Electrodynamics II

Lecture Notes

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Textbooks.

1. * Introduction to Electrodynamics
by David Griffiths, Prentice-Hall, Englewood Cliffs N.J., 1981.

2. * Classical Electromagnetic Radiation
by J. B. Marion & M. A. Heald
Academic Press, 1980.

3. Electricity & Magnetism
by E. M. Purcell, McGraw-Hill, Toronto, 1965.

4. Electromagnetism
by. P. Lorrain & D. R. Corson
W. H. Freeman & Co., San Francisco, 1979.

5. Principles of Electromagnetism
by M. Schwartz, McGraw-Hill, Toronto, 1972.

6. Feynman Lectures on Physics
by. R. P. Feynman, Addison-Wesley, Toronto, 1966.

7 Classical Electrodynamics
by. J. D. Jackson, John Wiley & Sons, Toronto, 1975.

8. Electromagnetic Fields
by R. K. Wangsness, John Wiley & Sons, Toronto, 1979.

9. Electromagnetic Theory
by D. R. Frankl, Prentice-Hall, 1986.

Spring Schedule

Lecture Topic	Date
Coulomb Force	Jan. 2
Electric Field	Jan. 4
Gauss Law	Jan. 6
Electrostatic Potential	Jan. 9
Image Charges	Jan. 11
Laplace Eqn. (Rect. coords.)	Jan. 13
Laplace Eqn. (Spher. coords.)	Jan. 16
Magnetic Fields	Jan. 18
Faraday Effect & Displace. Curr.	Jan. 20
Dielectrics	Jan. 23
$\vec{B}$ & $\vec{H}$	Jan. 25
Ohm's Law & Poynting Vector	Jan. 27
Test 1	Feb. 3
Scalar Wave Eqn.	Jan. 30
Poynting Vector for Complex Fields	Feb. 1
Plane Waves in Cond. Media	Feb. 3
Reflection & Refraction - Normal Inc.	Feb. 6
Reflection & Refraction - Oblique Inc.	Feb. 8
General Waveguide Eqns.	Feb. 10
Rect. Waveguide	Feb. 20
Coaxial Waveguide	Feb. 22
Test 2.	Mar. 1
Retarded Potentials	Feb. 24
Electric Dipole Radiation	Feb. 27
Magnetic Dipole Radiation	Mar. 3
Linear Antenna	Mar. 6
Liénard-Wiechert Potentials	Mar. 8.

Lecture Topic	Date
Point Charge in Motion	Mar. 10, 13
Relativity	Mar. 15
Origin of Magnetism	Mar. 17
Field Transformations	Mar. 22
	+ 3 lectures

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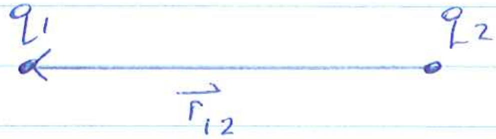
I Electric Fields

Coulomb's Law

Consider two charges separated by distance r_{12}:



Force on q_1 is $\vec{F}_1 = \dfrac{k\, q_1 q_2}{r_{12}^2}\, \hat{r}_{12}$

Force on q_2 is $\vec{F}_2 = \dfrac{k\, q_2 q_1}{r_{21}^2}\, \hat{r}_{21}$

$\Rightarrow \vec{F}_1 = -\vec{F}_2$

k is constant of proportionality

Experimental Observations

1) Electric charges come in integral multiples of electron charge $(-e)$ and/or the proton charge $(+e)$

2) $|$proton charge$|$ = $|$electron charge$|$

3) Fractional charges $\left(-\frac{e}{3}, +\frac{2}{3}e\right)$ have been postulated for quarks but never observed.

Units

1) MKS

$k = 9 \times 10^9$ Nt meter^2/Coulomb^2

F = force measured in Newtons
  1 Nt = kg. meter/sec^2

r = distance measured in meters

q = charge measured in Coulombs.

Charge of electron = 1.6×10^{-19} Coul. (Expt. Result)

1 Coulomb = 6.25×10^{18} electron charges.

2) CGS

$k = 1$ (no units)

F = force measured in dynes
  1 dyne = gm cm/sec^2

$= 10^{-5}$ Nt.

r = distance measured in cm.

q = charge measured in esu

1 esu = $\sqrt{\text{dyne cm}^2}$

Relation Between Coulomb + esu

Force between 2 charges q separated by distance r is $F = \frac{kq^2}{r^2}$.

$$F(\text{Newtons}) = F(\text{Newtons})$$

$$10^{-5} F(\text{dynes}) = F(\text{Newtons})$$

$$10^{-5} \frac{[q(\text{esu})]^2}{[r(\text{cm})]^2} = k \frac{[q(\text{Coul})]^2}{[r(\text{meters})]^2}$$

$$\frac{10^{-5} [q(\text{esu})]^2}{10^{4} [r(\text{meters})]^2} = k \frac{[q(\text{Coul})]^2}{[r(\text{meters})]^2}$$

$$[q(\text{esu})]^2 = 10^9 k [q(\text{Coul})]^2$$

$$q(\text{esu}) = \sqrt{10^9 k}\; q(\text{Coul}).$$

Now $k = 9 \times 10^9$ Nt meter^2/Coul^2

$$\boxed{\therefore\ q(\text{esu}) = 3 \times 10^9\, q(\text{Coul})}$$

Here we have omitted the units of k which will be inserted when doing calculations.

$\therefore$ charge of electron $= 3 \times 10^{9} \times 1.6 \times 10^{-19}$

$= 4.8 \times 10^{-10}$ esu.

Example

Find force on charges +2 esu, +5 esu separated by distance of 10 cm.

$q_1 = +2$ esu $\qquad\qquad$ $q_2 = +5$ esu

$\xleftarrow{\quad r = 10 \text{ cm.} \quad}\!\!\rightarrow$

Solution 1 (CGS)

$$F = \frac{q_1 q_2}{r^2}$$

$$= \frac{(+2 \text{ esu})(+5 \text{ esu})}{(10 \text{ cm})^2}$$

$$= \frac{1}{10} \text{ dyne.}$$

$$F = 10^{-6} \text{ Nt.}$$

Solution 2 (MKS)

$$F = \frac{k\, q_1 q_2}{r^2}$$

$q_1 = 2$ esu

$= \dfrac{2}{3 \times 10^{9}}$ Coul.

$= 6.67 \times 10^{-10}$ Coul.

$q_2 = 5$ esu

$= 1.67 \times 10^{-9}$ Coul.

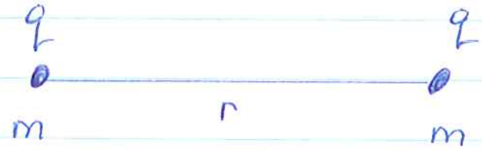
$$F = \frac{9 \times 10^{9} \frac{\text{Nt meter}^2}{\text{Coul}^2} \times 6.67 \times 10^{-10}\,\text{Coul} \times 1.67 \times 10^{-9}\,\text{Coul}}{(.10 \text{ meter})^2}$$

$\therefore F = 10^{-6}$ Nt.

In this course CGS units will be used to avoid having to write k.

Examples.

1) What charge must be on 2 equal 1 gm. masses to balance the gravitational force?



Gravitational Force = Coulomb Force.

$$\frac{G m^2}{r^2} = \frac{q^2}{r^2}.$$

$$q = \sqrt{G}\ m$$

$$= \left(6.67 \times 10^{-8} \frac{\text{dyne cm}^2}{\text{gm}^2}\right)^{1/2} 1 \text{ gm}.$$

$$= 2.6 \times 10^{-4} \sqrt{\text{dyne cm}^2}$$

$\therefore$ charge is $q = 2.6 \times 10^{-4}$ esu.

2) Find equation of motion of two particles having equal mass m and charge q. Initially the separation distance is $2x_0$.

Solution

Let particle positions be $x(t)$ and $-x(t)$.

Eqn. of motion: $m\ddot{x} = \dfrac{q^2}{(2x)^2}$

$$m\ddot{x}\,\dot{x} = \frac{q^2}{4}\,\frac{\dot{x}}{x^2}$$

$$\int_0^t m\,\ddot{x}\,\dot{x}\,dt = \int_0^t \frac{q^2}{4}\,\frac{\dot{x}}{x^2}\,dt$$

$$m\left(\frac{\dot{x}^2}{2} - 0\right) = \frac{q^2}{4}\left(-\frac{1}{x} + \frac{1}{x_0}\right).$$

$$m\frac{\dot{x}^2}{2} = \frac{q^2}{4}\left(\frac{1}{x_0} - \frac{1}{x}\right)$$

$$\dot{x} = \left\{\frac{q^2}{2m}\left(\frac{1}{x_0} - \frac{1}{x}\right)\right\}^{1/2}$$

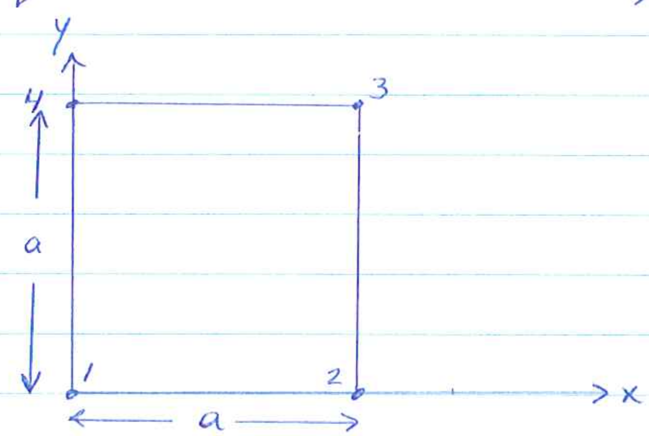
$\therefore$ at an infinite distance apart, particle speed is $\sqrt{\dfrac{q^2}{2mx_0}}$.

Finally to find $x(t)$, one must solve:

$$\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{1}{x_0} - \frac{1}{x}}} = \sqrt{\frac{q^2}{2m}} \int_0^t dt$$

$$= \sqrt{\frac{q^2}{2m}}\, t.$$

3) Four charges q_i $(i=1,2,3,4)$ are fixed on 4 corners of a square. Find the force felt by q_1 due to the other charges.



$q_1 = q$

$q_2 = 3q$

$q_3 = -2q$

$q_4 = q$

Force on q_1

Force on q_1 exerted by q_2 is $\frac{q(3q)}{a^2}(-\hat{x})$

Force on " " q_3 " $\frac{q(-2q)}{(\sqrt{2}a)^2}\left(-\frac{\hat{x}}{\sqrt{2}} - \frac{\hat{y}}{\sqrt{2}}\right)$

" " q_4 " $\frac{q\,q}{a^2}(-\hat{y})$

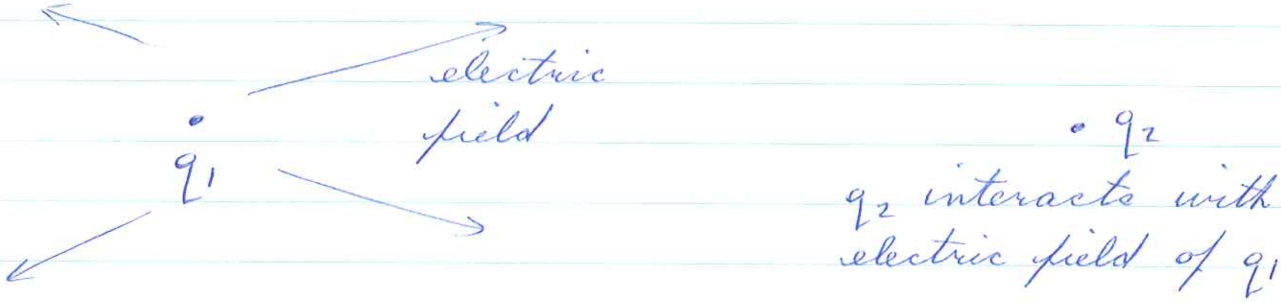
$\therefore$ total Force felt by q_1 is

$$\vec{F}_{TOT} = \frac{q^2}{a^2} \left\{ \left(-3 + \frac{1}{\sqrt{2}} \right) \hat{x} + \left(-1 + \frac{1}{\sqrt{2}} \right) \hat{y} \right\}.$$

Electric Field

Question: How does a charge know about other charges very far away?

Answer: Charges send messages called electric fields.

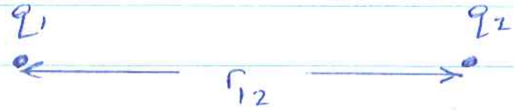


What happens if q_1 moves?

Answer: The message or electric field changes. The new message travels at the speed of light; that we will discuss later. For the time being we discuss static charges, i.e. ones that don't move.

Interaction of Two Charges

Consider once again 2 charges q_1 & q_2 separated by distance r_{12}.



Force on q_1 is $\vec{F} = \dfrac{q_1 q_2}{r_{12}^2} \hat{r}_{12}$

$$= q_1 \left(\frac{q_2 \hat{r}_{12}}{r_{12}^2} \right)$$

$$= q_1 \vec{E}(\vec{r}_1)$$

where $\boxed{\vec{E}(\vec{r}_1) \equiv \dfrac{q_2 \hat{r}_{12}}{r_{12}^2}}$ is called the electric field.

$\therefore$ the electric field is the force a unit charge feels.

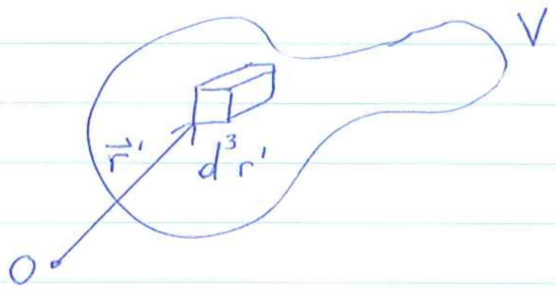
Electric Field From n Charges.

Consider next n charges q_i at positions $\vec{r}_i$ $i = 1, 2, \ldots, n$.

Electric field at position $\vec{r}$ is $\boxed{\vec{E}(\vec{r}) = \displaystyle\sum_{i=1}^{n} q_i \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}}$

Electric Field of Charge Distribution

Consider a volume V filled with charge density ρ.



Consider a volume element d^3r' located at $\vec{r}\,'$.
Charge inside d^3r' is $\rho(\vec{r}\,')\,d^3r'$.

$\therefore$ electric field at $\vec{r}$ due to charge $\rho(\vec{r}\,')\,d^3r'$ is:

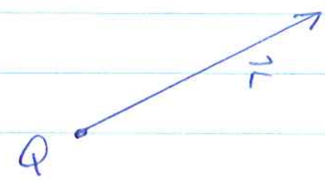
$$d\vec{E}(\vec{r}) = \rho(\vec{r}\,')\,d^3r' \,\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$\therefore$ electric field at $\vec{r}$ due to charge inside V is:

$$\vec{E}(\vec{r}) = \int_V \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,\rho(\vec{r}\,')\,d^3r'$$

Examples.

1) Find electric field of point charge Q located at the origin.



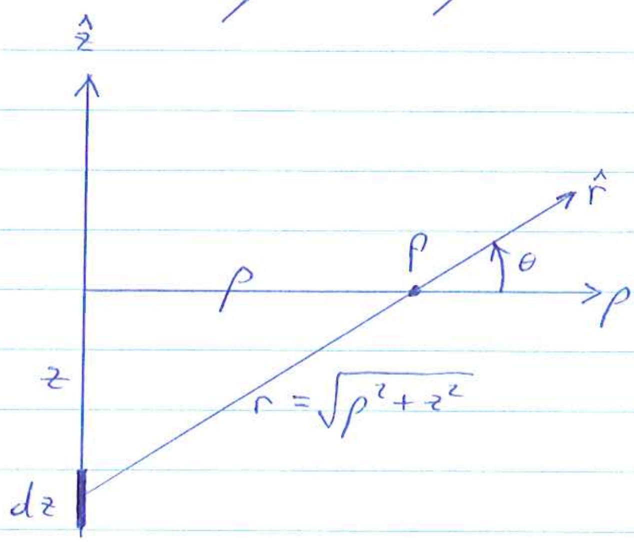
$$\vec{E}(\vec{r}) = \frac{Q}{r^2}\,\hat{r}$$

$$= \frac{Q\,\vec{r}}{r^3} \quad \text{where } \vec{r} = (x, y, z)$$

$$r^2 = x^2 + y^2 + z^2$$

$\therefore$ a unit charge placed at $\vec{r}$ experiences a force $\frac{Q}{r^3}\,\vec{r}$.

2) Find electric field due to infinite line of linear charge density λ.



By symmetry $\vec{E}$ is cylindrically symmetric about the line of charge or z axis. $\therefore$ $\vec{E}$ depends only on the radial vector $\vec{\rho}$ from the wire.

Consider a small length dz at position z on the wire.
$\therefore$ charge on length dz is $\lambda\, dz$.

$\therefore$ electric field due to $\lambda\, dz$ at P is:

$$d\vec{E} = \frac{\lambda\, dz}{r^2}\,\hat{r}$$

$$= \frac{\lambda\, dz}{r^2}\left(\cos\theta\;\hat{\rho} + \sin\theta\;\hat{z}\right)$$

$$= \frac{\lambda\, dz}{r^2}\left(\frac{\rho}{r}\,\hat{\rho} + \frac{z}{r}\,\hat{z}\right)$$

$$= \lambda\, dz\;\frac{(\rho\,\hat{\rho} + z\,\hat{z})}{(\rho^2 + z^2)^{3/2}}$$

$\therefore$ electric field at P of infinite line of charge is

$$\vec{E} = \int_{-\infty}^{\infty} \lambda\, dz\;\frac{(\rho\,\hat{\rho} + z\,\hat{z})}{(\rho^2 + z^2)^{3/2}}$$

$$= \lambda\left\{\hat{\rho}\,\rho\int_{-\infty}^{\infty}\frac{dz}{(\rho^2+z^2)^{3/2}} + \hat{z}\int_{-\infty}^{\infty}\frac{z\,dz}{(\rho^2+z^2)^{3/2}}\right\}$$

$$\vec{E} = \lambda \left\{ \hat{\rho}\,\rho \left[\int_{-\infty}^{0} \frac{dz}{(\rho^2+z^2)^{3/2}} + \int_{0}^{\infty} \frac{dz}{(\rho^2+z^2)^{3/2}} \right] \right.$$

$$\left. + \hat{z} \left[\int_{-\infty}^{0} \frac{z\,dz}{(\rho^2+z^2)^{3/2}} + \int_{0}^{\infty} \frac{z\,dz}{(\rho^2+z^2)^{3/2}} \right] \right\}$$

$$= \lambda \left\{ \hat{\rho}\,\rho \left[\int_{0}^{\infty} \frac{dz'}{(\rho^2+z'^2)^{3/2}} + \int_{0}^{\infty} \frac{dz}{(\rho^2+z^2)^{3/2}} \right] \right.$$

$$\left. + \hat{z} \left[-\int_{-0}^{\infty} \frac{z'\,dz'}{(\rho^2+z'^2)^{3/2}} + \int_{0}^{\infty} \frac{z\,dz}{(\rho^2+z^2)^{3/2}} \right] \right\}$$

where $z' \equiv -z$.

$$= 2\lambda\,\hat{\rho}\,\rho \int_{0}^{\infty} \frac{dz}{(\rho^2+z^2)^{3/2}}$$

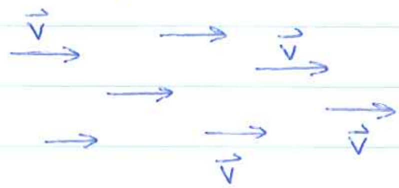
$$= \hat{\rho}\; 2\lambda\rho \int_{0}^{\pi/2} \frac{\rho \sec^2\theta\, d\theta}{\rho^3 \sec^3\theta} \qquad \begin{aligned} z &= \rho\tan\theta \\ dz &= \rho\sec^2\theta\, d\theta \end{aligned}$$

$$= \hat{\rho}\; \frac{2\lambda\rho}{\rho^2} \int_{0}^{\pi/2} \cos\theta\, d\theta \qquad \begin{aligned} z = 0 &\Rightarrow \theta = 0 \\ z = \infty &\Rightarrow \theta = \frac{\pi}{2} \end{aligned}$$

$$\therefore \vec{E} = \frac{2\lambda}{\rho}\,\hat{\rho}$$

Flux

Consider a river where the water flows with uniform velocity. $\vec{v}$.

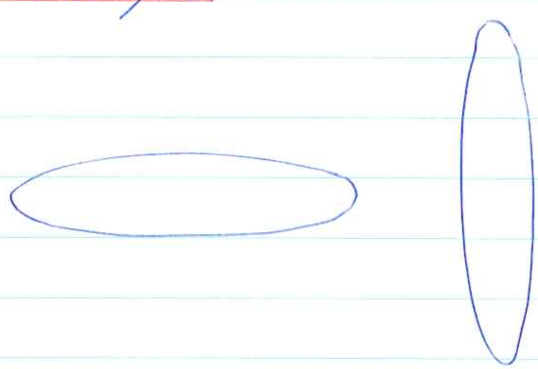


We would like to measure how much water passes through a hoop. A hoop is described by:

1) its area A

2) its orientation

Example.

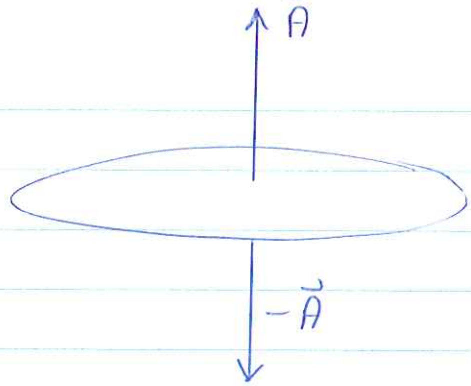


The above two hoops have the same area but different orientation. Both, the hoop area and orientation can be described by a vector $\vec{A}$ where

1) $|\vec{A}|$ = area of hoop

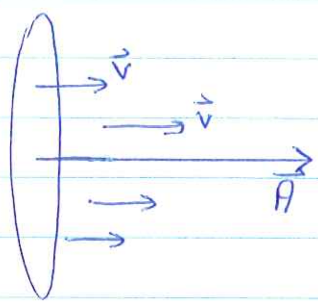
2) $\vec{A}$ points in direction $\perp$ to hoop's surface. Of course there are two surfaces and hence two opposing directions so we must be careful here.

i.e.



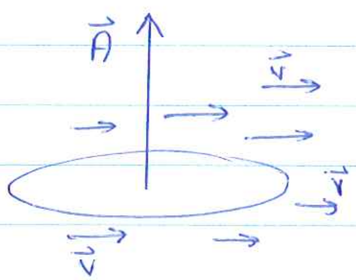
Now let's measure how much water gets through the hoop.

Case 1: Hoop area vector $\vec{A} \parallel \vec{v}$



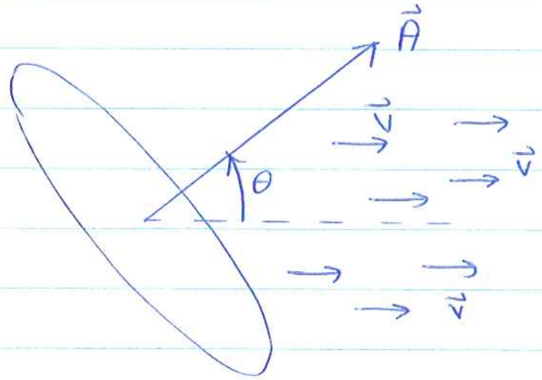
Flow rate or flux of water through hoop $\Phi = n v A$. (n = water density)

Case 2: $\vec{A} \perp \vec{v}$



Flux $\Phi = 0$.

Case 3



Flux $\Phi = n v A \cos\theta$

$= n \vec{v} \cdot \vec{A}$

$= \vec{J} \cdot \vec{A}$

where $\vec{J} = n\vec{v}$ is current density.

Suppose now that $\vec{v}$ & hence $\vec{J}$ isn't uniform. We then divide the hoop into small area elements $d\vec{a}$.

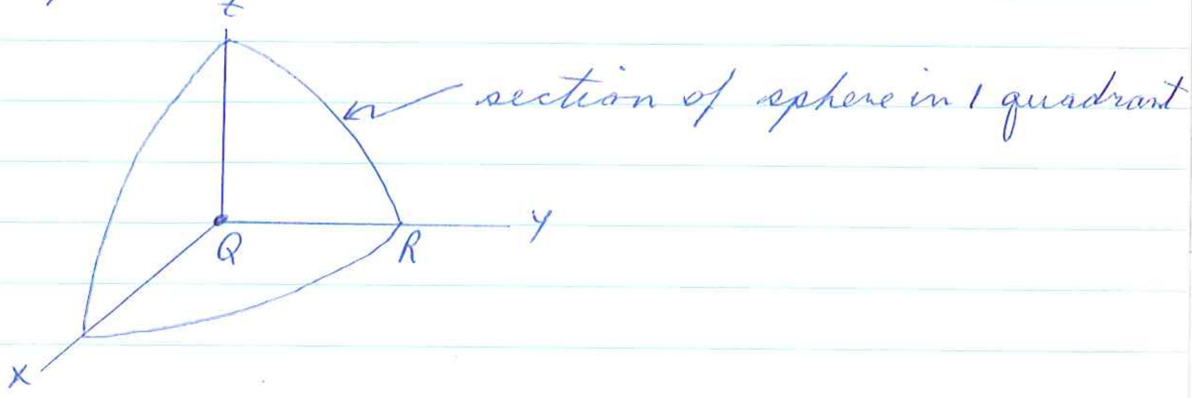
Flux through $d\vec{a}$ is $d\Phi = \vec{J} \cdot d\vec{a}$

Total flux through hoop $\Phi = \int_{\substack{\text{hoop} \\ \text{area}}} \vec{J} \cdot d\vec{a}$

Mathematically we say the flux of vector $\vec{J}$ through area A is
$$\boxed{\Phi = \int_A \vec{J} \cdot d\vec{a}.}$$

Flux due to a Point Charge

Find the electric field flux through the surface of a sphere of radius R due to a point charge Q at the sphere's center.



