= Np \sum_{i=0}^{L} \frac{(i+1)\,L!}{i!\,(L-i)!} p^i q^{L-i} \quad \text{where } L \equiv N-1$$

$$= Np \left\{ \sum_{i=1}^{L} \frac{L!}{(i-1)!\,(L-i)!} p^i q^{L-i} + \underbrace{\sum_{i=0}^{L} \frac{L!}{i!\,(L-i)!} p^i q^{L-i}}_{=(p+q)^L = 1} \right\}$$

Setting $j \equiv i-1$ & $M \equiv L-1$, one can show

$$\sum_{i=1}^{L} \frac{L!}{(i-1)!(L-i)!} p^i q^{L-i} = Lp = (N-1)p$$

$$\therefore \sum_{n=0}^{N} n^2 P_N(n) = Np\left\{(N-1)p+1\right\}$$

$$\& \quad \sigma^2 = Np\left\{(N-1)p+1\right\} - (Np)^2$$

$$= Np\left[Np - p + 1 - Np\right]$$

$$= Np(1-p)$$

$$\therefore \sigma^2 = Npq$$

5 a) $\bar{x} = 75 \qquad \sigma = 8$

$$P(x > 85\%) = \frac{1}{\sqrt{2\pi}} \int_{85}^{100} \frac{e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2}}{\sigma} dx$$

$$\simeq \frac{1}{\sqrt{2\pi}} \int_{85}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2}}{\sigma} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{1.25}^{\infty} e^{-z^2/2} dz \quad \text{where } z \equiv \frac{x-\bar{x}}{\sigma}$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_{0}^{1.25} e^{-z^2/2} dz$$

$$= .50 - .3944$$

$\therefore P(x > 85\%) = 10.5\%.$

b) Let middle 400 students have grades from $\bar{x} - \omega$ to $\bar{x} + \omega$.

$$P\left(x \in [\bar{x}-\omega, \bar{x}+\omega]\right) = \frac{400}{500} = \frac{1}{\sqrt{2\pi}} \int_{\bar{x}-\omega}^{\bar{x}+\omega} \frac{e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2}}{\sigma} dx$$

$$0.80 = \frac{1}{\sqrt{2\pi}} \int_{-\omega/\sigma}^{\omega/\sigma} e^{-z^2/2} dz \quad \text{where } z \equiv \frac{x-\bar{x}}{\sigma}$$

$$0.40 = \frac{1}{\sqrt{2\pi}} \int_{0}^{\omega/\sigma} e^{-z^2/2} dz$$

$$\therefore \frac{\omega}{\sigma} = 1.28$$

$$\omega = 10.2$$

$\therefore$ middle 400 students have grades between 65% and 85%.