Electric field of Q is $\vec{E}(\vec{r}) = \dfrac{Q}{r^2}\,\hat{r}$.

Flux of electric field through sphere $\Phi = \int_{\substack{\text{surface}\\ \text{of sphere}}} \vec{E} \cdot d\vec{a}$.

We shall specify that $d\vec{a}$ points outward i.e. $d\vec{a} \parallel +\hat{r}$ rather than inward so that $\Phi > 0$.

$$\Rightarrow \vec{E} \cdot d\vec{a} = E\, da.$$

$$\Phi = \int E\, da.$$

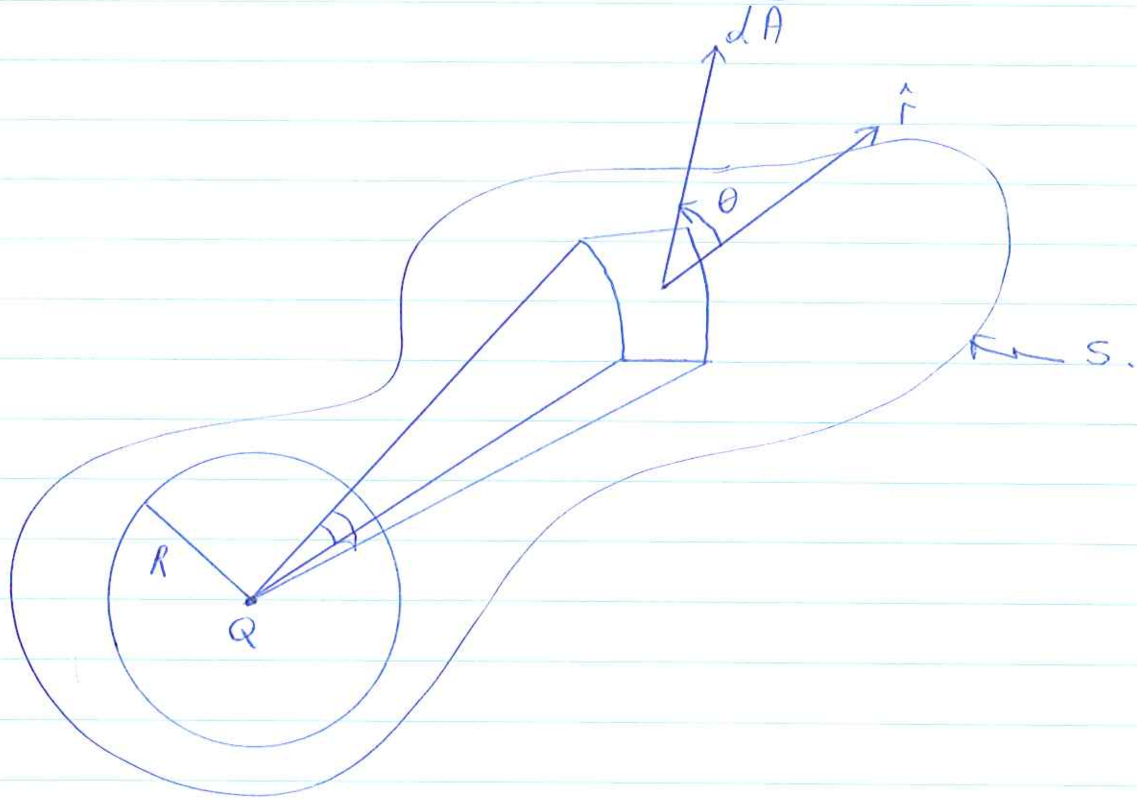
Next E is pulled outside the integral since it is constant on the sphere's surface. (i.e. E is independent of any angular coordinate)

$$\Phi = E(R) \int_{\substack{\text{surface}\\ \text{of sphere}}} da.$$

$$= \frac{Q}{R^2} \, 4\pi R^2$$

$\therefore \; \Phi = 4\pi Q$ is flux of electric field from charge Q at origin through sphere centered about origin.

Next consider the flux of a point charge through a surface S of arbitrary shape.



Surface S encloses a sphere of radius R centered about Q. Consider a small cone beginning at the origin and intersecting area da of sphere of radius R and dA of surface S.

Flux through da is $\vec{E}(\vec{R}) \cdot d\vec{a} = E(R)\, da$.

$$\left(E(R) = \frac{Q}{R^2} \right)$$

Flux through area dA is $\vec{E}(r) \cdot d\vec{A}$

$$= E(r)\, dA \cos\theta$$

$$= \left(E(R) \frac{R^2}{r^2} \right) dA \cos\theta$$

Area dA is larger than da by two factors.

1) ratio of distances squared $\left(\frac{r}{R}\right)^2$

2) $\frac{1}{\cos\theta}$ due to its inclination

$$\therefore dA = \left(\frac{r}{R}\right)^2 \frac{1}{\cos\theta} da.$$

$\therefore$ flux through area dA is $\left(E(R)\frac{R^2}{r^2}\right)\left(\frac{r^2}{R^2}\frac{1}{\cos\theta} da\right)\cos\theta$

$$= E(R)\, da.$$

$\therefore$ flux through dA on arbitrary surface S

$=$ flux through da on surface of sphere.

If surface S is divided into area elements dA, we see that flux through each area element dA is the same as the flux through a different part of a sphere.

$\therefore$ flux through arbitrary surface S

$=$ flux through a sphere centered about charge

$$\therefore \int_{\substack{\text{arbitrary} \\ \text{closed surface}}} \vec{E}\cdot d\vec{A} = \int_{\substack{\text{surface of} \\ \text{sphere}}} \vec{E}\cdot d\vec{a}$$

$$= 4\pi Q.$$

Electric fields and hence fluxes of various charges are additive.

$$\boxed{\therefore \int_{\substack{\text{arb.}\\ \text{surface}}} \vec{E} \cdot d\vec{a} = 4\pi Q}$$ Gauss Law

where Q is net charge enclosed by surface and $d\vec{a}$ points outward from the surface.

Differential Form of Gauss Law.

If surface S encloses a charge density ρ, then Gauss law generalizes to:

$$\int_{\substack{\text{arb.}\\ \text{surface}}} \vec{E} \cdot d\vec{a} = 4\pi \int_V \rho \, dV.$$

Using the divergence theorem, a surface integral can be converted into a volume integral.

$$\int_V \nabla \cdot \vec{E} \, dV = 4\pi \int_V \rho \, dV$$

$$\Rightarrow \boxed{\nabla \cdot \vec{E} = 4\pi \rho.}$$ Differential Form of Gauss Law

Physical Interpretation

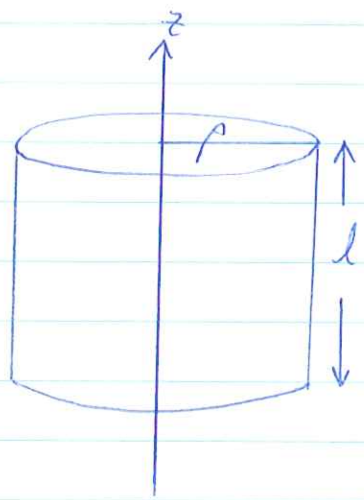
ρ = amount of charge in unit volume

Gauss Law: flux through surface S
= 4π × charge enclosed by S.

$\boxed{\text{If S is surface around unit volume, we see that } \nabla\cdot\vec{E} = \text{electric field flux out of unit volume.}}$

Examples.

1. Find electric field of infinite line of linear charge density λ.



By symmetry $\vec{E} \parallel \hat{\rho}$ and $E = E(\rho)$.
Consider a cylindrical can of length l and radius ρ centered about line of charge.

$$\int_{\substack{\text{surface}\\ \text{of can}}} \vec{E}\cdot d\vec{a} = 4\pi \underbrace{\int_V \rho \, dV}_{\text{charge enclosed by can}}$$

$$\underbrace{\int_{\text{Top of Can}} \vec{E}\cdot d\vec{a}}_{=0 \text{ since } \vec{E}\perp d\vec{a}} + \int_{\substack{\text{side of}\\ \text{can}}} \vec{E}\cdot d\vec{a} + \underbrace{\int_{\substack{\text{Bottom}\\ \text{of can}}} \vec{E}\cdot d\vec{a}}_{=0 \text{ since } \vec{E}\perp d\vec{a}} = 4\pi\lambda l.$$

$$\int_{\substack{\text{side of}\\ \text{can}}} \vec{E}\cdot d\vec{a} = 4\pi\lambda l$$

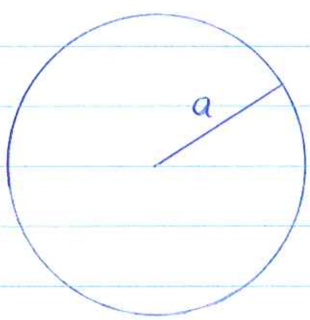
$$E(\rho)\int da = 4\pi\lambda l.$$

$$E(\rho)\, 2\pi\rho l = 4\pi\lambda l$$

$$E(\rho) = \frac{2\lambda}{\rho}$$

$$\boxed{\therefore\ \vec{E}(\rho) = \frac{2\lambda}{\rho}\,\hat{\rho}}$$

2) Find electric field of uniformly charged sphere (charge density ρ_0) of radius a.



By symmetry $\vec{E} \parallel \hat{r}$ and $E = E(r)$.

Consider a sphere of radius r.

Case 1: $r < a$

$$\int_S \vec{E}\cdot d\vec{a} = 4\pi\int_V \rho\, dV$$

$$E(r)\int da = 4\pi\,\rho_0\,\frac{4}{3}\pi r^3.$$

$$E(r)\, 4\pi r^2 = 4\pi \rho_0 \frac{4}{3}\pi r^3$$

$$E(r) = \frac{4}{3}\pi \rho_0 r$$

$$\therefore \vec{E}(r) = \frac{4}{3}\pi \rho_0 \vec{r}.$$

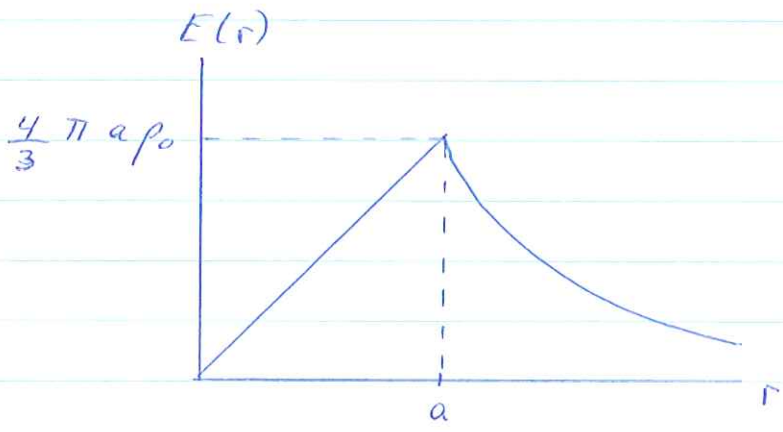
Case 2: $r > a$.

$$\int_S \vec{E} \cdot d\vec{a} = 4\pi \int_V \rho \, dV.$$

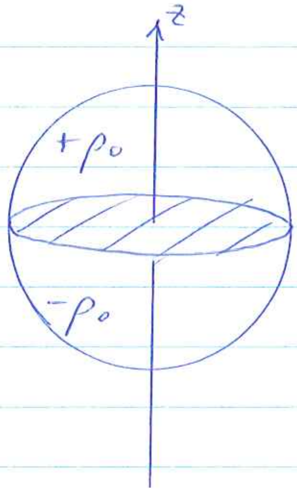
$$E(r)\, 4\pi r^2 = 4\pi \frac{4}{3}\pi a^3 \rho_0$$

$$E(r) = \frac{4}{3} \frac{\pi a^3 \rho_0}{r^2}$$

$$\vec{E}(r) = \frac{4\pi a^3 \rho_0}{3 r^2} \hat{r}.$$



3) Find electric field due to a sphere whose upper half has uniform charge density $+\rho_0$ and bottom half has charge density $-\rho_0$.



Gauss Law $\int_S \vec{E} \cdot d\vec{a} = 4\pi \int_V \rho \, dV$

Suppose S is a sphere of radius r.
Unfortunately in this case, $\vec{E} \nparallel \hat{r}$.

$$\therefore \int \vec{E} \cdot d\vec{a} \neq E \int da.$$

$\therefore$ Gauss law cannot be easily evaluated using symmetry arguments.

$\Rightarrow$ Gauss law is useful only for problems having high degree of symmetry.

Electrostatic Potential

Electric field at $\vec{r}$ due to charges q_i $i = 1, 2, \ldots n$ at positions $\vec{r}_i$ is:

$$\vec{E}(\vec{r}) = \sum_i q_i \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}$$

$$= -\nabla \sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|}$$

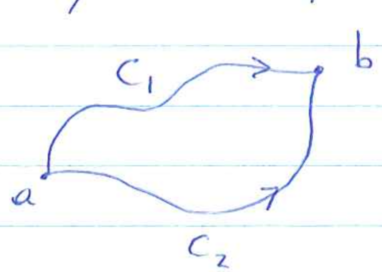
Exercise: Prove this last step.

$\therefore$ $\vec{E} = -\nabla \Phi$ where $\Phi \equiv \sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|}$ is called the electrostatic potential.

Integrating $\vec{E} = -\nabla \Phi$ we get:

$$\Phi(b) - \Phi(a) = -\int_a^b \vec{E} \cdot d\vec{l}$$

The integral $\int_a^b \vec{E} \cdot d\vec{l}$ depends only on the value of the potential Φ at the endpoints a and b. It is independent of the path taken when going from a to b.



i.e. $\int_{C_1} \vec{E} \cdot d\vec{l} = \int_{C_2} \vec{E} \cdot d\vec{l}$.

Work Done in Moving Charge

We shall compute the work needed to move charge q from position a to position b. A Coulomb force $\vec{F} = q\vec{E}$ exerts itself on the charge. Hence to overcome this force, an equal but opposite force $-\vec{F}$ must be applied.

$$\therefore \text{Work} \quad W = -\int_a^b \vec{F} \cdot d\vec{l}$$

$$= -q\int_a^b \vec{E} \cdot d\vec{l}$$

$$= q\left[\Phi_b - \Phi_a\right]$$

$\therefore$ work depends only on the potential evaluated at the endpoints a and b; and not on the path taken. Forces such as the Coulomb and Gravitational forces having this property are said to be conservative.

Note that the potential difference between two points is the work that must be done to move a unit charge between points a & b.

Examples.

1) Find potential due to a charge q_1 at the origin.

Electric field at $\vec{r}$ is $\vec{E}(\vec{r}) = \frac{q_1}{r^2}\hat{r}$.

$$\Phi_b - \Phi_a = -\int_a^b \vec{E} \cdot d\vec{l}.$$

This potential difference is the amount of work needed to move a unit charge from a to b in the presence of charge q_1. We shall evaluate the potential at position $\vec{r}$ and use ∞ as our reference point.

$$\Phi(\vec{r}) - \Phi(\infty) = -\int_\infty^{\vec{r}} \frac{q_1}{r^2}\hat{r} \cdot d\vec{l}$$

The integral is independent of the path taken.
$\therefore$ we choose the radial path such that $d\vec{l} = d\vec{r}$.

$$\Phi(r) - \Phi(\infty) = -\int_\infty^r \frac{q_1}{r^2}\,dr$$

$$= \frac{q_1}{r}$$

We set $\Phi(\infty) = 0$ since the interesting variables such as electric field, work done are independent of this constant reference value.

$$\therefore \Phi(r) = \frac{q_1}{r}$$

Next suppose that a charge $q_2 = 10^3$ esu is brought to within 10cm. of charge $q_1 = 10^2$ esu. How much work is done or what is potential energy of the system?

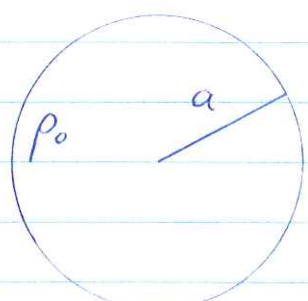
Work done $W = q_2 \Phi(r)$

$$= \frac{q_2 q_1}{r}$$

$$= \frac{10^3 \text{ esu} \cdot 10^2 \text{ esu}}{10 \text{ cm.}}$$

$$\therefore W = 10^4 \text{ erg.} \qquad \left(\text{Recall } 1 \text{ dyne} = \frac{1 \text{ esu}^2}{\text{cm}^2}\right)$$

2) Find potential everywhere of uniformly charged sphere (charge density ρ_0) of radius a. Set the potential to 0 at infinity.



$r < a \qquad \vec{E} = \frac{4}{3}\pi \rho_0 \vec{r}$

$r > a \qquad \vec{E} = \frac{4\pi a^3 \rho_0}{3 r^2}\hat{r}$

First consider a point $r > a$.

$$\Phi(r) - \Phi(\infty) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$$

$$= -\int_{\infty}^{r} \frac{4\pi a^3 \rho_0}{3 r^2} \hat{r} \cdot d\vec{l}$$

We may choose any path between ∞ & r, we desire so let's choose the radial path. $\therefore d\vec{l} = dr\,\hat{r}$.

$$\Phi(r) = -\int_{\infty}^{r} \frac{4\pi}{3} a^3 \rho_0 \frac{dr}{r^2}$$

$$= -\frac{4\pi}{3} a^3 \rho_0 \left[\frac{-1}{r}\right]_{\infty}^{r}$$

$$= \frac{4\pi}{3} a^3 \rho_0 \frac{1}{r}.$$

Next consider the region $r < a$.

$$\Phi(r) - \Phi(a) = -\int_a^r \vec{E} \cdot d\vec{l}$$

$$= -\int_a^r \frac{4\pi}{3} \rho_0 \, \vec{r} \cdot d\vec{l}$$

$$= -\int_a^r \frac{4}{3}\pi \rho_0 \, r \, dr$$

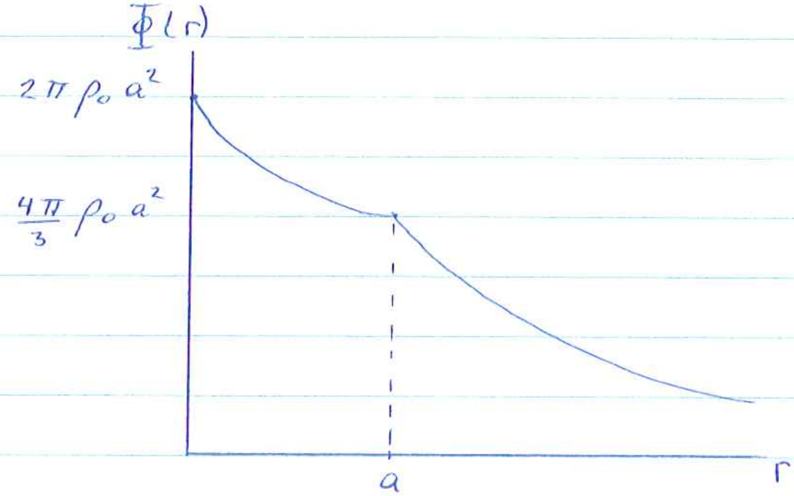
$$= -\frac{4}{3}\pi \rho_0 \left[\frac{r^2}{2}\right]_a^r$$

$$= -\frac{4}{3}\pi \rho_0 \frac{(r^2 - a^2)}{2}$$

$$\therefore \Phi(r) = \Phi(a) - \frac{4}{3}\pi \rho_0 \frac{(r^2 - a^2)}{2}$$

$$= \frac{4\pi}{3} a^2 \rho_0 - \frac{2\pi}{3}\rho_0 r^2 + \frac{2\pi}{3}\rho_0 a^2$$

$$= \frac{2\pi}{3}\rho_0 (3a^2 - r^2).$$



Comment

The potential Φ is a scalar quantity which is inherently simpler than the electric field $\vec{E} = (E_x, E_y, E_z)$ which is a vector. We shall therefore next see if we can find the potential directly from the charge distribution skipping over the computation of $\vec{E}$. Once Φ is known, then trivially $\vec{E} = -\nabla \Phi$.

Relation of Φ & ρ.

Gauss Law $\quad \nabla \cdot \vec{E} = 4\pi \rho \qquad (1)$

Electric Field $\quad \vec{E} = -\nabla \Phi \qquad (2)$

Substitute (2) into (1) and we get:

$$\boxed{\nabla^2 \Phi = -4\pi \rho \quad \text{Poisson's Equation}}$$

If no charges are present $\rho = 0$ and we obtain:

$$\boxed{\nabla^2 \Phi = 0 \quad \text{Laplace Equation}}$$

Hence the potential Φ is found directly from the charge density if we solve Poisson or Laplace equation. To solve these differential equations one must specify appropriate boundary conditions.

We shall first discuss the so called method of images. First however we need the following theorem.

Theorem

If Φ is a solution of $\nabla^2 \Phi = -4\pi\rho$ in a region V bounded by a surface S, then this solution is unique if either

1) the value Φ is specified on S

2) the normal derivative of Φ, i.e. $\frac{\partial \Phi}{\partial n}$ is given on S

Proof: Suppose there are two solutions Φ_1 & Φ_2.

Let $\phi = \Phi_1 - \Phi_2$

Then $\nabla^2 \phi = \nabla^2 \Phi_1 - \nabla^2 \Phi_2$

$= 0$

and either ϕ or $\frac{\partial \phi}{\partial n}$ is zero on S.

Next we use the following identity. (Proof - exercise)

$$\int_V \left[f \nabla^2 g + (\nabla f)\cdot(\nabla g) \right] dV = \int_S f \nabla g \cdot \hat{n} \, da$$

Set $f = g = \phi$

$$\int_V \left[\underbrace{\phi \nabla^2 \phi}_{=0} + (\nabla \phi)^2 \right] dV = \underbrace{\int_S \phi \nabla \phi \cdot \hat{n} \, da}_{=0 \text{ due to B.C.'s on } \phi}$$

$$\int_V (\nabla\phi)^2 \, dV = 0.$$

$\therefore$ $\nabla\phi = 0$ everywhere in V

$\phi = \text{const.}$

If $\phi = 0$ on S then $\phi = 0$ everywhere and $\Phi_1 = \Phi_2$.

If $\dfrac{\partial\phi}{\partial n} = 0$ on S then $\phi = \text{const.}$ everywhere. In this case Φ_1 & Φ_2 represent same physical situation however since electric fields $-\nabla\Phi$ are identical.

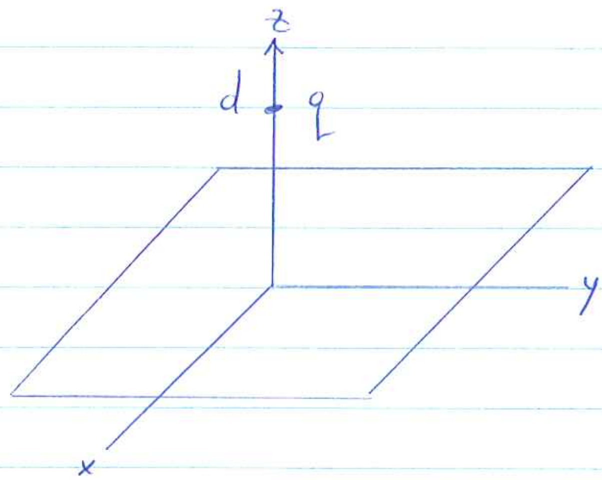
$\therefore$ solution to $\nabla^2\Phi = -4\pi\rho$ is unique.

Method of Images

Frequently we must solve $\nabla^2 \Phi = -4\pi\rho$ subject to boundary conditions that can be simulated by placement of additional charges called image charges. The potential in the original problem is then simply the potential of the image charges.

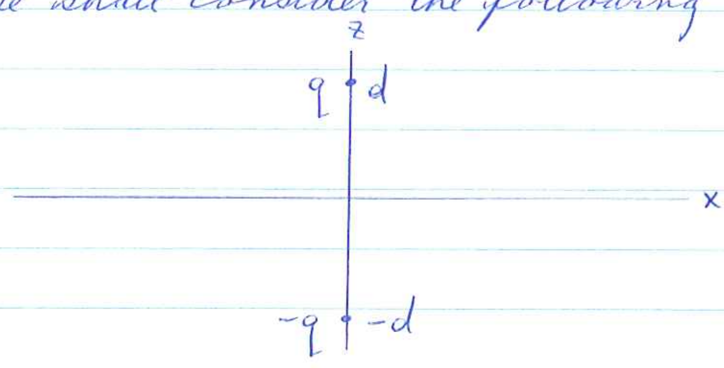
Example 1

Find potential everywhere of charge q at distance d above an infinite conducting plane.



i.e. Solve $\nabla^2 \Phi = -4\pi\rho$ in region $z > 0$ subject to boundary condition $\phi = 0$ on xy plane.

We shall consider the following problem instead.



In the region $z>0$, the potential of these two charges satisfies the same equation $\nabla^2 \Phi = -4\pi \rho$ as the charge q above the conducting plane. Furthermore, by symmetry, the new problem has the same boundary condition $\Phi = 0$ on the xy plane. From the theorem, we know that there is a unique solution to Poisson's equation. $\Longrightarrow$ solution of new problem in region $z>0$ is solution of original problem.

Potential due to q & $-q$ is:

$$\Phi(\vec{r}) = \frac{q}{|\vec{r} - (0,0,d)|} - \frac{q}{|\vec{r} - (0,0,-d)|}$$

$$= q\left\{ \frac{1}{(x^2+y^2+(z-d)^2)^{1/2}} - \frac{1}{(x^2+y^2+(z+d)^2)^{1/2}} \right\}$$

$\therefore$ potential for charge above conducting plane is:

$$\Phi(x,y,z>0) = q\left\{ [x^2+y^2+(z-d)]^{-1/2} - [x^2+y^2+(z+d)^2]^{-1/2} \right\}$$

Electric field of charge above conducting plane is:

$$\vec{E} = -\nabla \Phi$$

$$= -q\left\{ -\frac{1}{2}[x^2+y^2+(z-d)^2]^{-3/2}(2x, 2y, 2(z-d)) - \left(-\frac{1}{2}\right)[x^2+y^2+(z+d)^2]^{-3/2}(2x,2y,2(z+d)) \right\}$$

$$\vec{E} = q\left\{\frac{(x, y, z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{(x, y, z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}$$

Charge density on xy plane is

$$\sigma(x,y) = \frac{E_z(z=0)}{4\pi}$$

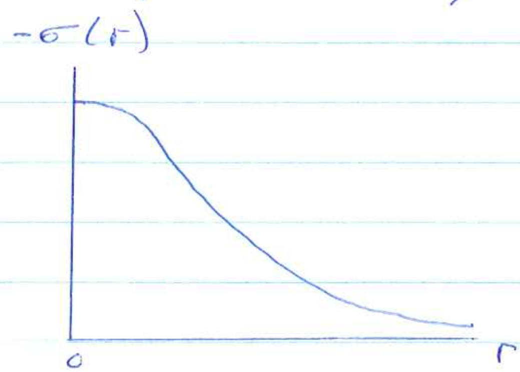
$$= \frac{q}{4\pi}\left\{\frac{-d}{(x^2+y^2+d^2)^{3/2}} - \frac{d}{(x^2+y^2+d^2)^{3/2}}\right\}$$

$$= -\frac{qd}{2\pi}(x^2+y^2+d^2)^{-3/2}$$

Let $r = \sqrt{x^2+y^2}$

$$\sigma(r) = -\frac{qd}{2\pi}(r^2+d^2)^{-3/2}$$

As one intuitively expects, the induced charge density on the conducting plane depends only on the radial distance from the origin.



Total induced charge on plane is:

$$q_{induced} = \int_{\substack{\text{infinite} \\ \text{plane}}} \sigma \, da$$

$$= \int_0^\infty \sigma(r) \, 2\pi r \, dr$$

$$= -\frac{qd}{2\pi} \int_0^\infty \frac{2\pi r \, dr}{(r^2 + d^2)^{3/2}}$$

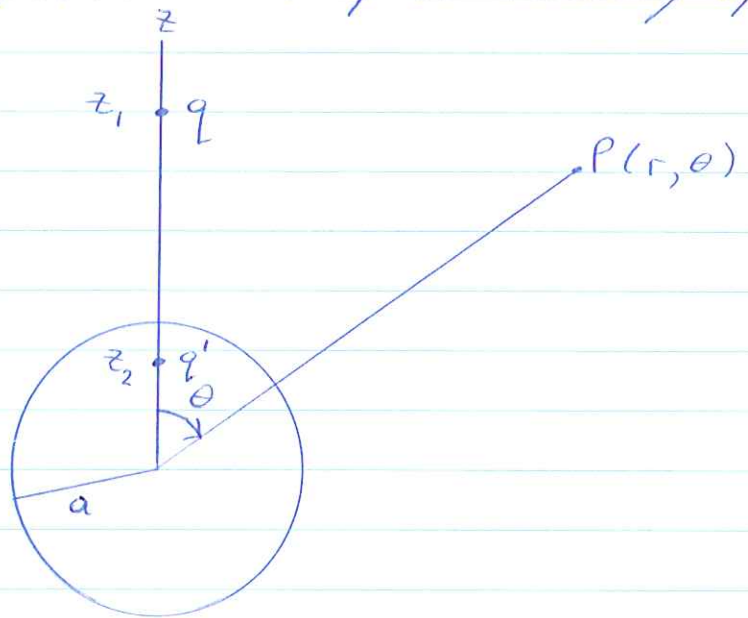
$$= -qd \left[-(r^2 + d^2)^{-1/2} \right]_0^\infty$$

$$\therefore q_{induced} = -q.$$

This also makes sense since we expect no flux to be able to leak into a conductor.

Example 2

Find potential everywhere of charge q at distance z_1 from center of conducting sphere.



We shall attempt to find an image charge q' such that the sphere is at 0 potential. The symmetry of the problem suggests that image charge $q'<0$ should be inside sphere on z axis at z_2.

Note that we only need concern ourselves with the 2 dimensional problem (i.e. r & θ or x & z) since the potential is unchanged if the sphere is rotated about z axis.

Potential of q & q' at $P(r,\theta)$.

$$\Phi(r,\theta) = \frac{q}{(r^2 + z_1^2 - 2rz_1\cos\theta)^{1/2}} + \frac{q'}{(r^2 + z_2^2 - 2rz_2\cos\theta)^{1/2}}$$

We require $\Phi(a,\theta)=0 \ \forall \theta$.

$$0 = \frac{q}{(a^2+z_1^2-2az_1\cos\theta)^{1/2}} + \frac{q'}{(a^2+z_2^2-2az_2\cos\theta)^{1/2}}$$

$$a^2+z_2^2-2az_2\cos\theta = \left(\frac{q'}{q}\right)^2 (a^2+z_1^2-2az_1\cos\theta).$$

Equating coefficient of $\cos\theta$ gives:

$$-2az_2 = -2az_1\left(\frac{q'}{q}\right)^2.$$

$$\left(\frac{q'}{q}\right)^2 = \frac{z_2}{z_1} \qquad (1)$$

Equating constant terms gives:

$$a^2+z_2^2 = \left(\frac{q'}{q}\right)^2 (a^2+z_1^2)$$

$$= \frac{z_2}{z_1}(a^2+z_1^2) \qquad \text{using (1).}$$

$$z_2^2 - z_2\frac{(a^2+z_1^2)}{z_1} + a^2 = 0.$$

$$\left(z_2 - z_1\right)\left(z_2 - \frac{a^2}{z_1}\right) = 0.$$

$$\therefore z_2 = z_1, \frac{a^2}{z_1}$$

We forget $z_2 = z_1$ since this implies $q' = -q$.

Substitute $z_2 = \frac{a^2}{z_1}$ in (1).

$$\left(\frac{q'}{q}\right)^2 = \frac{a^2}{z_1^2}$$

$$\therefore\ q' = -\frac{a}{z_1} q$$

Hence image charge $-\frac{a}{z_1} q$ at position $\frac{a^2}{z_1}$ on the z axis does the trick.

Laplace Equation

Solution in Cartesian Coordinates

We shall solve $\nabla^2 \Phi = 0$ in 2 dimensional Cartesian coordinates.

$$\therefore \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

Using the method of separation of variables, we let $\Phi(x, y) = X(x)\,Y(y)$.

$$\therefore Y(y) \frac{d^2 X}{dx^2} + X(x) \frac{d^2 Y}{dy^2} = 0.$$

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0.$$

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2}$$

The left side of this equation depends only on x.
" right " " y.

$$\therefore \frac{1}{X} \frac{d^2 X}{dx^2} = K \quad \text{where } K = \text{constant}$$

and $\dfrac{1}{Y} \dfrac{d^2 Y}{dy^2} = -K.$

Case 1: $K=0$ $\qquad \dfrac{1}{X}\dfrac{d^2X}{dx^2}=0$

$$\frac{d^2X}{dx^2}=0$$

$$X(x)=Ax+B \qquad A, B \text{ are constants}$$

also $\dfrac{1}{Y}\dfrac{d^2Y}{dy^2}=0$

$$Y(y)=Cy+D$$

$$\Rightarrow \Phi(x,y)=(Ax+B)(Cy+D)$$

Case 2: $K<0$ $\quad$ i.e. $K=-k^2$

$$\frac{1}{X}\frac{d^2X}{dx^2}=-k^2$$

$$\frac{d^2X}{dx^2}=-k^2X$$

$$X(x)=A\sin kx+B\cos kx$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2}=k^2$$

$$\frac{d^2Y}{dy^2}=k^2Y$$

$$Y(y)=Ce^{ky}+De^{-ky}$$

$$\Rightarrow \Phi(x,y)=(A\sin kx+B\cos kx)(Ce^{ky}+De^{-ky})$$

Case 3: $K > 0$ i.e. $K = k^2$

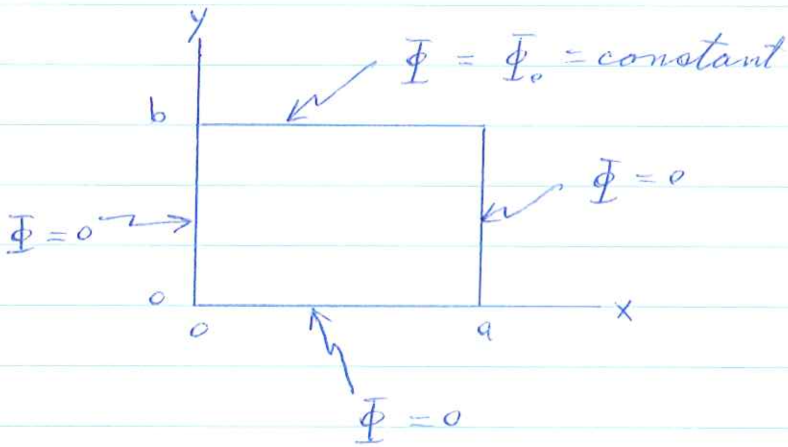
Similar to case 2 one can show

$$\Phi(x,y) = (A e^{kx} + B e^{-kx})(C \sin ky + D \cos ky)$$

The constants A, B, C, D, k are evaluated using the boundary conditions in the problem.

Example

Find the potential everywhere inside an infinitely long rectangular tube shown below.



To see which of cases 1, 2 or 3 apply we refer to the boundary conditions.

Since $\Phi(0,y) = \Phi(a,y) = 0 \;\; \forall y$ we must have

$$X(0) = X(a) = 0.$$

Case 1: $X(0) = X(a) = 0 \Rightarrow X(x) = 0 \Rightarrow \Phi(x,y) = 0$

But $\Phi(x, b) = \Phi_0$ $\therefore$ this case is impossible.

Case 3: $X(x) = A e^{kx} + B e^{-kx}$

$X(0) = 0 \Rightarrow A + B = 0 \quad (1)$

$X(a) = 0 \Rightarrow A e^{ka} + B e^{-ka} = 0 \quad (2)$

$(1) \,\&\, (2) \Rightarrow A = B = 0$

$\therefore X(x) = 0$

$\Phi(x, y) = 0$

$\therefore$ case 3 isn't possible

Case 2: $X(x) = A \cos kx + B \sin kx$

$X(0) = 0 \Rightarrow A = 0$

$\therefore X(x) = B \sin kx.$

$X(a) = 0 \Rightarrow B \sin ka = 0.$

$ka = n\pi \quad n \in \mathbb{Z}$ (integers)

There are infinitely many k values possible that we label by n, i.e. $k_n = \frac{n\pi}{a}$.

Similarly $X_n(x) = B \sin k_n x$.

Since $X_n(x) = -X_{-n}(x)$ we shall restrict $n \in \mathbb{N}$ since we are only interested in linearly independent solutions.

Next: $Y(y) = Ce^{ky} + De^{-ky}$

$\Phi(x,0)=0 \Rightarrow Y(0)=0 \Rightarrow \quad C + D = 0.$

$$D = -C$$

$$\therefore Y(y) = C(e^{ky} - e^{-ky})$$

Since $k = \frac{n\pi}{a}$, we also need to label Y by n.

$$\therefore Y_n(y) = C(e^{k_n y} - e^{-k_n y})$$

The most general possible solution is a linear combination of $X_n(x)\,Y_n(y)$.

$$\Phi(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\left(C_n\left(e^{n\pi y/a} - e^{-n\pi y/a}\right)\right)$$

$$= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \qquad (3)$$

where we have incorporated $2C_n$ into B_n.

B_n are found using the remaining boundary condition $\Phi(x,b) = \Phi_0$.

$$\Phi_0 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$$

$$\int_0^a \Phi_0 \sin\frac{m\pi x}{a}\,dx = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi b}{a} \int_0^a \sin\frac{m\pi x}{a} \sin\frac{n\pi x}{a}\,dx$$

(4)

Evaluation of $\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}\,dx$

Case 1: $m \neq n$

$$\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a} = \frac{1}{2}\left\{\cos(m-n)\frac{\pi x}{a} - \cos(m+n)\frac{\pi x}{a}\right\}$$

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}\,dx = \frac{1}{2}\left\{\frac{\sin(m-n)\pi x/a}{(m-n)\pi/a} - \frac{\sin(m+n)\pi x/a}{(m+n)\pi/a}\right\}_0^a$$

$$= 0$$

Case 2: $m = n$

$$\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a} = \sin^2\frac{m\pi x}{a}$$

$$= \frac{1}{2}\left[1 - \cos\frac{2m\pi x}{a}\right]$$

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}\,dx = \frac{1}{2}\int_0^a\left[1 - \cos\frac{2m\pi x}{a}\right]dx.$$

$$= \frac{1}{2}\left[x - \frac{\sin 2m\pi x/a}{2m\pi/a}\right]_0^a$$

$$= \frac{a}{2}$$

$$\therefore \int_0^a \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}\,dx = \frac{a}{2}\,\delta_{nm}$$

Using this result, (4) becomes:

$$\Phi_0 \int_0^a \sin\frac{m\pi x}{a}\,dx = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi b}{a}\;\frac{a}{2}\,\delta_{nm}$$

$$\Phi_0 \left[\frac{-\cos\frac{m\pi x}{a}}{m\pi/a}\right]_0^a = B_m \frac{a}{2}\sinh\frac{m\pi b}{a}$$

$$\frac{a\,\Phi_0}{m\pi}\left\{-\cos m\pi + 1\right\} = B_m \frac{a}{2}\sinh\frac{m\pi b}{a}$$

$$\frac{a\,\Phi_0}{m\pi}\left\{-(-1)^m + 1\right\} = B_m \frac{a}{2}\sinh\frac{m\pi b}{a}$$

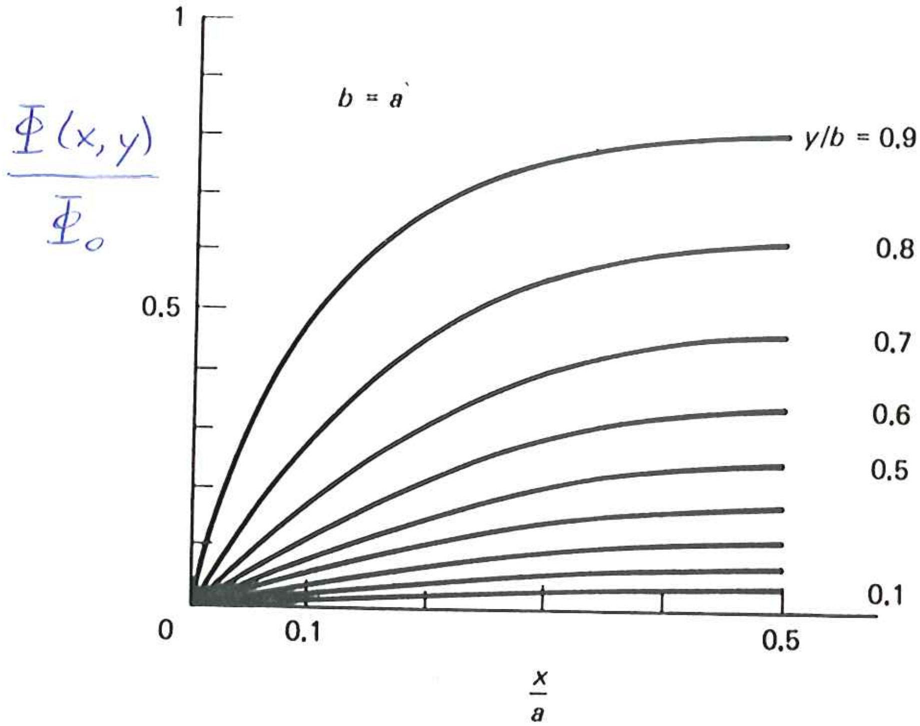
$$B_m = \frac{2\Phi_0}{m\pi}\,\frac{(1-(-1)^m)}{\sinh\frac{m\pi b}{a}}$$

or $B_m = 0$ if m is even

$$B_m = \frac{4\Phi_0}{m\pi\sinh\frac{m\pi b}{a}} \quad \text{if } m \text{ is odd.}$$

Substitute B_m into (3) finally gives:

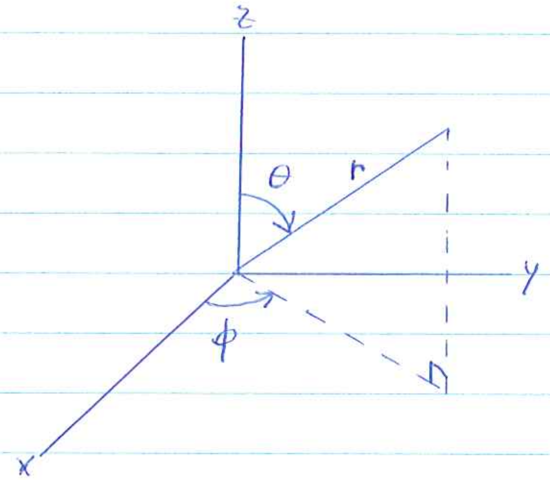
$$\boxed{\Phi(x,y) = \frac{2\Phi_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\,\frac{1}{\sinh\frac{n\pi b}{a}}\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}}$$



$\frac{\Phi(x,y)}{\Phi_0}$
1
0.5
0
$b = a$
$y/b = 0.9$
0.8
0.7
0.6
0.5
0.1
0.1
0.5
$\frac{x}{a}$

Solution of Laplace's Equation In Spherical Coordinates

Spherical Coordinates



$z = r\cos\theta$

$y = r\sin\theta\sin\phi$

$x = r\sin\theta\cos\phi$

Laplace's Equation

$$0 = \nabla^2 \Phi(r,\theta,\phi)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \qquad (1)$$

We shall consider the case of azimuthal symmetry, i.e. problem is symmetric w.r.t. rotation about z axis or $\Phi = \Phi(r,\theta)$.

We shall solve (1) using separation of variables.
Let $\Phi(r, \theta) = R(r)\, P(\theta)$.

$$0 = \frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right) \qquad (1')$$

The first term depends only on r, while the second depends only on θ.

$$\therefore \frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right) = -l(l+1) = \text{constant} \qquad (2)$$

We can rewrite this using $x = \cos\theta$ and see that this is Legendre's differential equation.

$$x = \cos\theta$$

$$\frac{dx}{d\theta} = -\sin\theta$$

$$\frac{d}{d\theta} = -\sin\theta\,\frac{d}{dx}.$$

$$\therefore (2) \Longrightarrow \frac{1}{P\sin\theta}(-\sin\theta)\frac{d}{dx}\left(\sin\theta(-\sin\theta)\frac{dP}{dx}\right) = -l(l+1)$$

$$\frac{d}{dx}\left(\sin^2\theta\,\frac{dP}{dx}\right) = -l(l+1)\,P(x)$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)\,P(x) = 0.$$

$$\boxed{(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + l(l+1)P(x) = 0. \quad \text{Legendre Diff. Eqn.}}$$

This equation has well defined solutions $P_l(x)$ known as Legendre polynomials for $x \in [-1, 1]$ only if l is a natural number.

Legendre Polynomials

$P_l(x)$ is a polynomial of degree l.

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_l(x) = \frac{1}{2^l\, l!}\frac{d^l (x^2-1)^l}{dx^l} \quad \text{Rodriguez Formula}$$

$$\int_{-1}^{1} P_l(x)\, P_{l'}(x)\, dx = \frac{2}{2l+1}\delta_{ll'} \quad \text{Orthogonality Relation}$$

Determination of $R(r)$

Using (1') & (2) we have the following.

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - l(l+1)R = 0.$$

The solution is $R(r) = A r^{l} + B r^{-(l+1)}$

Proof:
$$\frac{dR}{dr} = A l r^{l-1} - B(l+1) r^{-l-2}$$

$$r^2 \frac{dR}{dr} = A l r^{l+1} - B(l+1) r^{-l}$$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = A l(l+1) r^{l} + B l(l+1) r^{-l-1}$$

$$= l(l+1) R(r)$$

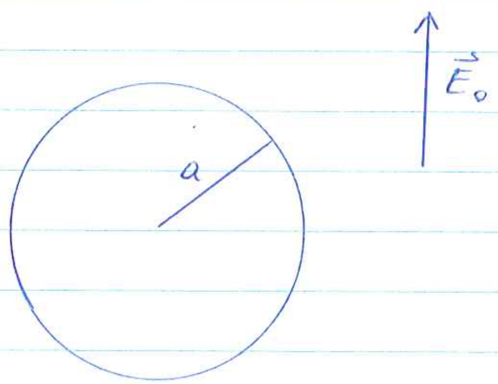
$$\therefore \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - l(l+1)R = 0.$$

General solution of (1') is:

$$\boxed{\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^{l} + B_l r^{-(l+1)}\right) P_l(\cos\theta)}$$

Example

Find potential exterior to a conducting sphere placed in a uniform electric field.



Let z axis be parallel to electric field at infinity.

At infinity $\quad \vec{E} = E_0 \hat{z}$

$$\begin{aligned} \Phi &= -E_0 z \\ &= -E_0 r \cos\theta \\ &= -E_0 r P_1(\cos\theta) \end{aligned}$$

General Solution $\quad \Phi = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$

As $r \to \infty$, $\Phi \to \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

Using boundary condition at infinity we get:

$A_l = 0 \;\; \forall\, l \neq 1$ and $A_1 = -E_0$.

$\therefore \; \Phi(r,\theta) = -E_0 r P_1(\cos\theta) + \sum_l B_l r^{-(l+1)} P_l(\cos\theta)$

To find B_ℓ, use boundary condition at $r = a$.

$$0 = \Phi(a, \theta)$$

$$= -E_0 \, a \, P_1(\cos\theta) + \sum_\ell B_\ell \, a^{-(\ell+1)} P_\ell(\cos\theta)$$

Setting coefficients of $P_\ell(\cos\theta)$ to zero we get:

$$B_\ell = 0 \quad \forall \ \ell \neq 1$$

$$-E_0 \, a + B_1 \, a^{-2} = 0$$

$$B_1 = E_0 \, a^3$$

$$\therefore \ \Phi(r, \theta) = -E_0 \, r \, P_1(\cos\theta) + E_0 \, a^3 \, r^{-2} \, P_1(\cos\theta)$$

$$\boxed{\Phi(r, \theta) = -E_0\left(r - \frac{a^3}{r^2}\right) P_1(\cos\theta)}$$

Electric Field

$$\vec{E} = -\nabla \Phi$$

$$= -\left(\hat{r} \, \frac{\partial \Phi}{\partial r} + \frac{1}{r} \, \frac{\partial \Phi}{\partial \theta} \, \hat{\theta}\right)$$

$$E_r = -\frac{\partial \Phi}{\partial r} = E_0\left(1 + \frac{2a^3}{r^3}\right) \cos\theta$$

$$E_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} = -E_0\left(1 - \frac{a^3}{r^3}\right) \sin\theta$$

Surface charge density induced on sphere is

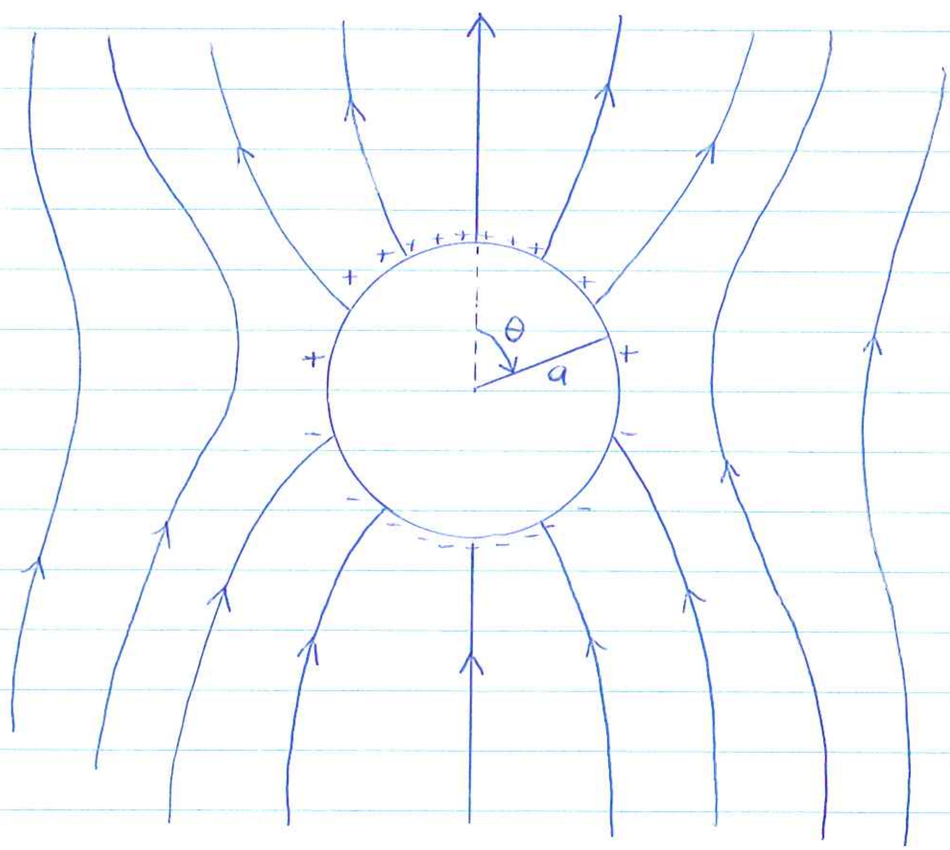
$$\sigma = \frac{E_r(r=a)}{4\pi}$$

$$= \frac{E_0}{4\pi}\left(1 + \frac{2a^3}{a^3}\right)\cos\theta$$

$$= \frac{3E_0}{4\pi}\cos\theta$$

Exercise: Show $\displaystyle\int_{\substack{\text{surface}\\ \text{of sphere}}} \sigma \, da = 0.$

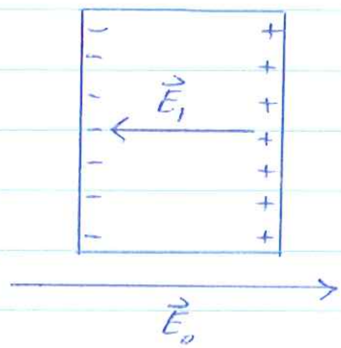
Electric Field



Assignment 1

1. Electrical Conductors

An ideal conductor is a material having an unlimited supply of free charges. Consider a conductor placed in an external electric field $\vec{E}_0$.



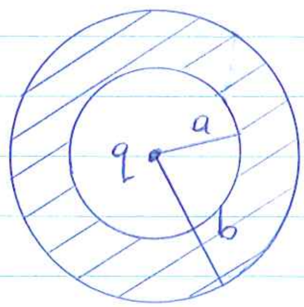
Inside the conductor, $\vec{E}_0$ drives positive charge to the right surface and negative charge to the left surface. These so called induced charges produce a field $\vec{E}_1$ opposing $\vec{E}_0$. Charges continue to move until $\vec{E}_1$ exactly cancels $\vec{E}_0$. The time for this movement of charge to occur is extremely short. Hence we conclude that inside a conductor the electric field is 0.

a) Show the charge density $\rho = 0$ inside conductor.

b) Show the potential Φ is constant inside a conductor.

c) Show that just outside a conductor $\vec{E}$ is perpendicular to the surface and equals $4\pi\sigma$ where σ = surface charge density.

2. A charge q sits in a spherical hollow inside a spherical conductor.



a) Find $\vec{E}$ everywhere.

b) What are charge densities on conductor surfaces?

c) Find potential everywhere taking $\Phi = 0$ at ∞. ~~position of charge q~~.

3. Consider a sphere of radius a of uniform charge density ρ_0.

a) Find $\vec{E}$ everywhere.

b) Find potential everywhere taking $\Phi = 0$ at origin.

c) $a = 2$ cm, $\rho_0 = \frac{3}{2\pi}$ esu/cm^3

i) What is total charge on sphere?

ii) What is electric field 10 cm from sphere center?
" " potential " " in statvolts? 1 statvolt = 1 esu/cm.

iii) A charge of 5 esu is moved from infinity to within 10 cm. from sphere center.

What is work done in ergs in moving charge?

What is force in dynes between charge and sphere at the final position?

4. For a time independent or static situation we showed

$$-\int_a^b \vec{E}\cdot d\vec{l} = \Phi(b) - \Phi(a)$$

depends only on the endpoints a & b.

a) Show that for any closed path $\oint \vec{E}\cdot d\vec{l} = 0$.

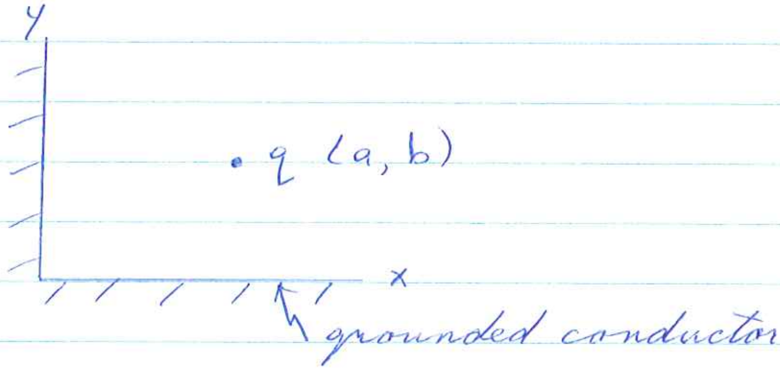
b) Using Stoke's Theorem $\oint \vec{E}\cdot d\vec{l} = 0 \Rightarrow \nabla \times \vec{E} = 0$.

c) $-\int_a^b \vec{E}\cdot d\vec{l} = \Phi(b) - \Phi(a)$ was derived by integrating $\vec{E} = -\nabla \Phi$. Using Cartesian coordinates show this implies $\nabla \times \vec{E} = 0$.

d) Sketch a vector field $\vec{A}$ for which $\oint \vec{A}\cdot d\vec{l} \neq 0$.

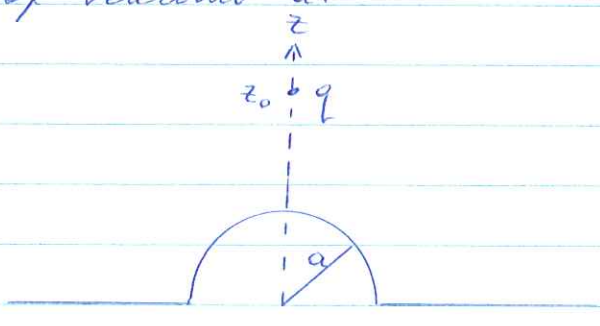
5. Consider an empty 3 dimensional rectangular cube having all sides at 0 potential. What is potential inside cube and how do you know this is the only possible answer?

6. An infinite conducting sheet is bent into a 90° corner. A point charge q is placed near the corner as shown.



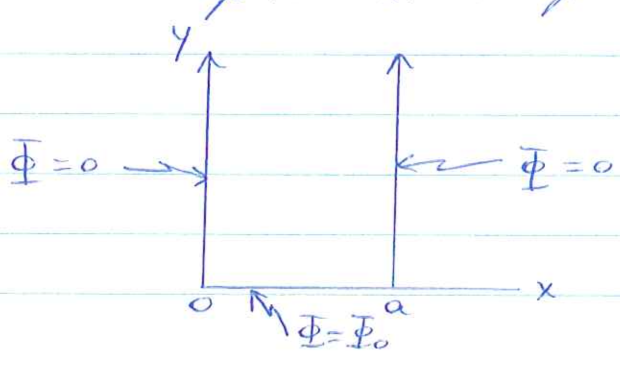
Find the potential everywhere.

7. An infinite conducting sheet has a hemispherical bubble of radius a.



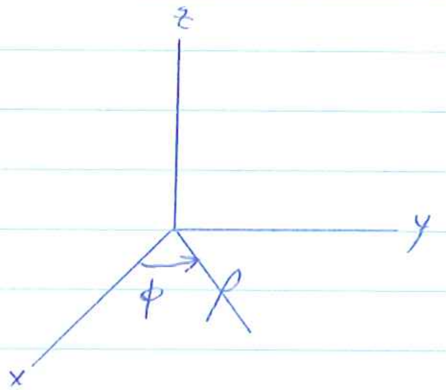
Find potential everywhere.

8. An infinitely deep trough has its two sides at 0 potential and its bottom at Φ_0. Find potential everywhere in trough.



9. Laplace equation in cylindrical coordinates is

$$0 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$



$z = z$

$y = \rho \sin\phi$

$x = \rho \cos\phi$

Consider the case where Φ is independent of z.

a) Let $\Phi = R(\rho)\,Q(\phi)$ and find differential eqns. for R + Q.

b) Constant $= 0$ Show $R = A \ln\rho + B$
$Q = C\phi + D$

Constant $k^2 > 0$ Show $R = A\rho^{k} + B\rho^{-k}$
$Q = C\cos k\phi + D\sin k\phi$

Constant $-k^2 < 0$ Show $Q = Ce^{k\phi} + De^{-k\phi}$

c) Suppose $\Phi(\rho,\phi) = \Phi(\rho, \phi + 2\pi) \Rightarrow Q(\phi) = Q(\phi + 2\pi)$.
Show that:

i) $k = n$ an integer and

ii) $\Phi(\rho,\phi) = A\ln\rho + B + \sum_{n=1}^{\infty}(A_n\rho^n + B_n\rho^{-n})(C_n\cos n\phi + D_n\sin n\phi)$

II Magnetic Fields

A charge q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ experiences the Lorentz force

$$\vec{F} = q\,\frac{\vec{v}}{c} \times \vec{B}$$

Magnetic fields are produced by moving charges or currents. Ampere observed that the integral of $\vec{B}$ around a loop was proportional to the current passing through the loop.

i.e. $$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\, I_{\substack{\text{enclosed}\\ \text{by loop}}}$$ Ampere's Law

To obtain the differential form of Ampere's Law we use Stoke's Thm. on the left side and write the current using the current density $\vec{J}$.

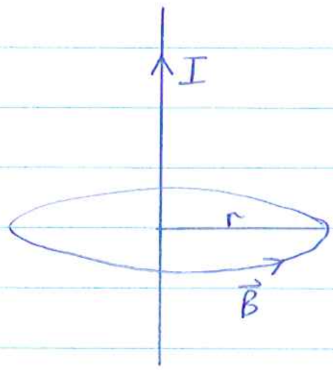
$$\int_{\text{surface}} \nabla \times \vec{B}\cdot d\vec{a} = \frac{4\pi}{c}\int_{\text{surface}} \vec{J}\cdot d\vec{a}$$

Since this is true for any area we may conclude

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J}$$ Differential Form of Ampere's Law

Examples

1. Find field due to infinite straight wire carrying current I.



The direction of $\vec{B}$ is given by the right hand rule. (Thumb ~~(fingers)~~ point along $\vec{I}$ and fingers ~~(thumb)~~ point along $\vec{B}$.) Therefore $\vec{B}$ is a vector tangent to a circle in the plane perpendicular to the wire.

Consider a circle of radius r around the wire as shown above.

$$\oint_{\substack{\text{circle of} \\ \text{radius } r}} \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I$$

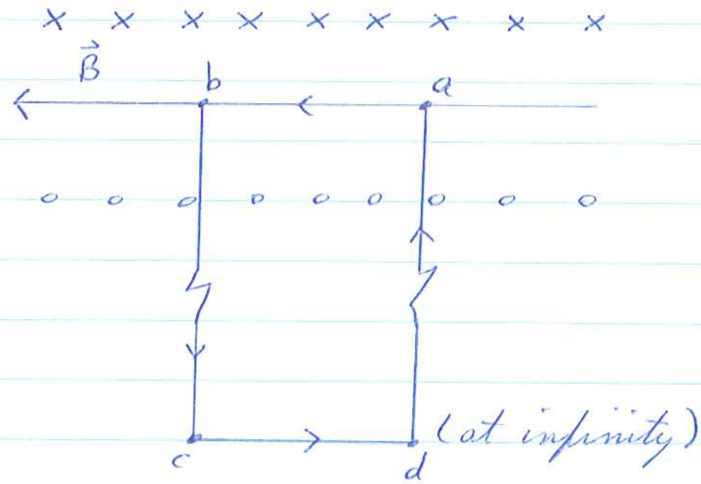
By symmetry $B = B(r)$.

$$\therefore \; B(r) \oint dl = \frac{4\pi}{c} I$$

$$B(r)\, 2\pi r = \frac{4\pi}{c} I$$

$$\boxed{B(r) = \frac{2I}{rc}}$$

2. Consider a solenoid having N turns/cm each carrying current I. Find $\vec{B}$ on axis neglecting fringing or end effects.



× ≡ current into page
o ≡ current out of page

Using the right hand rule, $\vec{B}$ points horizontally to the left. Let's integrate $\vec{B}$ around the loop abcda.

$$\oint_{abcda} \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I_{enclosed}$$

$$\int_a^b \vec{B} \cdot d\vec{l} + \underbrace{\int_b^c \vec{B} \cdot d\vec{l}}_{=0 \ \vec{B} \perp d\vec{l}} + \underbrace{\int_c^d \vec{B} \cdot d\vec{l}}_{\substack{=0 \\ \text{since} \\ \vec{B}(\infty)=0}} + \underbrace{\int_d^a \vec{B} \cdot d\vec{l}}_{=0 \ \vec{B} \perp d\vec{l}} = \frac{4\pi}{c} N l_{ab} I$$

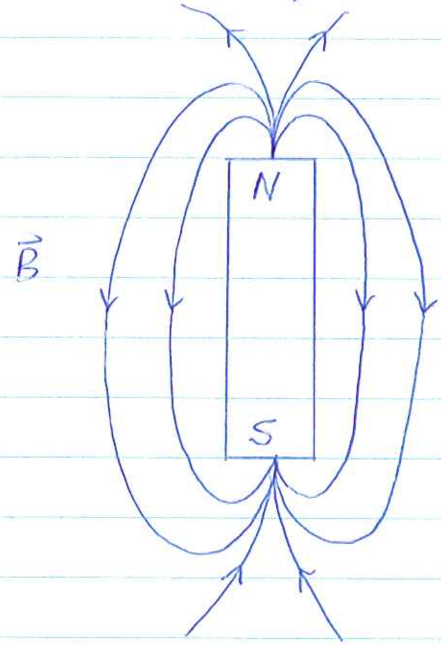
$$\int_a^b \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} N l_{ab} I$$

$$B \, l_{ab} = \frac{4\pi}{c} N l_{ab} I$$

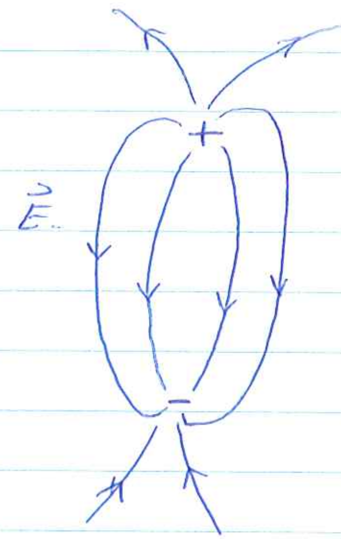
$$\boxed{B = \frac{4\pi}{c} N I}$$

Divergence of $\vec{B}$.

Field of Magnet

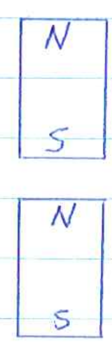


$\vec{E}$ Field of Electric Dipole

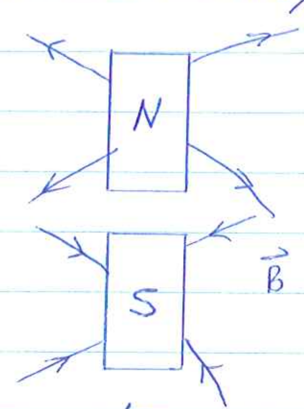


The field of a magnet resembles that of an electric dipole.

If we now cut the magnet in two, we may expect to isolate the N and S poles. Instead however we obtain 2 smaller magnets.



not



An isolated magnetic N or S pole (i.e. monopole) has never been observed.

i.e. $\displaystyle\int_{\text{surface of mag. anything}} \vec{B} \cdot d\vec{a} = 0$

If you ever see something for which this isn't true, go collect your Nobel!

Using the divergence theorem we can rewrite the last line as:

$$\int_V \nabla \cdot \vec{B} \, dV = 0. \quad \text{for any volume } V$$

$$\boxed{\nabla \cdot \vec{B} = 0}$$

This last statement merely says that the flux coming out of a unit volume is 0.

Vector Potential

Since $\nabla \cdot \vec{B} = 0$ we can write $\vec{B} = \nabla \times \vec{A}$ where $\vec{A}$ is called the vector potential.

Exercise: Verify using Cartesian coordinates that $\nabla \cdot \vec{B} = 0$ if $\vec{B} = \nabla \times \vec{A}$.

To find an expression for $\vec{A}$, let's first recall some electrostatics.

$$\nabla^2 \Phi = -4\pi \rho \quad \text{Poisson's Eqn.}$$

$$\Phi(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3r'$$

Let's begin with Ampere's Law.

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}$$

$$\nabla \times (\nabla \times \vec{A}) = \frac{4\pi}{c} \vec{J}$$

$$\nabla (\nabla . \vec{A}) - \nabla^2 \vec{A} = \frac{4\pi}{c} \vec{J}$$

We shall work using the Coulomb gauge $\nabla . \vec{A} = 0$. (Homework discusses this more.)

$$\therefore \nabla^2 \vec{A} = -\frac{4\pi}{c} \vec{J}.$$

Hence from the example of the potential in electrostatics, the solution of this equation is:

$$\boxed{\vec{A} = \frac{1}{c} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3 r'}$$

Determination of $\vec{B}$.

$$\vec{B} = \nabla \times \vec{A}$$

$$= \nabla \times \frac{1}{c} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3 r'$$

$$= -\frac{1}{c} \int \vec{J}(\vec{r}\,') \times \nabla (|\vec{r} - \vec{r}\,'|^{-1}) \, d^3 r'.$$

Now $\vec{r} = (x, y, z)$

$\vec{r}\,' = (x', y', z')$

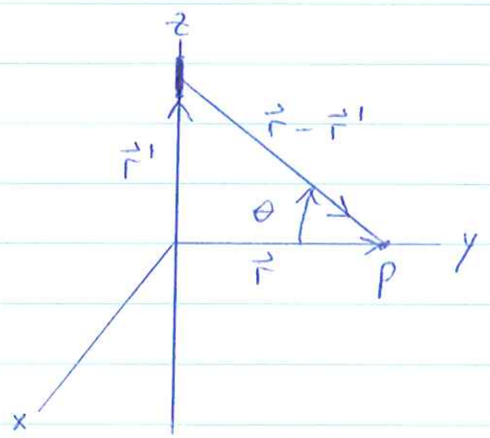
$$|\vec{r} - \vec{r}\,'| = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

Exercise: Show $\nabla\left(|\vec{r} - \vec{r}\,'|^{-1}\right) = -\dfrac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$

$$\therefore \boxed{\vec{B}(\vec{r}) = \frac{1}{c}\int \vec{J}(\vec{r}\,') \times \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}\, d^3r' \quad \text{Biot Savart Law}}$$

Examples

1. Find field due to infinite straight wire carrying current I.



Consider a current element at $\vec{r}\,'$. We shall compute $\vec{B}(P)$.

$\vec{r} = (0, y, 0)$

$\vec{r}\,' = (0, 0, z')$

$\therefore \vec{r} - \vec{r}\,' = (0, y, -z')$

Now $\vec{J}(\vec{r}\,') = \hat{z}\, I\, \delta(x')\, \delta(y')$ (wire is infinitely narrow)

$$\Rightarrow \int_{xy \text{ plane}} \vec{J} \cdot d\vec{a} = I$$

$$\vec{B}(P) = \frac{1}{c} \int_{\substack{\text{all}\\ \text{space}}} I\, \delta(x')\, \delta(y')\ \hat{z} \times \frac{(0, y, -z')}{(y^2 + z'^2)^{3/2}}\, dx'\, dy'\, dz'$$

$$= \frac{I}{c} \int_{-\infty}^{\infty} \hat{z} \times \frac{(0, y, -z')}{(y^2 + z'^2)^{3/2}}\, dz'$$

$$= \frac{I}{c} \int_{-\infty}^{\infty} \frac{(-y, 0, 0)}{(y^2 + z'^2)^{3/2}}\, dz'.$$

$$= -\frac{yI}{c}\ \hat{x} \int_{-\infty}^{\infty} \frac{dz'}{(y^2 + z'^2)^{3/2}}$$

let $\tan\theta = \frac{z'}{y}$ or $z' = y\tan\theta$.

$$dz' = y \sec^2\theta\, d\theta$$

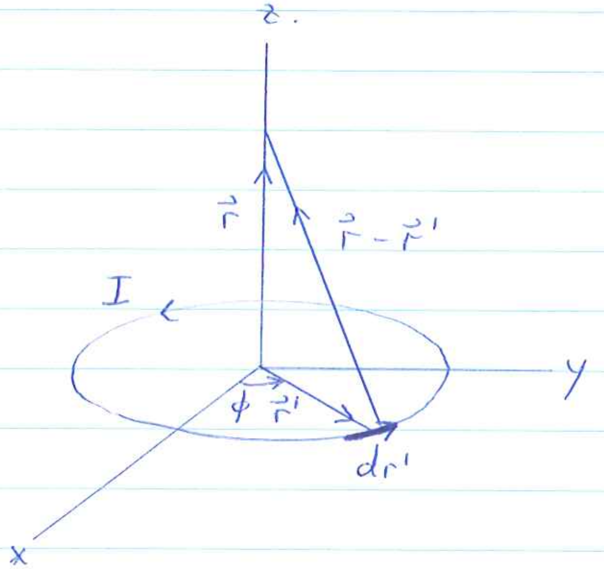
$$z' = -\infty \Rightarrow \theta = -\frac{\pi}{2}$$
$$z' = +\infty \Rightarrow \theta = +\frac{\pi}{2}.$$

$$\therefore\ \vec{B}(P) = -\frac{yI}{c}\ \hat{x} \int_{-\pi/2}^{\pi/2} \frac{y \sec^2\theta}{y^3 \sec^3\theta}\, d\theta.$$

$$= -\frac{I}{yc}\ \hat{x} \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta.$$

$$\boxed{\vec{B}(P) = -\frac{2I}{yc}\ \hat{x}.}$$

2) Find $\vec{B}$ a distance z above center of hoop carrying current I. (radius = a)



$\vec{r} = (0, 0, z)$
$\vec{r}' = (a\cos\phi, a\sin\phi, 0)$.

$\vec{r}-\vec{r}' = (-a\cos\phi, -a\sin\phi, z)$.

$d\vec{r}' = (-a\sin\phi, a\cos\phi, 0)\, d\phi$.

We can integrate (Biot Savart Law) over the infinitely small wire giving:

$$\vec{B}(\vec{r}) = \frac{I}{c}\oint d\vec{r}' \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{B}(0,0,z) = \frac{I}{c}\int_0^{2\pi} (-a\sin\phi, a\cos\phi, 0)\, d\phi \times \frac{(-a\cos\phi, -a\sin\phi, z)}{(a^2+z^2)^{3/2}}$$

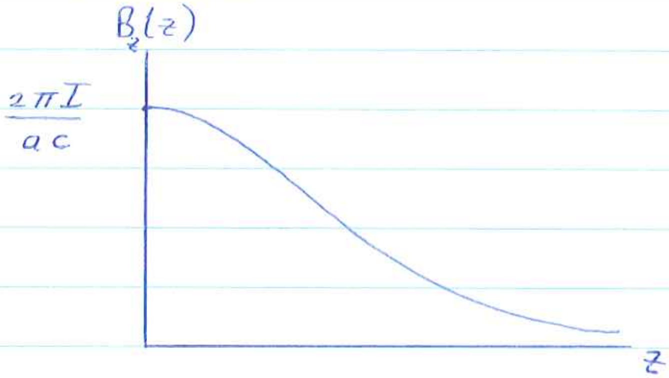
$$dr' \times (\vec{r}-\vec{r}') = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\phi\, d\phi & a\cos\phi\, d\phi & 0 \\ -a\cos\phi & -a\sin\phi & z \end{vmatrix}$$

$$= (az\cos\phi\, d\phi,\ az\sin\phi\, d\phi,\ a^2\, d\phi).$$

$$\vec{B}(0,0,z) = \frac{I}{c}\int_0^{2\pi} \frac{(az\cos\phi,\ az\sin\phi,\ a^2)}{(a^2+z^2)^{3/2}}\, d\phi.$$

$$\vec{B}(0,0,z) = \frac{I}{c}\frac{a^2}{(a^2+z^2)^{3/2}}\int_0^{2\pi} d\phi \,.\quad \hat{z}$$

$$\boxed{\vec{B}(0,0,z) = \frac{2\pi I a^2}{c(a^2+z^2)^{3/2}}\hat{z}}$$



Electrostatic + Magnetostatic Fields

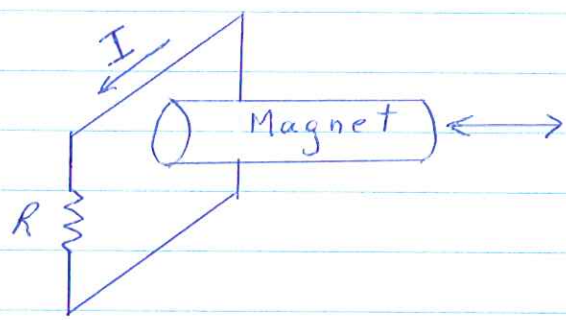
$\nabla \cdot \vec{E} = 4\pi \rho$ $\qquad\qquad$ $\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{E} = 0$ $\qquad\qquad$ $\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}$

All our results to date are derivable from these four equations.

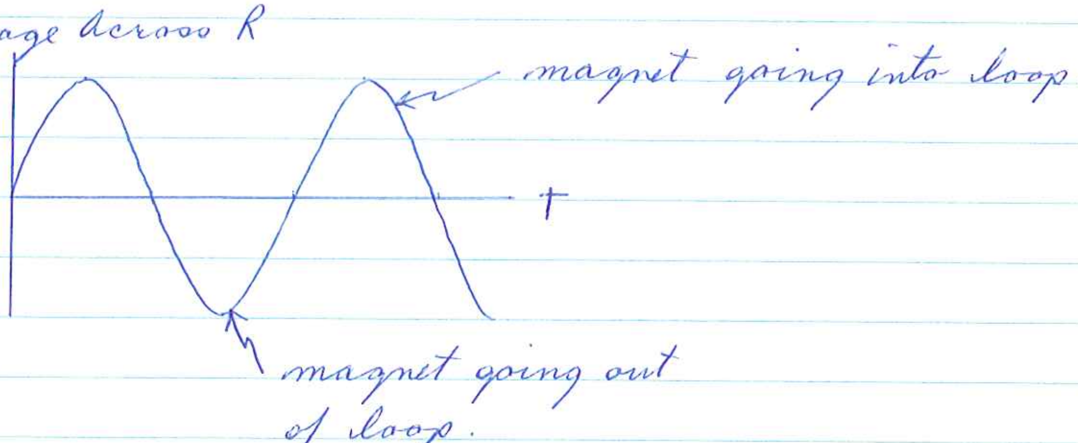
Time Dependent Fields

Faraday Effect
(Generator Principle)



A magnet is moved in and out of a loop of wire. This induces current to flow in the loop.

Voltage Across R



Energy is dissipated by the resistor. However magnetic fields cannot do any work. (Homework) Therefore the changing $\vec{B}$ field must create some rotating field $\vec{G}$ that drives a current in the loop. Since an electric field $\vec{E}$ can also drive a current, we let $\vec{G} \propto \vec{E}$.

Faraday's Law

The integral of the electric field around a closed loop is proportional to the changing flux of magnetic field through the loop.

$$\therefore \oint_{\substack{\text{around}\\ \text{loop}}} \vec{E} \cdot d\vec{l} = -\frac{1}{c}\frac{\partial}{\partial t}\int_{\substack{\text{area of}\\ \text{loop}}} \vec{B} \cdot d\vec{a}$$

Proportionality Constant is $-1/c$. Officially I haven't told you that c is the speed of light. I'll prove this in a few weeks. For now just think of c as some constant that was experimentally measured to be 3×10^{10} cm/sec.

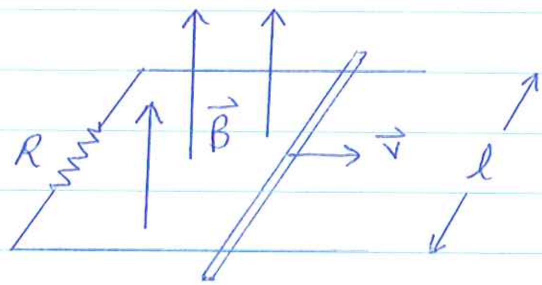
We can write Faraday's Law in differential form using Stokes Theorem.

$$\int_{\substack{\text{area of}\\ \text{loop}}} \nabla \times \vec{E} \cdot d\vec{a} = -\frac{1}{c}\int_{\substack{\text{area of}\\ \text{loop}}} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

$$\boxed{\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad \text{Differential Form of } \sout{\text{Gauss}} \text{ Faraday Law}}$$

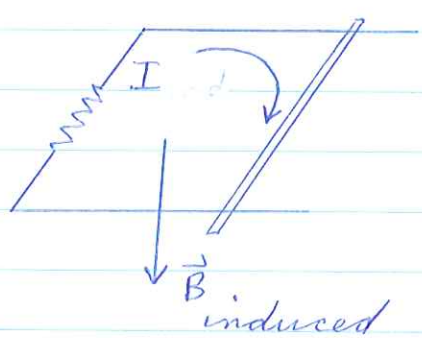
Example

Consider a uniform field $\vec{B}$ in the vertical direction. A loop in the x y plane has a resistor R on one side and a conducting bar that is moved with velocity $\vec{v}$ on the other side. What is the current flowing through the resistor?



Direction of Induced Current

The induced current flows in a direction that its magnetic field flux opposes any changing field flux. As the bar moves, the flux through the loop increases. Therefore the field of the induced current points in the downward direction.



Voltage across resistor $IR = \left| \oint \vec{E} \cdot d\vec{l} \right|$

$$= \left| -\frac{1}{c} \frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{a} \right|$$

$$= \frac{1}{c} \, B \, l \, v$$

$\therefore$ induced current $I = \dfrac{B l v}{R c}$

Continuity Equation

Let $\vec{J}$ be current density coming out of a volume V enclosed by surface S.

Current coming out of V is $I = \int_S \vec{J} \cdot d\vec{a}$

Since charge is conserved $I = -\dfrac{dQ}{dt}$ where

$Q = \int_V \rho \, dV$ is the net charge enclosed in V.

Equating the two expressions for current gives the following.

$$\int_S \vec{J} \cdot d\vec{a} = -\frac{\partial}{\partial t} \int_V \rho \, dV$$

Using the divergence theorem the left side can be rewritten to give:

$$\int_V \nabla \cdot \vec{J} \, dV = - \int_V \frac{\partial \rho}{\partial t} \, dV$$

Since this is true for any volume V, we must have that:

$$\boxed{\nabla \cdot \vec{J} = - \frac{\partial \rho}{\partial t} \qquad \text{Continuity Equation}}$$

Therefore the continuity equation is just a mathematical statement that electric charge is conserved. $\nabla \cdot \vec{J}$ is the current coming out of a unit volume and $-\frac{\partial \rho}{\partial t}$ is the change in charge of a unit volume.

Case of Static Fields

By definition $\frac{\partial \rho}{\partial t} = 0. \Rightarrow \nabla \cdot \vec{J} = 0$ (Continuity Eqn.)

This is consistent with Ampere's law since

$$\nabla \cdot \vec{J} = \nabla \cdot \frac{c}{4\pi} (\nabla \times \vec{B}) = 0.$$

Time Dependent Case

Now $\frac{\partial \rho}{\partial t} \neq 0. \Rightarrow \nabla . \vec{J} \neq 0.$

But Ampere's law still implies $\nabla . \vec{J} = 0$!

$\Rightarrow$ Something's Wrong!

Solution

Maxwell proposed that when generalizing Ampere's law for static fields to time dependent fields, an extra term $\frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ must be added.

i.e. $$\boxed{\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}}$$

The continuity equation is now obeyed since:

$$\nabla . \vec{J} = \nabla . \left\{ \frac{c}{4\pi}\left[\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \right]\right\}$$

$$= \frac{c}{4\pi}\nabla . (\nabla \times \vec{B}) - \frac{1}{4\pi}\frac{\partial}{\partial t}(\nabla . \vec{E})$$

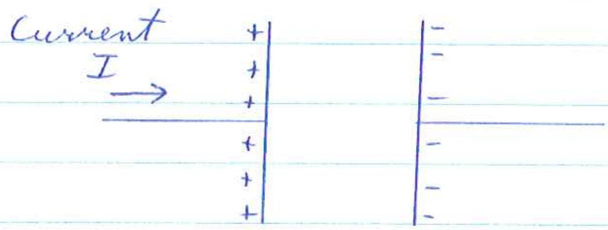
$$= 0 - \frac{1}{4\pi} 4\pi \frac{\partial \rho}{\partial t} \quad \text{using Gauss law}$$

$\therefore \nabla . \vec{J} = -\frac{\partial \rho}{\partial t}.$

Hence $\frac{1}{4\pi}\frac{\partial \vec{E}}{\partial t}$ acts as a current and is called the displacement current.

Example

Consider a capacitor.



Obviously no D.C. current may flow through the capacitor.

Electric field between plates
$$E = 4\pi\sigma = 4\pi \frac{Q}{A}$$

(Q = charge, A = plate area, σ = charge density)

$$\frac{\partial E}{\partial t} = 4\pi \frac{1}{A}\frac{\partial Q}{\partial t}.$$

$$\frac{1}{4\pi}\frac{\partial E}{\partial t} = \frac{1}{A} I$$

$\therefore$ we see that between plates the current density

$$J = \frac{1}{4\pi}\frac{\partial E}{\partial t}.$$

Hence between the capacitor plates the current is due to a changing electric field and not by movement of charge. (since there is just empty space between two plates.)

Maxwell Equations

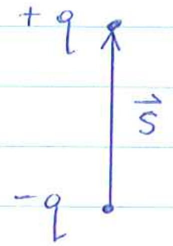
$$\nabla \cdot \vec{E} = 4\pi \rho \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

These are the equations that are obeyed by time dependent electric and magnetic fields.

Electric Fields in Matter

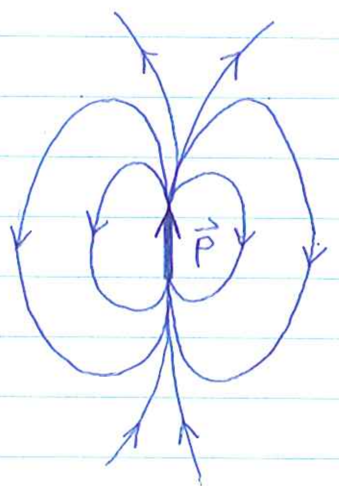
Electric Dipole



An ideal dipole $\vec{p} \equiv \lim_{\substack{q \to \infty \\ s \to 0}} q\vec{s}$ such that p remains finite. One can show the following for a dipole at the origin.

Electric Field $\vec{E}(\vec{r}) = -\dfrac{\vec{p}}{r^3} + 3\vec{r}\,\dfrac{(\vec{p}\cdot\vec{r})}{r^5}$

$\vec{E}(\vec{r})$



Potential $\Phi(\vec{r}) = \dfrac{\vec{p}\cdot\vec{r}}{r^3}$

Polarization

Consider a material composed of atoms placed in an external field $\vec{E}$. The field attracts positive charge and repels negative charge. The material can be pictured as consisting of many dipoles as shown below.

$\vec{E} = 0$ $\quad$ $\vec{p}$ points in any direction

$\vec{E} \neq 0$ $\quad$ $\vec{p}$ aligns with $\vec{E}$

We shall find the potential of such a polarized material. First we define the polarization $\vec{P}$ to be the dipole moment per unit volume.

Potential at $\vec{r}$ of a dipole $\vec{p}$ at $\vec{r}\,'$ is

$$\Phi(\vec{r}) = \frac{(\vec{r} - \vec{r}\,')\cdot\vec{p}}{|\vec{r} - \vec{r}\,'|^3}$$

Potential at $\vec{r}$ due to dipole $\vec{P}\,d^3r'$ at $\vec{r}\,'$ is:

$$d\Phi(\vec{r}) = \frac{(\vec{r} - \vec{r}\,')\cdot\vec{P}(\vec{r}\,')\,d^3r'}{|\vec{r} - \vec{r}\,'|^3}$$

$$\Phi(\vec{r}) = \int \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \cdot \vec{P}(\vec{r}\,') d^3r'$$

$$= \int \vec{P} \cdot \nabla'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) d^3r'$$

Using the identity $\nabla' \cdot (f\vec{P}) = f(\nabla' \cdot \vec{P}) + \vec{P} \cdot (\nabla' f)$

with $f = \dfrac{1}{|\vec{r}-\vec{r}\,'|}$ we get:

$$\Phi(\vec{r}) = \int \nabla' \cdot \left(\frac{\vec{P}}{|\vec{r}-\vec{r}\,'|}\right) d^3r' - \int \frac{\nabla' \cdot \vec{P}}{|\vec{r}-\vec{r}\,'|} d^3r'$$

$$= \int_S \frac{\vec{P}}{|\vec{r}-\vec{r}\,'|} \cdot d\vec{a}\,' - \int_V \frac{\nabla' \cdot \vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} dV'$$

The first term looks like the potential of a surface charge density

$$\boxed{\sigma_b = \vec{P} \cdot \hat{n}}$$

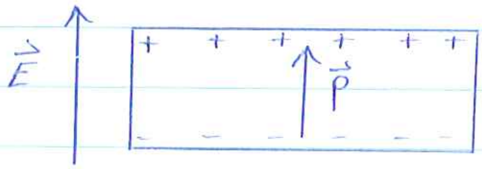
where $\hat{n}$ is the normal vector to the surface. The second term looks like the potential of a volume charge with charge density

$$\boxed{\rho_b = -\nabla \cdot \vec{P}.}$$

The subscript b has been added to indicate these charges are not free to wander around but are stuck in an atom or molecule.

Example.

Consider a slab of material placed in an electric field.



bound charge density $\rho_b = -\nabla\cdot\vec{P} = 0$ for uniform $\vec{P}$

top surface charge density $\sigma_{Top} = \vec{P}\cdot\hat{z} = +P$

bottom " " $\sigma_{Bot} = \vec{P}\cdot(-\hat{z}) = -P$

Gauss Law for Dielectric Media

Gauss Law $\nabla\cdot\vec{E} = 4\pi\rho$

$$= 4\pi(\rho_b + \rho_f)$$

where ρ_f denote charge that is free to move around, and ρ_b is charge density stuck in atom.

$$\nabla\cdot\vec{E} = 4\pi\rho_f + 4\pi(-\nabla\cdot\vec{P})$$

$$\nabla\cdot(\vec{E} + 4\pi\vec{P}) = 4\pi\rho_f$$

$\boxed{\nabla\cdot\vec{D} = 4\pi\rho_f}$ where $\vec{D} \equiv \vec{E} + 4\pi\vec{P}$ is called the electric displacement.

or $\boxed{\int_S \vec{D}\cdot d\vec{a} = 4\pi\, Q_{\text{free enclosed by } S}}$

Linear Dielectrics

A linear dielectric is one where the polarization is linearly proportional to the electric field.

i.e. $\vec{P} = \chi \vec{E}$ where the constant of proportionality χ is called the electric susceptibility.

$$\therefore \vec{D} = \vec{E} + 4\pi \chi \vec{E}$$

$$= (1 + 4\pi \chi) \vec{E}$$

$$= \epsilon \vec{E}$$ where $\epsilon \equiv 1 + 4\pi\chi$ is called the dielectric constant

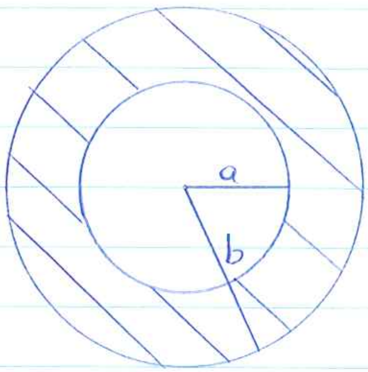
TABLE 4.2 Dielectric constants[a]

Material	Dielectric Constant
Vacuum	1
Helium	1.000068
Neon	1.00013
Hydrogen	1.00025
Argon	1.00051
Nitrogen	1.00055
Air	1.00059
Air (100 atm)	1.055
Polyethylene	2.26
Glass	4–7
Porcelain	6–8
Ethanol	24.3
Water	80.1
Water (0°C)	87.8
HCN (0°C)	158

Note that polar molecules such as H_2O have high dielectric constants.

Example

A metal conducting sphere of radius a has a charge Q. It is surrounded by a dielectric material of constant ϵ to radius b.



First let's find the electric displacement $\vec{D}$ using the modified Gauss law $\int_S \vec{D} \cdot d\vec{a} = 4\pi \int_V \rho_{free}\, dV$.

By symmetry $\vec{D} = D(r)\,\hat{r}$. Consider a sphere of radius r.

$r < a$ $\quad \int_S D \cdot d\vec{a} = 4\pi \int_V 0\, dV.$

$$D(r)\, 4\pi r^2 = 0.$$
$$D(r) = 0.$$

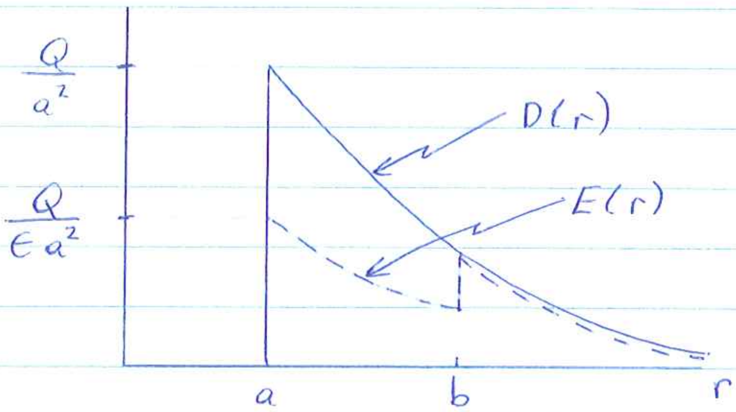
$r > a$ $\quad D(r)\, 4\pi r^2 = 4\pi Q.$

$$D(r) = \frac{Q}{r^2}$$
$$\vec{D}(r) = \frac{Q}{r^2}\,\hat{r}.$$

$\therefore$ electric field $\vec{E} = \dfrac{\vec{D}}{\epsilon} \Rightarrow$ $\quad r < a \quad \vec{E} = 0$

$\quad b > r > a \quad \vec{E} = \dfrac{Q}{\epsilon r^2}\,\hat{r}$

electric field $r > b$ $\quad \vec{E} = \dfrac{Q}{r^2}\,\hat{r}$.



Polarization $\quad \vec{P} = \dfrac{\epsilon - 1}{4\pi}\,\vec{E}$

$r < a \qquad \vec{P} = 0$

$a < r < b \qquad \vec{P} = \dfrac{\epsilon - 1}{4\pi\epsilon}\,\dfrac{Q}{r^2}\,\hat{r}$

$r > b \qquad \vec{P} = 0$

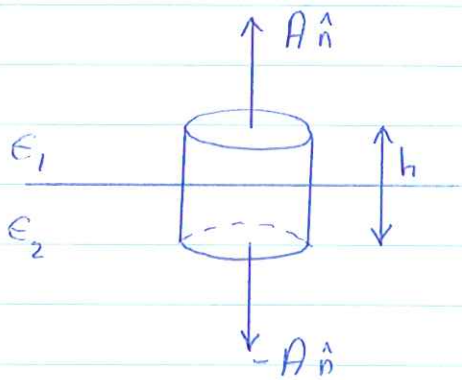
Exercise: Show $\quad \sigma_b(r = b) = \dfrac{\epsilon - 1}{\epsilon}\,\dfrac{Q}{4\pi b^2}$

$$\sigma_b(r = a) = -\frac{\epsilon - 1}{\epsilon}\,\frac{Q}{4\pi a^2}$$

Continuity Conditions at Interface of 2 Media

We wish to find how the components of $\vec{E}$ & $\vec{D}$ normal and tangent to the boundary are related in the different media.

First consider a pillbox of height h and area A straddling the boundary.



$$\int_{\substack{\text{pillbox}\\ \text{surface}}} \vec{D} \cdot d\vec{a} = 4\pi \underbrace{Q_{free}}_{=0}$$

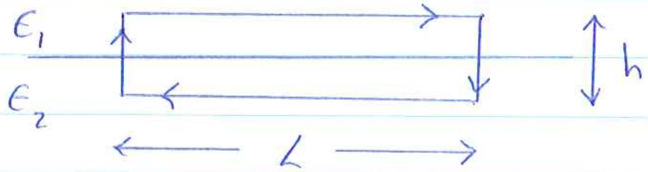
$= 0$ assuming no surface charge densities

In the limit $h \to 0$ we get: $(D_{1n} - D_{2n}) A = 0$

or $\boxed{D_{1n} = D_{2n}}$ (normal component of $\vec{D}$ is continuous at boundary)

or $\boxed{\epsilon_1 E_{1n} = \epsilon_2 E_{2n}}$

Next consider the following path around the boundary.



Faraday's Law $\oint \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{a}$

Surface enclosed by loop

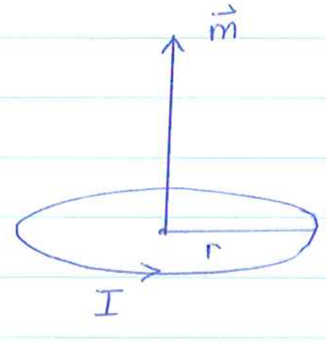
$$\lim_{h\to 0} \oint \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{\partial}{\partial t}\lim_{h\to 0}\int \vec{B}\cdot d\vec{a}$$

$$(E_{1t} - E_{2t})L = 0$$

$\boxed{E_{1t} = E_{2t}}$ (tangential component of $\vec{E}$ is continuous at boundary)

Magnetic Fields in Matter

Magnetic Dipole



magnetic dipole $\vec{m} = \dfrac{I\vec{A}}{c}$

$\vec{A}$ = area vector of current loop

Direction of $\vec{m}$ is determined by the right hand rule.
(Fingers point along current, thumb along $\vec{m}$.)
One can show the following for a dipole at the origin.

Magnetic Field $\vec{B}(\vec{r}) = -\dfrac{\vec{m}}{r^3} + \dfrac{3\vec{r}(\vec{m}\cdot\vec{r})}{r^5}$

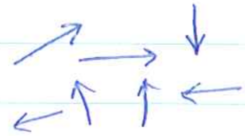
Exercise: Draw $\vec{B}$ and compare to $\vec{E}$ of electric dipole.

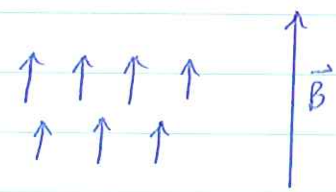
Vector Potential $\vec{A}(\vec{r}) = \dfrac{\vec{m}\times\vec{r}}{r^3}$

Exercise: Verify that $\vec{B} = \nabla \times \vec{A}$.

Magnetization

Consider a material composed of atoms placed in an external magnetic field $\vec{B}$. Each atom contains a small current loop (usually due to the electron) or magnetic dipole $\vec{m}$.





$\vec{B} = 0$
i.e. $\vec{m}$ points in any direction

$\vec{B} \neq 0$
$\vec{m} \parallel \vec{B}$

We shall find the vector potential of such a magnetized material. First we define the <u>magnetization</u> $\vec{M}$ to be the magnetic dipole moment per unit volume.

Vector potential at $\vec{r}$ of a magnetic dipole $\vec{m}$ at $\vec{r}\,'$ is:

$$\vec{A}(\vec{r}) = \vec{m} \times \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$

Potential at $\vec{r}$ due to mag. dipole $\vec{M}(\vec{r}\,')\, d^3r'$ at $\vec{r}\,'$ is:

$$d\vec{A}(\vec{r}) = \vec{M}(\vec{r}\,')\, d^3r' \times \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$

$$\vec{A}(\vec{r}) = \int_V \vec{M}(\vec{r}\,') \times \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3r'$$

$$= \int_V \vec{M}(\vec{r}\,') \times \nabla'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) d^3r'$$

Using the identity $\nabla' \times (f\vec{M}) = f\,\nabla' \times \vec{M} - \vec{M} \times \nabla' f$
where $f = \dfrac{1}{|\vec{r}-\vec{r}\,'|}$ we get:

$$\vec{A}(\vec{r}) = \int_V \frac{\nabla' \times \vec{M}}{|\vec{r}-\vec{r}\,'|}\, d^3r' - \int_V \nabla' \times \left(\frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) d^3r'$$

$$= \int_V \frac{\nabla' \times \vec{M}}{|\vec{r}-\vec{r}\,'|}\, d^3r' + \int_S \frac{\vec{M} \times \hat{n}}{|\vec{r}-\vec{r}\,'|}\, da$$

Exercise: Prove $\displaystyle\int_V \nabla \times \vec{F}\, dV = -\int_S \vec{F} \times d\vec{a}$

Hint: Replace $\vec{F}$ by $\vec{F} \times \vec{c}$ in divergence theorem where $\vec{c}$ is a constant vector.

The first term looks like the vector potential of a volume current density

$$\vec{J}_b = c\,(\nabla \times \vec{M})$$

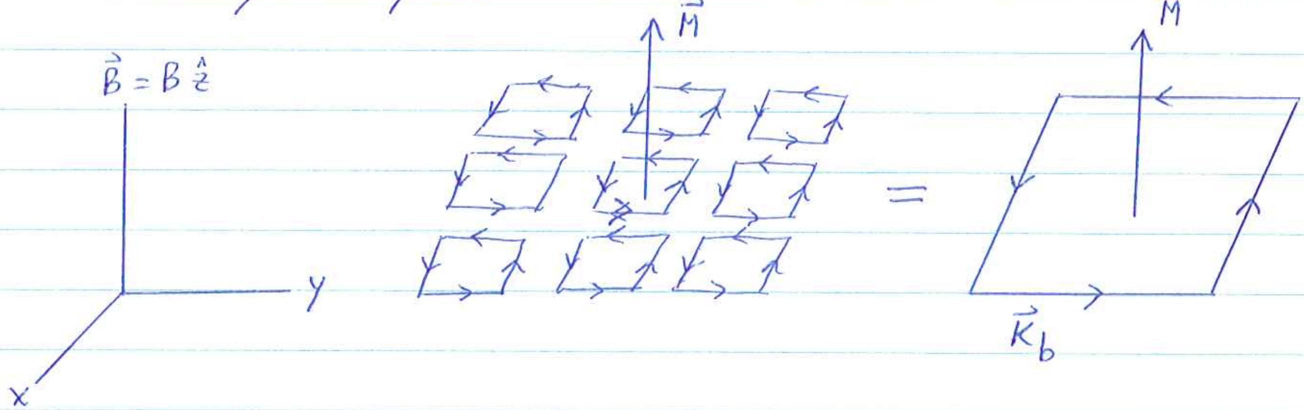
The second term looks like the vector potential of a surface current density

$$\vec{K}_b = c\,\vec{M} \times \hat{n}$$

The subscript b has been written to indicate the currents are bound to the atom.

Example

Consider a 2 dim. slab of material perpendicular to a magnetic field.



For uniform magnetization:

volume current density $\vec{J}_b = c(\nabla \times \vec{M}) = 0$

surface current density $\vec{K}_b = c(\vec{M} \times \hat{n}) \neq 0$.

Ampere's Law For Magnetic Materials

Total Current density $\vec{J} = \vec{J}_b + \vec{J}_f$ ← due to power supply

↑ due to magnetization

$$\nabla \times \vec{B} = \frac{4\pi}{c}(\vec{J}_b + \vec{J}_f)$$

We consider static fields and therefore ignore the displacement current.

$$\nabla \times \vec{B} = \frac{4\pi}{c}\left(c\, \nabla \times \vec{M} + \vec{J}_f \right)$$

$$\nabla \times \left(\vec{B} - 4\pi \vec{M} \right) = \frac{4\pi}{c} \vec{J}_f$$

$$\nabla \times \vec{H} = \frac{4\pi}{c} \vec{J}_f \quad \text{where} \quad \vec{H} \equiv \vec{B} - 4\pi \vec{M}.$$

Significance of $\vec{H}$.

$\vec{H}$ depends on $\vec{J}_f$, the free current which we generate by our power supply. Therefore H can be more easily controlled than B.

Linear Magnetized Media

A linear magnetized media is one where the magnetization is linearly proportional to the field $\vec{H}$.

i.e. $\vec{M} = \chi_m \vec{H}$ where χ_m is called the magnetic susceptibility, χ_m = constant for linear media

$$\therefore \vec{H} = \vec{B} - 4\pi \chi_m \vec{H}$$

$$\vec{B} = (1 + 4\pi \chi_m)\vec{H}$$
$$= \mu \vec{H}$$ where $\mu \equiv 1 + 4\pi \chi_m$ is called the permeability.

TABLE 6.1 Magnetic susceptibilities[a]

Material	Magnetic Susceptibility
Diamagnetic:	
Bismuth	-16.5×10^{-5}
Gold	-3.0×10^{-5}
Silver	-2.4×10^{-5}
Copper	-0.96×10^{-5}
Water	-0.90×10^{-5}
Carbon Dioxide	-1.2×10^{-8}
Hydrogen	-0.22×10^{-8}
Paramagnetic:	
Oxygen	190×10^{-8}
Sodium	0.85×10^{-5}
Aluminum	2.1×10^{-5}
Tungsten	7.8×10^{-5}
Gadolinium	$48{,}000 \times 10^{-5}$

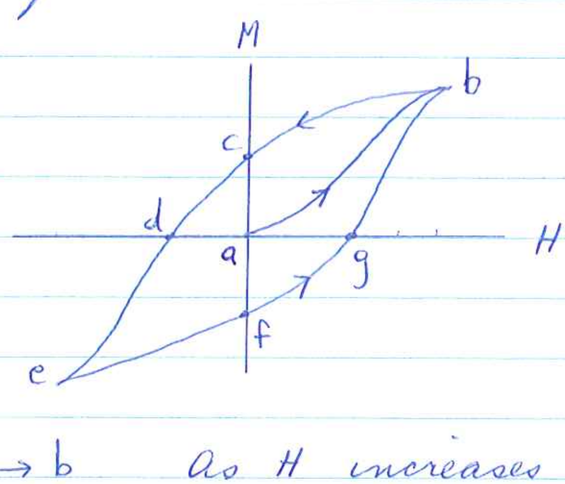
Nonlinear Magnetized Media (Ferromagnetism)

Magnetization $\vec{M} \not\parallel \vec{H}$ or $\mu \neq$ constant but $\mu = \mu(H)$.

In fact it is possible to have $\vec{M} \neq 0$ if $\vec{H} = 0$, i.e. a permanent magnet. The "atomic current loops" exert forces on each other to keep the alignment

Hysteresis

We shall study the behaviour of a nonlinear material. We begin at a with an unmagnetized sample.



$a \to b$ As H increases so does M until saturation is reached at b. At b all current loops are aligned.

$b \to c \to d \to e$ As H is turned off and reversed, M decreases but more slowly at first since "atomic current loops" try to keep original alignment. At e we have saturation - all current loops are aligned in direction opposite to that at b.

$e \to f \to g \to b$ As H increases so does M but at first more slowly.

Comments

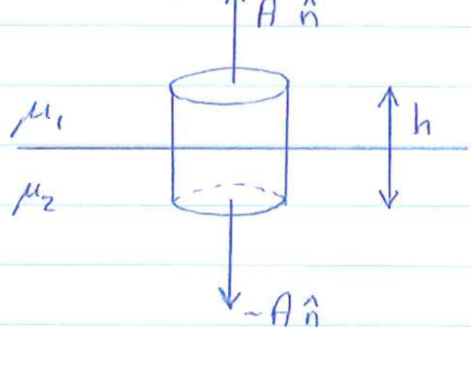
1. At c & f, material is a permanent magnet.

2. The magnetization $\vec{M}$ depends on the history of the substance. ∴ the above figure is called a hysteresis loop.

Continuity Conditions at Interface of 2 Media

We wish to find how the components of $\vec{B}$ & $\vec{H}$ normal and tangent to the boundary are related in the different magnetized media.

First consider a pillbox of height h and area A straddling the boundary.



$$\nabla \cdot \vec{B} = 0.$$

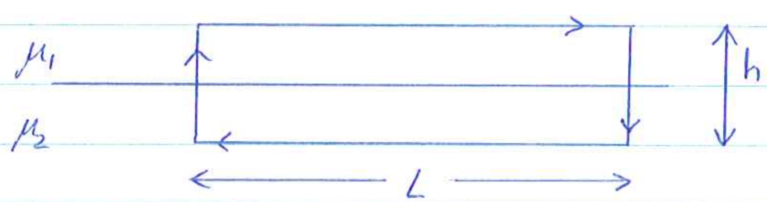
$$\int_{\substack{\text{pillbox}\\ \text{surface}}} \vec{B} \cdot d\vec{a} = 0$$

In the limit $h \to 0$ we get $(B_{1n} - B_{2n})A = 0.$

or $\boxed{B_{1n} = B_{2n}}$ (normal component of $\vec{B}$ is continuous at boundary)

or $\boxed{\mu_1 H_{1n} = \mu_2 H_{2n}}$

Next consider the following path around the boundary



$$\nabla \times \vec{H} = \frac{4\pi}{c}\vec{J}_{free} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

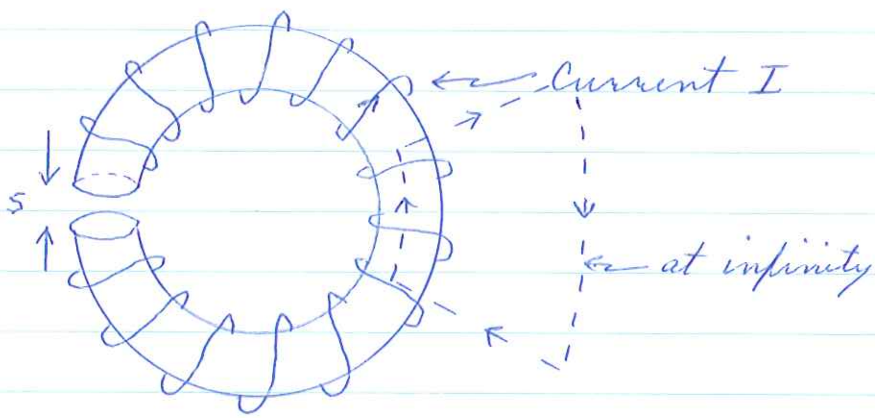
$$\lim_{h\to 0}\oint \vec{H}\cdot d\vec{l} = \lim_{h \to 0}\left\{ \frac{4\pi}{c}\underbrace{\int \vec{J}_{free}\cdot d\vec{a}}_{\substack{=0 \text{ provided there}\\ \text{are no surface}\\ \text{current densities}}} + \frac{1}{c}\frac{\partial}{\partial t}\int \vec{D}\cdot d\vec{a}\right\}$$

$$(H_{1t} - H_{2t})L = 0.$$

$\boxed{H_{1t} = H_{2t} \quad \text{or} \quad \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}}$ Tangential compon. of $\vec{H}$ is continuous

Example

Consider a circular piece of iron with N windings/cm around it. A gap s of area A exists. Find B & H in the gap.



First we find $\vec{H}$ using $\oint \vec{H} \cdot d\vec{l} = \frac{4\pi}{c} I_{free}$

around the dotted loop. By symmetry (for small gap) $\vec{H}$ points in direction $\perp$ to radius vector.

$$H\, l = \frac{4\pi}{c} N I l.$$

length of dotted arc in magnet

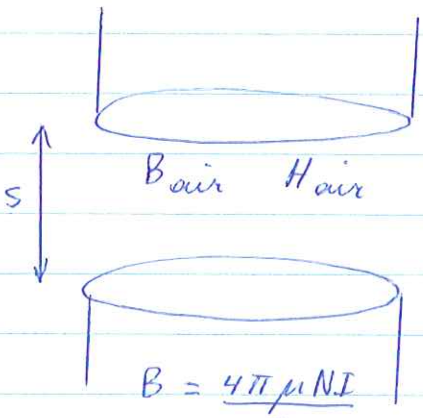
$$\therefore \; H = \frac{4\pi I N}{c} \quad \text{in magnet.}$$

Now $B = \mu H$

$$= \frac{4\pi \mu N I}{c}$$

Since $\mu \simeq 10^3$ for Fe, $B \gg H$ in magnet.

Air Gap.



$$B = \frac{4\pi \mu N I}{c}$$

$$H = \frac{4\pi N I}{c}$$

If s is small then $\vec{B}$ & $\vec{H}$ are nearly perpendicular to surface of magnet.

$$\therefore \quad B_{air} = B_{magnet} = \frac{4\pi \mu N I}{c}$$

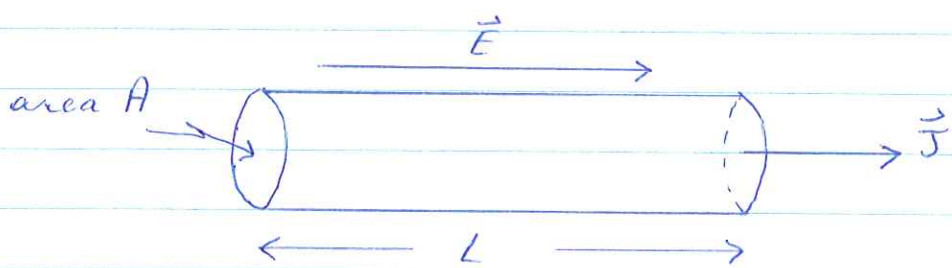
$$\mu_{air} \simeq 1 \Rightarrow H_{air} = B_{air} = \frac{4\pi \mu N I}{c}$$

Ohm's Law

For many materials, the current density $\vec{J}$ is linearly proportional to the electric field.

i.e. $\boxed{\vec{J} = \sigma \vec{E}}$ where σ is called the conductivity.

This is a statement of the familiar Ohm's law in disguise. To see this consider a current flowing through a cylinder of area A and length L.



Current flowing through cylinder $I = JA$

Electric potential difference or voltage between ends of cylinder

$$V = \int \vec{E} \cdot d\vec{l} = EL$$

$\therefore$ $J = \sigma E$ can be rewritten as:

$$\frac{I}{A} = \sigma \frac{V}{L}$$

$$I = \frac{\sigma A}{L} V$$

$\boxed{I = \frac{V}{R} \quad \text{where} \quad R \equiv \frac{L}{\sigma A}}$ is called the resistance

The resistance is also written as $R = \frac{\rho L}{A}$ where $\rho \equiv \frac{1}{\sigma}$ is called the resistivity. As intuitively expected the resistance is directly proportional to the wire length and inversely proportional to its area.

Energy of Static Fields

Energy stored in electric field is: (for nondielectric material)

$$U_E = \int_V \frac{E^2}{8\pi}\, dV$$

$\therefore \frac{E^2}{8\pi}$ is energy density of electric field.

Energy stored in magnetic field is: (for nonmagnetized material)

$$U_B = \int_V \frac{B^2}{8\pi}\, dV$$

$\therefore \frac{B^2}{8\pi}$ is energy density of magnetic field.

Material	Resistivity (ohm-meters)
Conductors:	
Silver	1.59×10^{-8}
Copper	1.67×10^{-8}
Gold	2.35×10^{-8}
Aluminum	2.65×10^{-8}
Nichrome	100×10^{-8}
Semiconductors:	
Salt water (saturated)	0.044
Germanium	0.46
Silicon	300–400 (depending on purity)
Insulators:	
Water (pure)	2.5×10^{5}
Wood	10^{8}–10^{11}
Glass	10^{10}–10^{14}
Quartz	10^{13}
Sulfur	2×10^{15}
Rubber	10^{13}–10^{16}

Energy of Time Dependent Fields

The Lorentz force on charge q is:

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

Work done by Lorentz force when charge q moves distance $\vec{v}\,dt$ is:

$$dW = \vec{F} \cdot \vec{v}\,dt$$

$$\frac{dW}{dt} = q\,\vec{E} \cdot \vec{v}$$

Similarly if we consider a charge distribution having a charge density ρ, then:

$$\frac{dW}{dt} = \int_V \rho\, dV\, \vec{E} \cdot \vec{v}$$

$$= \int_V \vec{J} \cdot \vec{E}\, dV \qquad \text{where } \vec{J} = \rho\vec{v}$$

$\therefore$ $\vec{J} \cdot \vec{E}$ is work done per unit time by Lorentz force on charge in unit volume.

We shall now rewrite $\frac{dW}{dt}$ using Maxwell's equations.

We begin with "Ampere's law."

$$\vec{J} = \frac{c}{4\pi}\left\{\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t}\right\}$$

$$\therefore \int_V \vec{E}\cdot\vec{J}\, dV = \frac{c}{4\pi}\int_V \left(\nabla\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\right)\cdot\vec{E}\, dV.$$

$$= \frac{c}{4\pi}\int_V \left[(\nabla\times\vec{H})\cdot\vec{E} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\cdot\vec{E}\right] dV$$

Now $\nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H})$

$$\int_V \vec{E}\cdot\vec{J}\, dV = \frac{c}{4\pi}\int_V \left[\vec{H}\cdot(\nabla\times\vec{E}) - \nabla\cdot(\vec{E}\times\vec{H}) - \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\cdot\vec{E}\right] dV$$

$$= \frac{c}{4\pi}\int_V \left[-\frac{1}{c}\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \nabla\cdot(\vec{E}\times\vec{H}) - \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\cdot\vec{E}\right] dV$$

using Faraday's law

$$\frac{1}{4\pi}\int_V \left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right) dV = -\int_V \vec{J}\cdot\vec{E}\, dV - \frac{c}{4\pi}\int_V \nabla\cdot(\vec{E}\times\vec{H})\, dV$$

For linear media $\vec{B} = \mu\vec{H}$, $\vec{D} = \epsilon\vec{E}$,

$\therefore\ \vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{H}\cdot\vec{B})$ and $\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D})$

Also we use the divergence theorem to rewrite the last term.

$$\boxed{\frac{\partial}{\partial t}\int_V \left(\frac{\vec{H}\cdot\vec{B}}{8\pi} + \frac{\vec{E}\cdot\vec{D}}{8\pi}\right) dV = -\int_V \vec{J}\cdot\vec{E}\, dV - \frac{c}{4\pi}\int_S (\vec{E}\times\vec{H})\cdot d\vec{a}}$$

This equation is a statement of energy conservation. Let's examine it term by term.

$$\frac{\partial}{\partial t}\int_V \frac{\vec{H}\cdot\vec{B}}{8\pi}\, dV$$ This is the rate of change of energy stored in the magnetic field.

$$\frac{\partial}{\partial t}\int_V \frac{\vec{E}\cdot\vec{D}}{8\pi}\, dV$$ This is the rate of change of energy stored in the electric field.

$$\int_V \vec{J}\cdot\vec{E}\, dV$$ This is the rate at which the Lorentz force does work.

$$\frac{c}{4\pi}\int_S \vec{E}\times\vec{H}\cdot d\vec{\sigma}$$ This is the rate at which energy is flowing out of volume V through surface S.

Hence the left side is rate of change of stored energy in volume V. This energy can do work or flow out through surface S enclosing volume

Poynting Vector

$$\boxed{\vec{S} \equiv \frac{c}{4\pi}\vec{E}\times\vec{H}}$$

$|\vec{S}|$ is the energy passing through a unit area per unit time.

$\vec{S}$ points in the direction of energy flow.

Question: How does electric & magnetic field energy flow?

Answer: Stay Tuned.

Assignment 2

1. Prove that magnetic fields can't do any work.

2. $\nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}$
However many vector potentials $\vec{A}$ give rise to the same magnetic field $\vec{B}$, eg. $\vec{A}$ & $\vec{A} + \nabla \chi$.
This freedom to choose $\nabla \chi$ permits us to arbitrarily choose the value of $\nabla \cdot \vec{A}$. In the Coulomb gauge $\nabla \cdot \vec{A} = 0$. Show that this is possible by finding χ such that:
$$\nabla \cdot (\vec{A} + \nabla \chi) = 0.$$

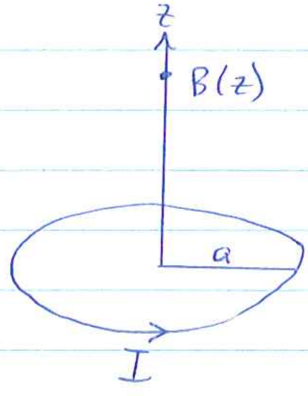
Hint: Remember solution to Poisson's Eqn.

For further reference we state the Lorentz Gauge.

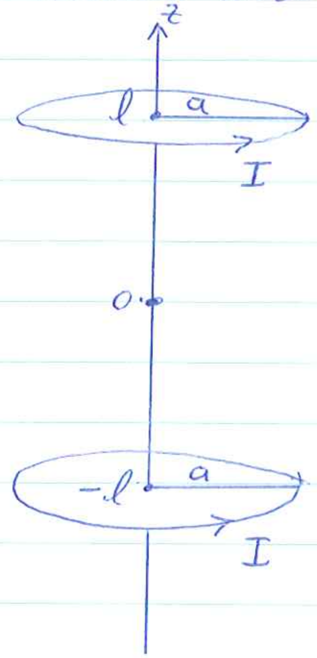
$$\nabla \cdot \vec{A} = -\frac{1}{c}\frac{\partial \Phi}{\partial t} \qquad \Phi = \text{electric potential}$$

3. The magnetic field at height z above a single loop of current I is:

$$\vec{B} = \frac{2\pi I a^2}{c(a^2 + z^2)^{3/2}}\hat{z}$$



It is important in many experiments to have a uniform field. Helmholtz coils are used for this purpose as shown below.



a) What is field on the z axis due to both coils?

b) Why are all odd derivatives of $B(z)$ at the origin equal to 0?

c) Show that the second derivative of $B(z)$ evaluated at the origin is 0 if $2l = a$.
(coil separation distance = coil radius)
This is the so called Helmholtz criterion.

d) $a = 30$ cm.
$I = 20$ amps.
turns in each coil $N = 50$
Units of magnetic field in cgs system are gauss.

$$1 \text{ gauss} = \frac{\text{esu/sec}}{\text{cm cm/sec}} = \frac{\text{esu}}{\text{cm}^2}$$

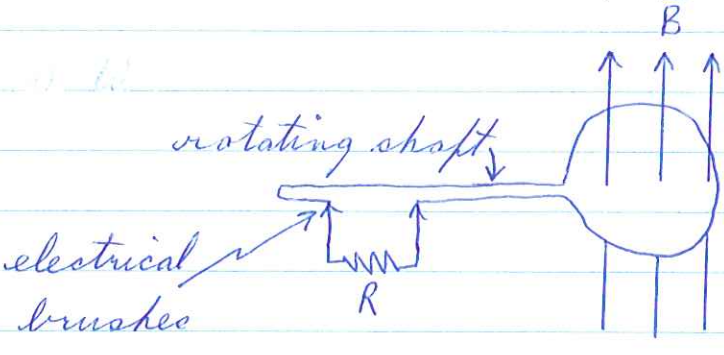
i) Convert current 20 amps to current in esu/sec.

Verify that $\dfrac{I(\text{esu/sec})}{c} = \dfrac{I(\text{amps})}{10}$

ii) What is field at center of Helmholtz coils?

iii) Earth's magnetic field is about 1/2 gauss. What current is needed to generate a field that can cancel the Earth's field?

4. Consider a loop of wire of area A at the end of a rotating shaft. The shaft rotates at angular frequency ω. The current loop is immersed in a uniform field $\vec{B}$.



a) What is flux through loop as a function of time?

b) What is voltage created by changing flux?

c) What is current flowing in resistor as a function of time and plot it?

5. Consider the example of the conducting sphere with charge Q surrounded by dielectric.

a) Find bound surface charge densities σ_b at $r = a$ & b, a = rad. of sphere, b = rad. of dielectric mat. surrounding sphere

b) What is electric potential at center of sphere taking potential to be 0 at infinity.

6. Ampere's equation + displacement current in material is:

$$\nabla \times \vec{H} = \frac{4\pi}{c} \vec{J}_{free} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

One may wonder why this last term isn't $\frac{1}{c}\frac{\partial \vec{E}}{\partial t}$.

Show that the above equation gives rise to the continuity equation of charge $\nabla \cdot \vec{J}_{free} = -\frac{\partial \rho}{\partial t}$.

7. The differential forms of Maxwell's equations in a nonpolarizable, nonmagnetizable media are:

$$\nabla \cdot \vec{E} = 4\pi \rho \qquad\qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad\qquad \nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

State the integral form of each and give a brief qualitative description of each of these "complicated mathematical equations".

Summary of Equations

Maxwell's Equations

$$\nabla \cdot \vec{D} = 4\pi\rho \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

Force on a Moving Charge

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

Constitutive Relations

$$\vec{D} = \vec{E} + 4\pi\vec{P}$$

$$\vec{H} = \vec{B} - 4\pi\vec{M}$$

Constitutive Relations for linear media

$$\vec{D} = \epsilon\vec{E}$$

$$\vec{B} = \mu\vec{H}$$

Continuity Equation

$$\nabla \cdot \vec{J} = -\frac{\partial\rho}{\partial t}$$

Ohm's Law

$$\vec{J} = \sigma\vec{E}$$

III Waves

Maxwell's Equations in Nonconducting Linear Media assuming no free charges or currents are:

$$\nabla \cdot \vec{E} = 0 \qquad (1) \qquad\qquad \nabla \cdot \vec{B} = 0 \qquad (3)$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad (2) \qquad\qquad \nabla \times \vec{B} = \frac{\epsilon\mu}{c}\frac{\partial \vec{E}}{\partial t} \qquad (4)$$

Taking the curl of (2) we get:

$$\nabla \times (\nabla \times \vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\epsilon\mu}{c}\frac{\partial \vec{E}}{\partial t}\right) \quad \text{using (4)}$$

$$\boxed{\nabla^2 \vec{E} - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0} \quad \text{using (1)}$$

Exercise: Show that $\boxed{\nabla^2 \vec{B} - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0}$

The above equations are so called wave equations. They have the form

$$\nabla^2 \psi - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

where $\psi = E_x, E_y, E_z, B_x, B_y$ or B_z.

Scalar Wave Equation

Consider the one dimensional form of the last equation.

$$\frac{\partial^2 \psi}{\partial z^2} - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

We shall show that a solution is $\psi = A\cos(kz - \omega t + \delta)$

$$\frac{\partial \psi}{\partial z} = -Ak\sin(kz - \omega t + \delta)$$

$$\frac{\partial^2 \psi}{\partial z^2} = -Ak^2\cos(kz - \omega t + \delta)$$

$$\frac{\partial \psi}{\partial t} = A\omega\sin(kz - \omega t + \delta)$$

$$\frac{\partial^2 \psi}{\partial t^2} = -A\omega^2\cos(kz - \omega t + \delta)$$

The wave equation then implies:

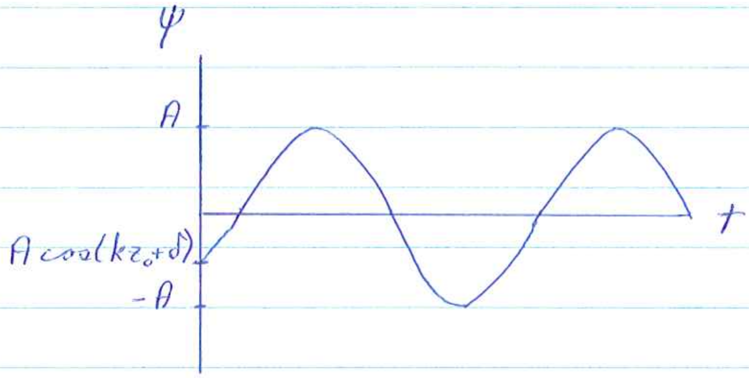
$$-k^2 - \frac{\epsilon\mu}{c^2}(-\omega^2) = 0.$$

$$\boxed{k = \sqrt{\epsilon\mu}\,\frac{\omega}{c}}$$

$\therefore$ $\psi = A\cos(kz - \omega t + \delta)$ is a solution of the wave equation provided that $k = \sqrt{\epsilon\mu}\,\frac{\omega}{c}$.

Interpretation of Solution

Consider an observer at a fixed position z_0 in space. Observer sees wave $\psi = A \cos(kz_0 - \omega t + \delta)$

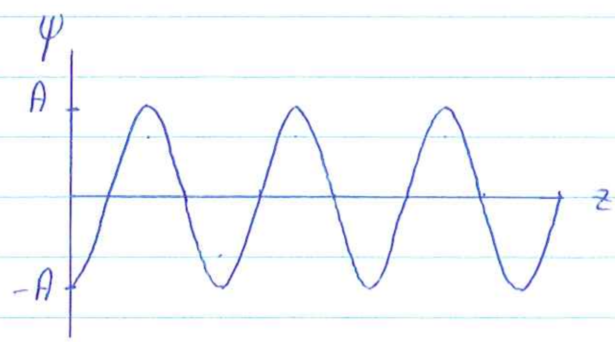


Period of wave $T = \frac{2\pi}{\omega}$

Frequency of wave $\nu = \frac{1}{T} = \frac{\omega}{2\pi}$

Amplitude of wave is A.

Next consider an observer who takes a snapshot of the wave at time t_0. Snapshot records wave $\psi = A \cos(kz - \omega t_0 + \delta)$.



Spatial period of wave or wavelength $\lambda = \frac{2\pi}{k}$.

Speed of Wave

To find the speed of the wave, we let the observer ride along with the wave. i.e. observer's position & time are such that ψ = const.

$$\Rightarrow \quad kz - \omega t + \delta = \text{const.}$$

$$k\frac{dz}{dt} - \omega = 0$$

$$\frac{dz}{dt} = \frac{\omega}{k}$$

∴ speed of wave also called phase velocity

$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon\mu}} \quad \text{using relation between } k \text{ \& } \omega.$$

We also see that $\sqrt{\epsilon\mu} \equiv n$ is index of refraction.

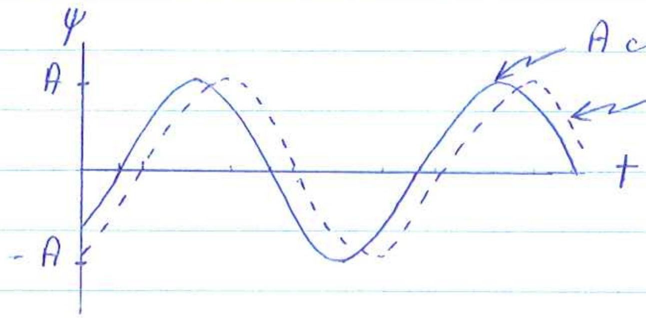
Direction of Wave

Since $\frac{dz}{dt} > 0$, wave propagates in $+\hat{z}$ direction.

One can also show that $\psi = A\cos(kz + \omega t + \delta)$ is a solution of the wave equation. This wave has the same amplitude, wavelength and frequency as the preceding wave but propagates in the $-\hat{z}$ direction.

Significance of δ

Consider an observer sitting at position z_0 and watching 2 waves $A\cos(kz_0 - \omega t)$ and $A\cos(kz_0 - \omega t + \delta)$ passing by.



$\therefore$ $A\cos(kz_0 - \omega t + \delta)$ lags behind $A\cos(kz_0 - \omega t)$ by a constant amount. δ is said to be the phase shift of $\cos(kz - \omega t)$ relative to $\cos(kz - \omega t + \delta)$.

Scalar Wave Eqn. in 3 Dimensions

$$\nabla^2 \psi - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

A solution of this is $\psi = A\cos(\vec{k}\cdot\vec{r} - \omega t)$ where:

$k = \sqrt{\epsilon\mu}\,\frac{\omega}{c}$ This follows from the wave eqn.

A = amplitude of wave

$\frac{2\pi}{|\vec{k}|}$ = wavelength

$\frac{\omega}{2\pi}$ = frequency

$\vec{k}$ points in direction of propagation

Plane Waves

Suppose $\vec{k} = k\hat{z}$. Then $\psi = A\cos(kz - \omega t)$ is a solution of the 3 dim. scalar wave equation.
For a given z, ψ has the same value for all x + y values. i.e. ψ is constant over entire xy plane

$A\cos(kz - \omega t)$ is therefore called a plane wave propagating in direction $\vec{k} = k\hat{z}$.

Similarly $\psi = A\cos(\vec{k}\cdot\vec{r} - \omega t)$ is a plane wave travelling in direction $\vec{k}$. ψ is constant in plane perpendicular to the propagation vector $\vec{k}$.

Superposition Principle

If ψ_1 + ψ_2 are solutions of the wave equation, then $\psi_1 + \psi_2$ is a solution.

Exercise: Prove the above.

General Solution

$$\psi = \sum_k A_k \cos(\vec{k}\cdot\vec{r} - \omega t) \quad \text{where } k = \sqrt{\epsilon\mu}\,\frac{\omega}{c}$$

or if k is continuous we have:

$$\psi = \int A(k)\cos(\vec{k}\cdot\vec{r} - \omega t)\,dk.$$

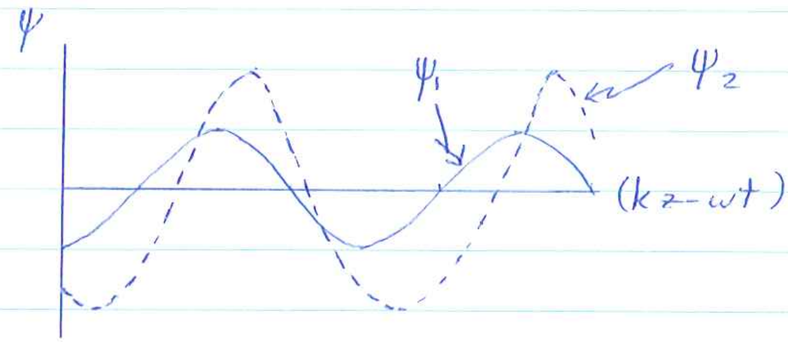
By Fourier analysis, we know that any complicated wave can be expressed as some linear combination of cosine or plane waves. ∴ study of plane waves is important.

Complex Notation

Suppose we wish to add the following two waves.

$$\psi_1 = A_1 \cos(kz - \omega t + \delta_1)$$

$$\psi_2 = A_2 \cos(kz - \omega t + \delta_2).$$



Since both waves have the same frequency & wavelength, so will the resultant wave $\psi = \psi_1 + \psi_2$.

Indeed one can show (after a ton of algebra) that

$$\psi = A \cos(kz - \omega t + \delta)$$

where $A = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos(\delta_1 - \delta_2)}$

$$\delta = \tan^{-1}\left[\frac{A_1 \sin\delta_1 + A_2 \sin\delta_2}{A_1 \cos\delta_1 + A_2 \cos\delta_2}\right]$$

Exercise: Try to show the above.

We shall show the above using a different method.

Note $\psi_1 = Re\left[A_1 e^{i(kz-\omega t+\delta_1)}\right]$

$$\psi_2 = Re\left[A_2 e^{i(kz-\omega t+\delta_2)}\right]$$

We shall denote the waves by their complex analogs, remembering that the actual part that we physically experience is the real part.

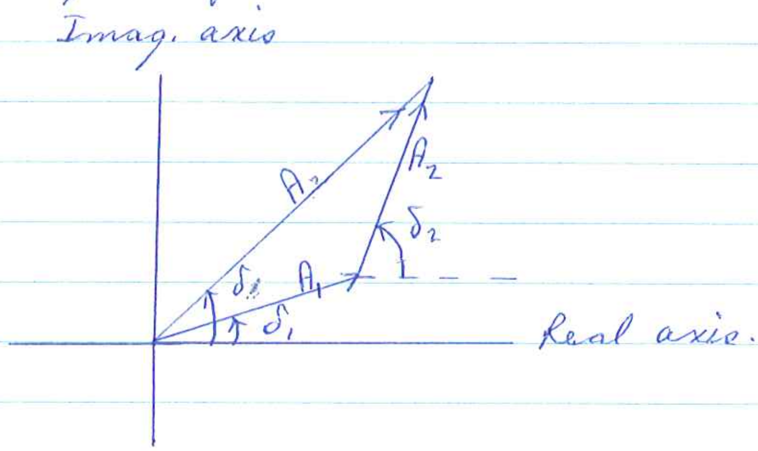
$$\psi_1 + \psi_2 = A_1 e^{i(kz-\omega t+\delta_1)} + A_2 e^{i(kz-\omega t+\delta_2)}$$

$$= \left(A_1 e^{i\delta_1} + A_2 e^{i\delta_2}\right) e^{i(kz-\omega t)}$$

Writing $\psi \equiv \psi_1 + \psi_2 = A e^{i\delta} e^{i(kz-\omega t)}$ we see that

$$A e^{i\delta} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2} \qquad (1)$$

In the complex plane, this is merely the addition of 2 complex quantities.



Equating real and imaginary parts of (1) gives:

$$A \cos\delta = A_1 \cos\delta_1 + A_2 \cos\delta_2$$
$$A \sin\delta = A_1 \sin\delta_1 + A_2 \sin\delta_2$$

$$\Longrightarrow \quad \tan\delta = \frac{A_1 \sin\delta_1 + A_2 \sin\delta_2}{A_1 \cos\delta_1 + A_2 \cos\delta_2}$$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_1 - \delta_2)}$$

Note that this complex notation greatly simplified the algebra!

Plane Wave in Complex Form.

$$\psi = A e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

This is a plane wave propagating in direction $\vec{k}$. We implicitly remember that we must always take the real part of ψ to find the field we physically experience.

Plane Wave Solution of Wave Equation

Wave Equation For $\vec{E}$ & $\vec{B}$

$$\nabla^2 \vec{E} - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\nabla^2 \vec{B} - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

One can show that the plane wave propagating in the $+\hat{z}$ direction

$$\vec{E} = \vec{E}_0 \, e^{i(kz-\omega t)}$$

$$\vec{B} = \vec{B}_0 \, e^{i(kz-\omega t)}$$

where $\vec{E}_0$ and $\vec{B}_0$ are constant vector amplitudes is a solution of the wave equation provided:

$$k^2 - \epsilon\mu \frac{\omega^2}{c^2} = 0$$

$$\text{or } k = \sqrt{\epsilon\mu}\,\frac{\omega}{c}$$

Exercise: Derive this last result.

In addition to satisfying the wave equation, the plane wave solution must satisfy the four Maxwell equations which were used to derive the wave equation.

$\nabla \cdot \vec{E} = 0 \implies \boxed{i\vec{k}\cdot\vec{E}_0 = 0}$

or in our case $\hat{z}\cdot\vec{E}_0 = 0$

$\therefore \vec{E}_0$ is a vector having no component in the direction of propagation. i.e. $\vec{E}_0 \perp \vec{k}$.

$\nabla \cdot \vec{B} = 0 \implies \boxed{i\vec{k}\cdot\vec{B}_0 = 0}$

or in our case $\hat{z}\cdot\vec{B}_0 = 0$.

$\therefore \vec{B}_0$ is a vector having no component in the direction of propagation, i.e. $\vec{B}_0 \perp \vec{k}$.

$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \implies i\,\vec{k}\times\vec{E}_0 = -\frac{1}{c}(-i\omega)\vec{B}_0$

$\boxed{\hat{k}\times\vec{E}_0 = \frac{1}{\sqrt{\epsilon\mu}}\vec{B}_0}$ using $k = \sqrt{\epsilon\mu}\,\frac{\omega}{c}$

$\therefore \hat{k}$, $\vec{E}_0$ and $\vec{B}_0$ are 3 vectors all perpendicular to each other. Also $|\vec{E}_0| = \frac{|\vec{B}_0|}{\sqrt{\epsilon\mu}}$

In vacuum $\epsilon = \mu = 1 \implies |\vec{E}_0| = |\vec{B}_0|$.

$\nabla \times \vec{B} = \frac{1}{c}\epsilon\mu\frac{\partial \vec{E}}{\partial t}$ yields no additional information.

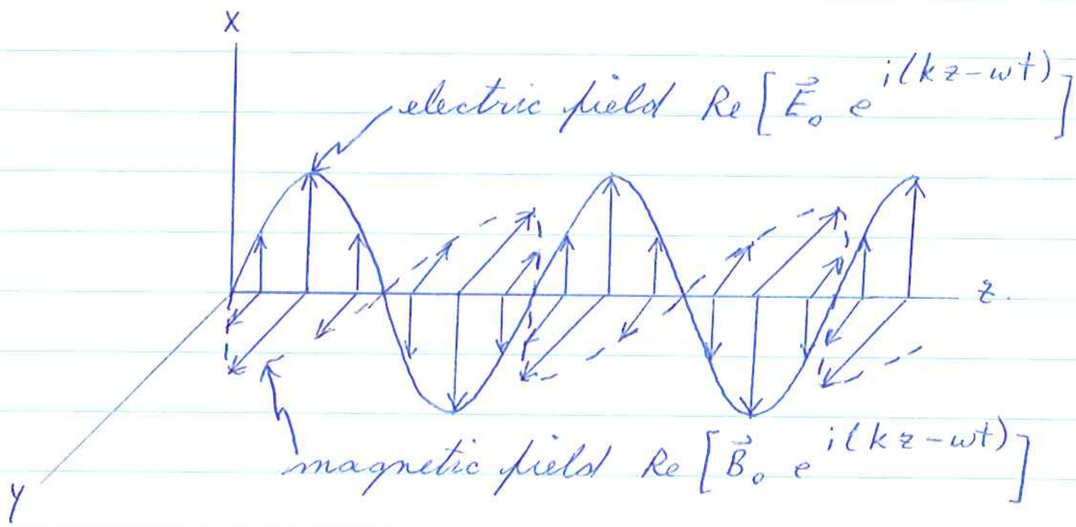
Exercise: Show this is indeed the case.

Hence there are two linearly independent plane wave solutions propagating in $\hat{k} = +\hat{z}$ direction.

1) $\vec{E}_0 \parallel \hat{x}$ & $\vec{B}_0 \parallel \hat{y}$

2) $\vec{E}_0 \parallel \hat{y}$ & $\vec{B}_0 \parallel \hat{x}$

Solution 1



Exercise: Sketch solution 2.

$\therefore$ solution of wave equation is an electromagnetic wave travelling at speed $\frac{c}{\sqrt{\epsilon\mu}}$. In vacuum $\epsilon = \mu = 1$,

speed of wave $= c =$ speed of light!

$\boxed{\text{Maxwell: Light is an electromagnetic wave!!}}$

Polarization

The wave of solution 1 is said to be linearly polarized along the $\hat{x}$ direction since $\vec{E}_0 \parallel \hat{x}$. Similarly the wave of solution 2 is said to be linearly polarized along $\hat{y}$ direction since $\vec{E}_0 \parallel \hat{y}$.

Transverse Nature of Light Wave

A light wave is similar to a wave propagating down a string. A string oscillates in 2 possible modes - up-down or side-to side. These 2 vibration directions are perpendicular or transverse to the wave propagation direction.

$\therefore$ light is called a transverse electromagnetic wave.

General Plane Wave Soln.

$$\vec{E} = E_0 \, \hat{n} \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\vec{B} = \sqrt{\epsilon\mu} \, E_0 \, (\hat{k} \times \hat{n}) \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$k = \sqrt{\epsilon\mu} \, \frac{\omega}{c}$$

This is a plane wave propagating along direction $\vec{k}$ linearly polarized in direction $\hat{n}$. For each $\vec{k}$, there are two independent polarization directions $\hat{n}$.

TABLE 8.1 The electromagnetic spectrum

Frequency (Hz)	Name of Radiation	Wavelength (m)
10^{22}		
10^{21}	Gamma rays	10^{-13}
10^{20}		10^{-12}
10^{19}		10^{-11}
10^{18}	*X* rays	10^{-10}
10^{17}		10^{-9}
10^{16}	Ultraviolet	10^{-8}
10^{15}		10^{-7}
10^{14}	Visible	10^{-6}
10^{13}	Infrared	10^{-5}
10^{12}		10^{-4}
10^{11}		10^{-3}
10^{10}	Microwave	10^{-2}
10^{9}		10^{-1}
10^{8}	TV, FM	1
10^{7}		10
10^{6}	Standard broadcast	10^{2}
10^{5}		10^{3}
10^{4}	Radiofrequency	10^{4}
10^{3}		10^{5}

Color	Wavelength (m)	Frequency (Hz)
Near ultraviolet	3.0×10^{-7}	10×10^{14}
Shortest visible blue	4.0×10^{-7}	7.5×10^{14}
Blue	4.6×10^{-7}	6.5×10^{14}
Green	5.4×10^{-7}	5.6×10^{14}
Yellow	5.9×10^{-7}	5.1×10^{14}
Orange	6.1×10^{-7}	4.9×10^{14}
Longest visible red	7.6×10^{-7}	3.9×10^{14}
Near infrared	10.0×10^{-7}	3.0×10^{14}

Poynting Vector For Complex Field Vectors

Poynting Vector $\vec{S} \equiv \frac{c}{4\pi} (\vec{E} \times \vec{H})$

$\vec{S}$ is the energy passing through a unit area per second. $\vec{S}$ was defined for real fields $\vec{E}$ + $\vec{H}$. Hence if we use complex fields, we should write:

$$\vec{S} = \frac{c}{4\pi} (Re\,\vec{E}) \times (Re\,\vec{H})$$

In the lab, $\vec{S}$ is measured at a fixed position by a detector such as a photodiode or photomultiplier. The fastest detectors respond in about 10^{-9} sec. This is much longer than the period of visible light $\sim 10^{-14}$ sec. Hence, in practice we measure the Poynting vector averaged over many wave periods which we denote by $\langle \vec{S} \rangle$.

$$\langle \vec{S} \rangle = \frac{c}{4\pi} \langle (Re\,\vec{E}) \times (Re\,\vec{H}) \rangle. \quad (1)$$

The plane wave seen by an observer at a fixed position has the form $\begin{cases} \vec{E} = \vec{E}_0 e^{-i\omega t} \\ \vec{H} = \vec{H}_0 e^{-i\omega t} \end{cases}$ where the amplitude vectors $\vec{E}_0$ + $\vec{H}_0$ are in general complex.

i.e. $\vec{E}_0 = \vec{E}_1 + i\vec{E}_2$

$\vec{H}_0 = \vec{H}_1 + i\vec{H}_2$

We shall now show that (1) is equivalent to

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \, Re \, (\vec{E} \times \vec{H}^*)$$

$$Re \, (\vec{E} \times \vec{H}^*) = Re \left\{ (\vec{E}_1 + i\vec{E}_2) e^{-i\omega t} \times (\vec{H}_1 - i\vec{H}_2) e^{i\omega t} \right\}$$

$$= Re \left\{ (\vec{E}_1 + i\vec{E}_2) \times (\vec{H}_1 - i\vec{H}_2) \right\}$$

$$= \vec{E}_1 \times \vec{H}_1 + \vec{E}_2 \times \vec{H}_2$$

$$Re \, \vec{E} = Re \left\{ (\vec{E}_1 + i\vec{E}_2) e^{-i\omega t} \right\}$$

$$= Re \left\{ (\vec{E}_1 + i\vec{E}_2)(\cos \omega t - i \sin \omega t) \right\}$$

$$= \vec{E}_1 \cos \omega t + \vec{E}_2 \sin \omega t$$

Similarly $Re \, \vec{H} = \vec{H}_1 \cos \omega t + \vec{H}_2 \sin \omega t$

$$\langle (Re \, \vec{E}) \times (Re \, \vec{H}) \rangle = \langle (\vec{E}_1 \cos \omega t + \vec{E}_2 \sin \omega t) \times (\vec{H}_1 \cos \omega t + \vec{H}_2 \sin \omega t) \rangle$$

$$= \vec{E}_1 \times \vec{H}_1 \underbrace{\langle \cos^2 \omega t \rangle}_{= \frac{1}{2}}$$

$$+ (\vec{E}_1 \times \vec{H}_2 + \vec{E}_2 \times \vec{H}_1) \underbrace{\langle \cos \omega t \sin \omega t \rangle}_{= 0}$$

$$+ \vec{E}_2 \times \vec{H}_2 \underbrace{\langle \sin^2 \omega t \rangle}_{= \frac{1}{2}}$$

$$= \frac{1}{2} \left(\vec{E}_1 \times \vec{H}_1 + \vec{E}_2 \times \vec{H}_2 \right)$$

$\therefore \langle (Re\,\vec{E}) \times (Re\,\vec{H}) \rangle = \frac{1}{2} Re\,(\vec{E} \times \vec{H}^*)$ using earlier result

$$\therefore \langle \vec{S} \rangle = \frac{c}{8\pi} Re\,(\vec{E} \times \vec{H}^*).$$

Exercise: Show for linear media, that the energy density averaged over many periods is

$$\langle u \rangle = \frac{1}{16\pi}\left(\vec{E}\cdot\vec{D}^* + \vec{B}\cdot\vec{H}^*\right)$$

where the fields are complex.

Example

Find the relation between the Poynting vector and the energy density for a plane wave travelling in vacuum. $(\epsilon = \mu = 1)$

Fields of plane wave travelling in $\hat{k}$ direction are:

$$\vec{E} = \hat{n}\, E_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\vec{B} = (\hat{k} \times \hat{n})\, E_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Magnetic field energy density $\langle u_M \rangle = \frac{1}{16\pi} \vec{B}\cdot\vec{B}^*$

$$= \frac{E_0^2}{16\pi}$$

Electric field energy density $\langle u_E \rangle = \frac{1}{16\pi} \vec{E}\cdot\vec{E}^* = \frac{E_0^2}{16\pi}$

$$\therefore \langle u_E \rangle = \langle u_M \rangle$$

$\therefore$ total field energy density $\langle U \rangle = \langle U_M \rangle + \langle U_E \rangle$

$$= \frac{E_0^2}{8\pi}$$

Poynting vector $\langle \vec{S} \rangle = \frac{c}{8\pi} \, Re \, (\vec{E} \times \vec{B}^*)$

$$= \frac{c}{8\pi} E_0^2 \left[\hat{n} \times (\hat{k} \times \hat{n}) \right]$$

$$= \frac{c}{8\pi} E_0^2 \left[\hat{k} \, (\hat{n} \cdot \hat{n}) - \hat{n} \, (\hat{n} \cdot \hat{k}) \right]$$

$$= \frac{c}{8\pi} E_0^2 \, \hat{k}$$

$$\boxed{\therefore \langle \vec{S} \rangle = c \langle U \rangle \hat{k}}$$

The right side is the energy density of the wave times the wave speed c. This is simply the energy passing through a unit area per second which is just the Poynting vector.

Waves in a Conducting Media

We shall examine wave propagation in a linear conducting material in the absence of any net free charge. Maxwell's equations then are as follows.

$$\nabla\cdot\vec{E} = 0 \quad (1) \qquad \nabla\cdot\vec{B} = 0 \quad (3)$$

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \quad (2) \qquad \nabla\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t} \quad (4)$$

where $\vec{D} = \epsilon\vec{E}$, $\vec{B} = \mu\vec{H}$, $\vec{J} = \sigma\vec{E}$ (5).

Taking the curl of (2) we get:

$$\nabla\times(\nabla\times\vec{E}) = \frac{-1}{c}\frac{\partial}{\partial t}(\nabla\times\vec{B})$$

$$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = \frac{-1}{c}\frac{\partial}{\partial t}\mu\left(\frac{4\pi}{c}\sigma\vec{E} + \frac{\epsilon}{c}\frac{\partial\vec{E}}{\partial t}\right)$$

using (4) + (5)

$$-\nabla^2\vec{E} = -\frac{4\pi\mu\sigma}{c^2}\frac{\partial\vec{E}}{\partial t} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{using (1)}$$

$$\boxed{\nabla^2\vec{E} - \frac{4\pi\mu\sigma}{c^2}\frac{\partial\vec{E}}{\partial t} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0} \quad (6)$$

Exercise: Show $\boxed{\nabla^2\vec{B} - \frac{4\pi\mu\sigma}{c^2}\frac{\partial\vec{B}}{\partial t} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0}$

Consider a plane wave propagating in the z direction.

i.e. let $\vec{E} = \vec{E}_0 \, e^{i(kz - \omega t)}$

$$\vec{B} = \vec{B}_0 \, e^{i(kz-\omega t)}$$

Substituting the above in (6) yields:

$$-k^2 \vec{E}_0 - \frac{4\pi\mu\sigma}{c^2}(-i\omega)\vec{E}_0 - \frac{\epsilon\mu}{c^2}(-\omega^2)\vec{E}_0 = 0$$

$$-k^2 + i\frac{4\pi\mu\sigma\omega}{c^2} + \epsilon\mu\frac{\omega^2}{c^2} = 0.$$

$$k^2 = \epsilon\mu\frac{\omega^2}{c^2} + i\frac{4\pi\mu\sigma\omega}{c^2}$$

$$= \frac{\mu\epsilon\omega^2}{c^2}\left\{1 + i\frac{4\pi\sigma}{\epsilon\omega}\right\}$$

$\therefore$ k is complex, i.e. $k = \alpha + i\beta$

One can show that:

$$\alpha = \frac{\omega}{c}\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2} + 1\right]^{1/2}$$

$$\beta = \frac{\omega}{c}\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2} - 1\right]^{1/2}$$

One can also write $k = |k|e^{i\phi}$ where

$|k| = \sqrt{\alpha^2+\beta^2}$ and $\tan\phi = \dfrac{\beta}{\alpha}$.

Exercise: If the media is nonconducting ($\sigma = 0$) check that the above solution agrees with the result of the preceding lecture.

Next we substitute the plane wave solution into the four Maxwell's equations.

$$\nabla \cdot \vec{E} = 0 \implies \boxed{\vec{k} \cdot \vec{E}_0 = 0}$$
$$\nabla \cdot \vec{B} = 0 \implies \boxed{\vec{k} \cdot \vec{B}_0 = 0}$$

$\therefore$ wave is transverse to the propagation direction. (i.e. $\hat{k} \perp \vec{E}_0$ & $\hat{k} \perp \vec{B}_0$)

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \implies i\,\vec{k} \times \vec{E}_0 = -\frac{1}{c}(-i\omega)\vec{B}_0$$

$$\vec{k} \times \vec{E}_0 = \frac{\omega}{c}\vec{B}_0$$

$$\boxed{\vec{B}_0 = \frac{c}{\omega}\,\vec{k} \times \vec{E}_0}$$

Exercise: Show (4) yields same as preceding eqn.

Results

$$\vec{E} = \vec{E}_0\, e^{i(kz - \omega t)}$$

$$= \vec{E}_0\, e^{i[(\alpha + i\beta)z - \omega t]}$$

$$= \vec{E}_0\, e^{-\beta z}\, e^{i(\alpha z - \omega t)}$$

$$\vec{B} = \vec{B}_0 \, e^{i(kz-\omega t)}$$

$$= \frac{c}{\omega} (\vec{k} \times \vec{E}_0) \, e^{i[(\alpha + i\beta)z - \omega t]}$$

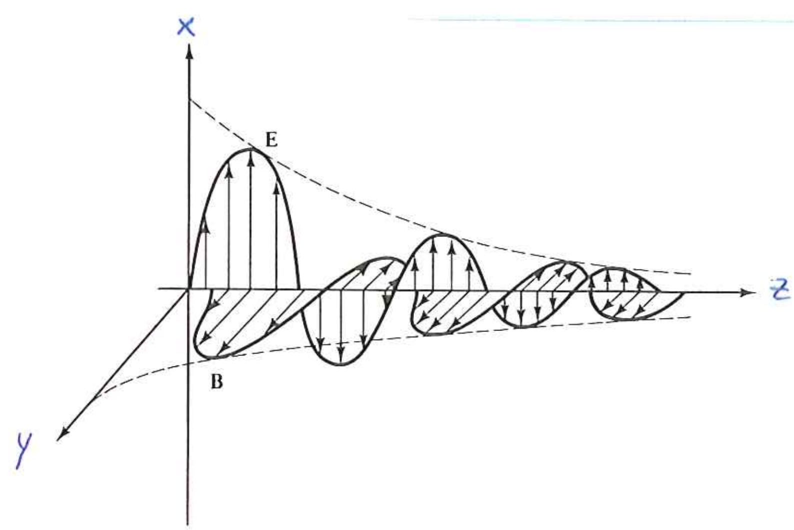
$$= \hat{k} \times \vec{E}_0 \; \frac{c}{\omega} |k| \, e^{i\phi} \, e^{-\beta z} \, e^{i(\alpha z - \omega t)}$$

Comments

1. $\vec{E}$ and $\vec{B}$ are out of phase by amount ϕ.

2. $|\vec{E}| \neq |\vec{B}|$

3. $\vec{E}$ and $\vec{B}$ get smaller due to $e^{-\beta z}$ as we get farther into the material, as shown below.



Good Conductor Limit

By good conductor we mean $\frac{4\pi\sigma}{\omega\epsilon} \gg 1$

This is true for most metals at frequencies below 10^{17} Hz. This is saying that the conduction current term in Maxwell's eqn. $\frac{4\pi}{c} J = \frac{4\pi}{c}\sigma E$ is much larger than the displacement current $\left|\frac{1}{c}\epsilon\frac{\partial \vec{E}}{\partial t}\right| = \frac{\epsilon\omega}{c}E$.

In this limit, α and β reduce to the following expressions.

$$\beta = \alpha = \frac{\omega}{c}\sqrt{\frac{\mu\epsilon}{2}}\sqrt{\frac{4\pi\sigma}{\omega\epsilon}}$$

$$= \frac{1}{c}\sqrt{2\pi\,\omega\mu\sigma}$$

$$|k| = \sqrt{\alpha^2+\beta^2}$$

$$= \frac{1}{c}\sqrt{4\pi\,\omega\mu\sigma}$$

$$\tan\phi = \frac{\beta}{\alpha} = 1$$

$$\therefore \phi = 45^\circ$$

Comments

1. $\vec{E}$ and $\vec{B}$ are out of phase by $\phi = 45^\circ$.

2. $|\vec{B}_0| \gg |\vec{E}_0|$

To see this let's start with equation resulting from Faradays law.

$$\vec{B}_0 = \frac{c}{\omega} \vec{k} \times \vec{E}_0$$

$$\therefore |\vec{B}_0| = \frac{c}{\omega} |k| \, |\vec{E}_0|$$

$$= \frac{c}{\omega} \frac{1}{c} \sqrt{4\pi \omega \mu \sigma} \, |\vec{E}_0|$$

$$= \sqrt{\frac{4\pi\sigma}{\omega \epsilon}} \sqrt{\mu \epsilon} \, |\vec{E}_0|$$

$$\gg |\vec{E}_0| \quad \text{since} \quad \frac{4\pi\sigma}{\omega\epsilon} \gg 1$$

$\therefore$ most of the wave energy is stored by the magnetic field.

3. Fields are given by the following.

$$\vec{E} = \vec{E}_0 \, e^{-\beta z} \, e^{i(\beta z - \omega t)}$$

$$\vec{B} = \hat{k} \times \vec{E}_0 \frac{c}{\omega} |k| \, e^{i\pi/4} \, e^{-\beta z} \, e^{i(\beta z - \omega t)}$$

Therefore $\vec{E}$ and $\vec{B}$ get smaller due to $e^{-\beta z}$ as we get farther into material. The amplitude is reduced by a factor $\frac{1}{e}$ in distance

$$\delta \equiv \frac{1}{\beta}$$

$$\boxed{\delta = \frac{c}{\sqrt{2\pi \sigma \mu \omega}}}$$

δ is called the skin depth.

At optical frequencies, in metals, δ is very small. This is of course what we expect since metals such as Ag, Au, Cu, Fe, Al etc. are opaque. Only at very low frequencies does δ get large as shown below for Cu.

$\nu = \omega/2\pi$	δ (mm)
10^3 Hz.	2.1
10^4	0.66
10^5	0.21
10^6	0.066

Reflection and Refraction

We shall study the behaviour of EM waves propagating from one medium to another. The fields obey the following conditions at the boundary.

1) Tangential component of $\vec{E}$ is continuous.

2) Normal " " $\vec{D}$ " "

3) Tangential " " $\vec{H}$ " "

4) Normal " " $\vec{B}$ " "

We shall consider materials that are:

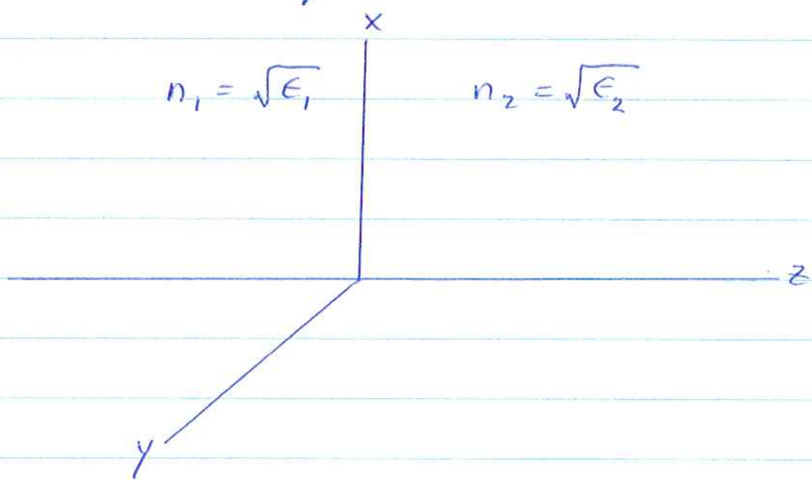
1) linear dielectrics

2) nonmagnetic (i.e. $\mu = 1 \Rightarrow \vec{B} = \vec{H}$)

3) nonconducting ($\sigma = 0$)

Reflection and Refraction at Normal Incidence

We shall first consider a plane wave propagating in the $+\hat{z}$ direction that is normally incident on a planar boundary.



The x axis is chosen to lie along the electric field of the incident wave.

Incident Wave

$$\vec{E}_0 = \hat{x}\, E_0\, e^{i(k_1 z - \omega t)}$$

$$\vec{B}_0 = (\hat{z} \times \hat{x})\, \sqrt{\epsilon_1}\, E_0\, e^{i(k_1 z - \omega t)}$$

$$= \hat{y}\, n_1 E_0\, e^{i(k_1 z - \omega t)}$$

where $k_1 = \sqrt{\epsilon_1}\, \frac{\omega}{c} = n_1 \frac{\omega}{c}$

At the boundary, part of the wave is reflected and the rest transmitted.

Let the reflected wave fields $\vec{E}_1$ & $\vec{B}_1$ be given by:

$$\vec{E}_1 = -\hat{x}\ E_1\ e^{-i(k_1 z + \omega t)}$$

$$\vec{B}_1 = (-\hat{z}) \times (-\hat{x})\ \sqrt{\epsilon_1}\ E_1\ e^{-i(k_1 z + \omega t)}$$

$$= \hat{y}\ n_1\ E_1\ e^{-i(k_1 z + \omega t)}$$

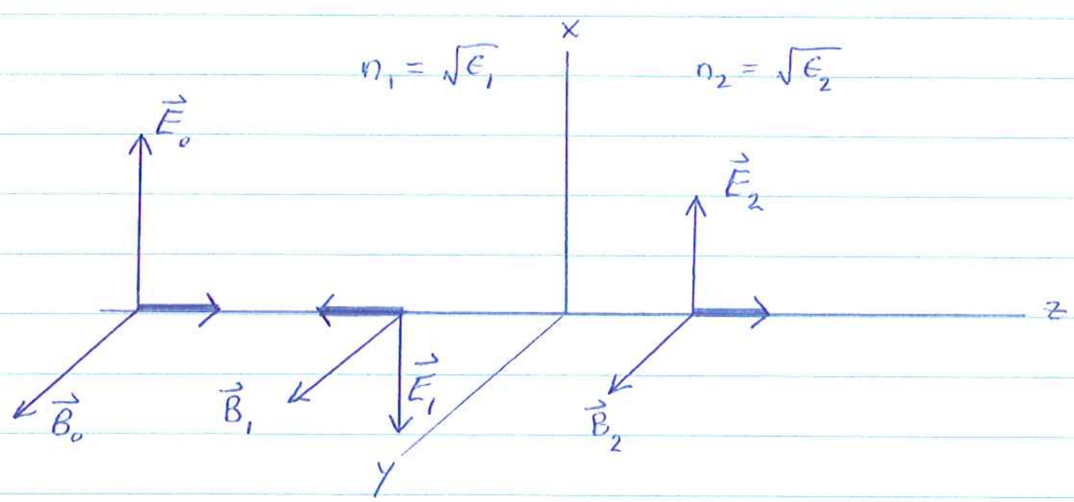
Similarly the transmitted wave is given by:

$$\vec{E}_2 = \hat{x}\ E_2\ e^{i(k_2 z - \omega t)}$$

$$\vec{B}_2 = \hat{y}\ n_2\ E_2\ e^{i(k_2 z - \omega t)}$$

where $k_2 = n_2 \frac{\omega}{c}$.

We have assigned the three waves the same frequency $\nu = \frac{\omega}{2\pi}$ since we observe that reflected and refracted light has the same frequency (i.e. same colour) as the incident light.



The reflected and transmitted field amplitudes $\vec{E}_1$ + $\vec{E}_2$ are found using the boundary conditions.

1) tangential component of $\vec{E}$ is continuous yields:

$$E_0 - E_1 = E_2 \qquad (1)$$

2) tangential component of $\vec{H}$ is continuous ($\vec{B} = \vec{H}$ since $\mu = 1$)

$$B_0 + B_1 = B_2$$

$$n_1 E_0 + n_1 E_1 = n_2 E_2 \qquad (2)$$

The other two boundary conditions are obeyed trivially since $\vec{B}_\perp = \vec{D}_\perp = 0$.

Equations (1) + (2) can be solved giving:

$$\boxed{\begin{aligned} E_1 &= \frac{n_2 - n_1}{n_2 + n_1} E_0 \\ E_2 &= \frac{2 n_1}{n_2 + n_1} E_0 \end{aligned}}$$

If $n_2 > n_1$, E_1 + E_0 are either both positive or both negative. They are said to be in phase with each other. If $n_2 < n_1$, E_1 + E_0 have opposite signs and are said to be out of phase.

Reflection & Transmission Coefficients

The reflection and transmission coefficients are found using a detector to measure the energies of the various beams. We therefore evaluate the time averaged Poynting vector

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \operatorname{Re} (\vec{E} \times \vec{H}^*)$$

Poynting Vector of Incident Wave

$$\langle \vec{S}_0 \rangle = \frac{c}{8\pi} \operatorname{Re} \left\{ \hat{x} E_0 e^{i(k_1 z - \omega t)} \times \hat{y}\, n_1 E_0 e^{-i(k_1 z - \omega t)} \right\}$$

$$= \hat{z} \frac{c}{8\pi} n_1 E_0^2$$

Poynting Vector of Reflected Wave

$$\langle \vec{S}_1 \rangle = -\hat{z} \frac{c}{8\pi} n_1 E_1^2$$

Poynting Vector of Transmitted Wave

$$\langle \vec{S}_2 \rangle = \hat{z} \frac{c}{8\pi} n_2 E_2^2$$

Reflection Coefficient $R \equiv \dfrac{\text{power of reflected beam/cm}^2}{\text{power of incident beam/cm}^2}$

$$= \frac{\langle \vec{S}_1 \rangle \cdot (-\hat{z})}{\langle \vec{S}_0 \rangle \cdot (\hat{z})}$$

$$R = \frac{E_1^2}{E_0^2}$$

$$\boxed{R = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2}$$

Transmission Coefficient $T \equiv \dfrac{\text{power of transmitted beam/cm}^2}{\text{power of incident beam/cm}^2}$

$$= \frac{\langle \vec{S}_2 \rangle \cdot \hat{z}}{\langle \vec{S}_0 \rangle \cdot \hat{z}}$$

$$= \frac{n_2 E_2^2}{n_1 E_0^2}$$

$$= \frac{n_2}{n_1}\left(\frac{2 n_1}{n_2 + n_1}\right)^2$$

$$\boxed{T = \frac{4 n_1 n_2}{(n_2 + n_1)^2}}$$

Exercise: Check that $R + T = 1$.

$R + T = 1$ is a statement of energy conservation. The sum of the reflected and transmitted beam energies equals the incident beam energy.

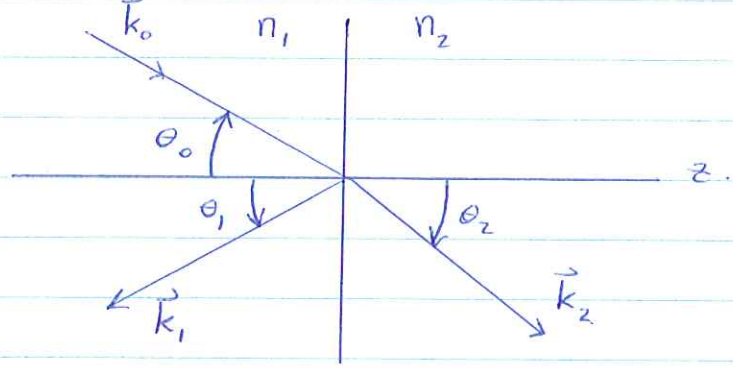
Example: Consider light passing from air into glass.

$$n_1 = n_{air} = 1 \qquad n_2 = n_{glass} = 1.5$$

reflectivity $R = \left(\dfrac{1.5 - 1}{1.5 + 1}\right)^2 = .04$ transmission $T = \dfrac{4 \cdot 1 \cdot (1.5)}{(1.5+1)^2} = .96$

Reflection and Refraction At Oblique Incidence

We shall now consider a plane wave incident at some angle θ_0 to a planar boundary.



$\vec{k}_0$ = wavevector of incident plane wave
$\vec{k}_1$ = " " reflected " "
$\vec{k}_2$ = " " transmitted " "

Incident Plane Wave

$$\vec{E}_0 = \vec{E}_{00}\, e^{i(\vec{k}_0 \cdot \vec{r} - \omega t)}$$

$$\vec{B}_0 = \sqrt{\epsilon_1}\; \hat{k}_0 \times \vec{E}_0$$

$$= \frac{n_1}{k_0}\; \vec{k}_0 \times \vec{E}_{00}\, e^{i(\vec{k}_0 \cdot \vec{r} - \omega t)}$$

where $\vec{E}_{00}$ is constant vector amplitude (independent of t + $\vec{r}$)

$$k_0 = n_1 \frac{\omega}{c}$$

Reflected Plane Wave

$$\vec{E}_1 = \vec{E}_{10}\, e^{i(\vec{k}_1 \cdot \vec{r} - \omega t)}$$

$$\vec{B}_1 = \frac{n_1}{k_1}\, \vec{k}_1 \times \vec{E}_{10}\, e^{i(\vec{k}_1 \cdot \vec{r} - \omega t)}$$

where $k_1 = n_1 \frac{\omega}{c} = k_0$

Transmitted Plane Wave

$$\vec{E}_2 = \vec{E}_{20}\, e^{i(\vec{k}_2 \cdot \vec{r} - \omega t)}$$

$$\vec{B}_2 = \frac{n_2}{k_2}\, \vec{k}_2 \times \vec{E}_{20}\, e^{i(\vec{k}_2 \cdot \vec{r} - \omega t)}$$

where $k_2 = n_2 \frac{\omega}{c}$.

To find $\vec{E}_{10}$, $\vec{E}_{20}$, $\vec{k}_1$, $\vec{k}_2$ we shall need the field boundary conditions.

Boundary Conditions

When using the boundary conditions for the field components $\perp$ & $\parallel$ to the boundary, we get expressions of the following form.

$$(\quad)\, e^{i(\vec{k}_0 \cdot \vec{r} - \omega t)} + (\quad)\, e^{i(\vec{k}_1 \cdot \vec{r} - \omega t)} = (\quad)\, e^{i(\vec{k}_2 \cdot \vec{r} - \omega t)}$$

The boundary conditions must be obeyed at all points on the boundary. i.e. at all $x\,y$ on $z = 0$ plane

This can only occur if:

$$\vec{k}_0 \cdot \vec{r} = \vec{k}_1 \cdot \vec{r} = \vec{k}_2 \cdot \vec{r} \quad \text{where } \vec{r} \text{ is pt. on } z=0 \text{ plane}$$
$$\text{i.e. } \vec{r} = (x, y, 0)$$

or $\quad k_{0x}x + k_{0y}y = k_{1x}x + k_{1y}y = k_{2x}x + k_{2y}y \quad \forall x, y$

$$\Rightarrow \begin{cases} k_{0x} = k_{1x} = k_{2x} \\ k_{0y} = k_{1y} = k_{2y} \end{cases}$$

We can choose the x & z axes so that $\vec{k}_0$ lies in the x-z plane.

$$\text{i.e. } \vec{k}_0 = (k_{0x}, 0, k_{0z})$$

Then from the preceding 2 equations we have that:

$$k_{1y} = k_{2y} = 0$$

$$\Rightarrow \begin{cases} \vec{k}_1 = (k_{1x}, 0, k_{1z}) \\ \vec{k}_2 = (k_{2x}, 0, k_{2z}) \end{cases}$$

$\therefore$ wavevectors of incident, reflected and transmitted waves all lie in the same plane called the incident plane.

Aside: One in general could assign different frequencies ω_0 ω_1 ω_2 to the 3 waves. But the boundary conditions must hold at all times. This implies $\omega_0 = \omega_1 = \omega_2$!

Next consider the x component of the incident and reflected wavevectors.

$$k_{0x} = k_{1x}$$

$$k_0 \sin\theta_0 = k_1 \sin\theta_1$$

But $k_0 = k_1 = n_1 \frac{\omega}{c} \Rightarrow \sin\theta_0 = \sin\theta_1$

$$\therefore \theta_0 = \theta_1$$

Law of Reflection: Angle of incidence equals angle of reflection.

Next consider the x component of the incident and transmitted wavevectors.

$$k_{0x} = k_{2x}$$

$$k_0 \sin\theta_0 = k_2 \sin\theta_2$$

$$n_1 \frac{\omega}{c} \sin\theta_0 = n_2 \frac{\omega}{c} \sin\theta_2$$

$$\boxed{n_1 \sin\theta_0 = n_2 \sin\theta_2}$$

This is Snell's Law which was empirically found in 1621.

Actual Application of Boundary Conditions

1) Perpendicular component of $\vec{D} = \epsilon \vec{E}$ is continuous.

$$\boxed{\epsilon_1 \left[\vec{E}_{00} + \vec{E}_{10} \right]_z = \epsilon_2 \left[\vec{E}_{20} \right]_z \qquad (1)}$$

2) Tangential component of $\vec{E}$ is continuous.

$$\boxed{\left[\vec{E}_{00} + \vec{E}_{10} \right]_{x,y} = \left[\vec{E}_{20} \right]_{x,y} \qquad (2)}$$

3) Perpendicular component of $\vec{B}$ is continuous.

$$n_1 \left[\hat{k}_0 \times \vec{E}_{00} \right]_z + n_1 \left[\hat{k}_1 \times \vec{E}_{10} \right]_z = n_2 \left[\hat{k}_2 \times \vec{E}_{20} \right]_z$$

4) Tangential component of $\vec{H}$ is continuous.
$\vec{B} = \vec{H}$ since $\mu = 1$.

$$n_1 \left[\hat{k}_0 \times \vec{E}_{00} \right]_{x,y} + n_1 \left[\hat{k}_1 \times \vec{E}_{10} \right]_{x,y} = n_2 \left[\hat{k}_2 \times \vec{E}_{20} \right]_{x,y}$$

These last 2 eqns. can be written as follows.

$$\boxed{n_1 (\hat{k}_0 \times \vec{E}_{00}) + n_1 (\hat{k}_1 \times \vec{E}_{10}) = n_2 \, \hat{k}_2 \times \vec{E}_{20} \qquad (3)}$$

Wavevectors

$$\vec{k}_0 = k_0(-\sin\theta_0, 0, \cos\theta_0)$$

$$\vec{k}_1 = k_1(-\sin\theta_1, 0, -\cos\theta_1)$$
$$= k_0(-\sin\theta_0, 0, -\cos\theta_0)$$

$$\vec{k}_2 = k_2(-\sin\theta_2, 0, \cos\theta_2)$$

Case 1

$\vec{E}_{00} = (0, E_{00}, 0)$
$\vec{E}_{00}$ lies perpendicular to the plane of incidence. Such a wave is called s polarized. Its reflection and transmission coefficients will be found as a HW problem.

Case 2

$\vec{E}_{00} = E_{00}(\cos\theta_0, 0, \sin\theta_0)$
$\vec{E}_{00}$ lies parallel to the plane of incidense. Such a wave is called p polarized.

We first use the wavevectors and field amplitudes to expand (3).

$$\hat{k}_0 \times \vec{E}_{00} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\theta_0 & 0 & \cos\theta_0 \\ E_{00}\cos\theta_0 & 0 & E_{00}\sin\theta_0 \end{vmatrix}$$

$$= (0, +E_{00}, 0).$$

$$\hat{k}_1 \times \vec{E}_{10} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta_0 & 0 & -\cos\theta_0 \\ E_{10x} & E_{10y} & E_{10z} \end{vmatrix}$$

$$= (E_{10y}\cos\theta_0,\ E_{10z}\sin\theta_0 - E_{10x}\cos\theta_0,\ -E_{10y}\sin\theta_0)$$

$$\hat{k}_2 \times \vec{E}_{20} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta_2 & 0 & \cos\theta_2 \\ E_{20x} & E_{20y} & E_{20z} \end{vmatrix}$$

$$= (-E_{20y}\cos\theta_2,\ E_{20z}\sin\theta_2 + E_{20x}\cos\theta_2,\ -E_{20y}\sin\theta_2)$$

$$\therefore (3) \Rightarrow \quad n_1(0,\ +E_{00},\ 0) + n_1(E_{10y}\cos\theta_0,\ E_{10z}\sin\theta_0 - E_{10x}\cos\theta_0,\ -E_{10y}\sin\theta_0)$$

$$= n_2(-E_{20y}\cos\theta_2,\ E_{20z}\sin\theta_2 + E_{20x}\cos\theta_2,\ -E_{20y}\sin\theta_2)$$

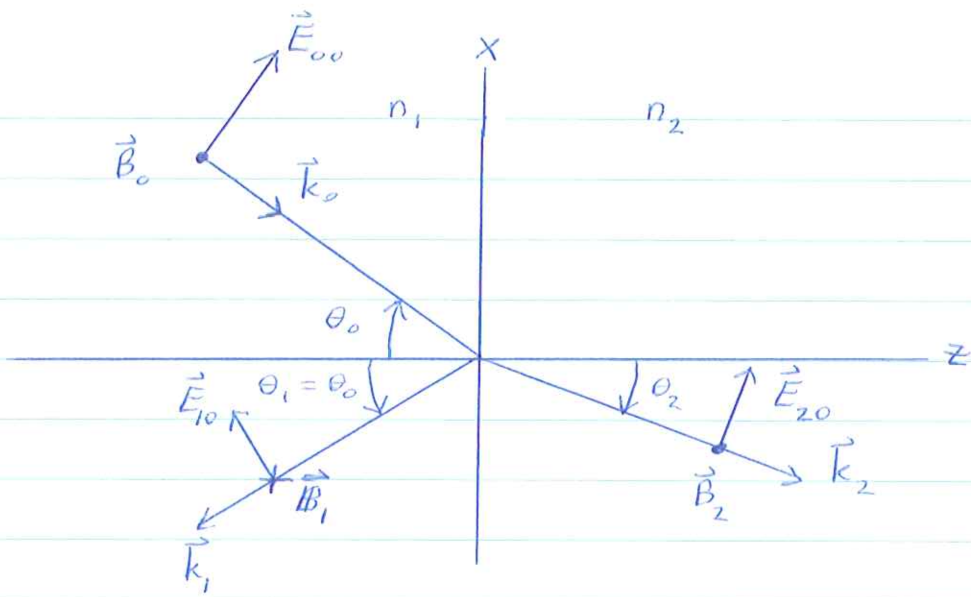
x component $\Rightarrow \quad n_1 \cos\theta_0\, E_{10y} = -n_2 \cos\theta_2\, E_{20y}$

z component $\Rightarrow \quad -n_1 \sin\theta_0\, E_{10y} = -n_2 \sin\theta_2\, E_{20y}$

$E_{10y} = E_{20y}$ using Snell's law

The only way for the x + z component equations to simultaneously hold is for $E_{10y} = E_{20y} = 0$.

$\therefore$ electric fields of incident, reflected and transmitted waves are all p polarized.



• $\vec{B}$ is out of page
x $\vec{B}$ is into page.

$$\vec{E}_{00} = E_{00}(\cos\theta_0, 0, \sin\theta_0)$$

$$\vec{E}_{10} = E_{10}(\cos\theta_0, 0, -\sin\theta_0)$$

$$\vec{E}_{20} = E_{20}(\cos\theta_2, 0, \sin\theta_2).$$

Boundary Condition (1) gives:

$$\epsilon_1 \left[E_{00} \sin\theta_0 - E_{10} \sin\theta_0 \right] = \epsilon_2 E_{20} \sin\theta_2$$

$$n_1^2 \sin\theta_0 \, (E_{00} - E_{10}) = n_2^2 \sin\theta_2 \, E_{20}.$$

$$(E_{00} - E_{10}) = \frac{n_2}{n_1} E_{20} \quad \text{using Snell's law}$$

$$E_{00} - E_{10} = \beta E_{20} \qquad \text{where } \beta \equiv \frac{n_2}{n_1} \qquad (4)$$

Boundary Condition (2) gives:

$$E_{00}\cos\theta_0 + E_{10}\cos\theta_0 = E_{20}\cos\theta_2$$

$$E_{00} + E_{10} = \alpha E_{20} \quad \text{where } \alpha \equiv \frac{\cos\theta_2}{\cos\theta_0}. \qquad (5)$$

(4) + (5) can be solved yielding the following.

$$\boxed{\begin{aligned} E_{10} &= \frac{\alpha - \beta}{\alpha + \beta} E_{00} \\ E_{20} &= \frac{2}{\alpha + \beta} E_{00} \end{aligned}}$$

One can show the above are consistent with the y component of eqn. (3).

To find the reflection and transmission coefficients we need the Poynting vector. $\langle \vec{S} \rangle = \frac{c}{8\pi} \mathrm{Re}(\vec{E}\times\vec{H}^*)$

Poynting Vector of Incident Wave

$$\langle \vec{S}_0 \rangle = \frac{c}{8\pi} n_1 E_{00}^2 \hat{k}_0$$

Poynting Vector of Reflected Wave

$$\langle \vec{S}_1 \rangle = \frac{c}{8\pi} n_1 E_{10}^2 \hat{k}_1.$$

Poynting Vector of Transmitted Wave

$$\langle \vec{S}_2 \rangle = \frac{c}{8\pi} n_2 E_{20}^2 \hat{k}_2$$

The power per unit area striking the boundary is $\vec{S} \cdot \hat{n}$ where $\hat{n}$ is the unit vector normal to the interface.

$\therefore$ incident intensity $I_0 = \langle \vec{S}_0 \rangle \cdot \hat{z}$

$$= \frac{c}{8\pi} n_1 E_{00}^2 \cos\theta_0$$

reflected intensity $I_1 = \langle \vec{S}_1 \rangle \cdot (-\hat{z})$

$$= \frac{c}{8\pi} n_1 E_{10}^2 \cos\theta_1$$

transmitted intensity $I_2 = \langle \vec{S}_2 \rangle \cdot \hat{z}$

$$= \frac{c}{8\pi} n_2 E_{20}^2 \cos\theta_2$$

Reflection Coefficient $R = \dfrac{I_1}{I_0}$

$$= \frac{E_{10}^2}{E_{00}^2}$$

$$\boxed{R = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2}$$

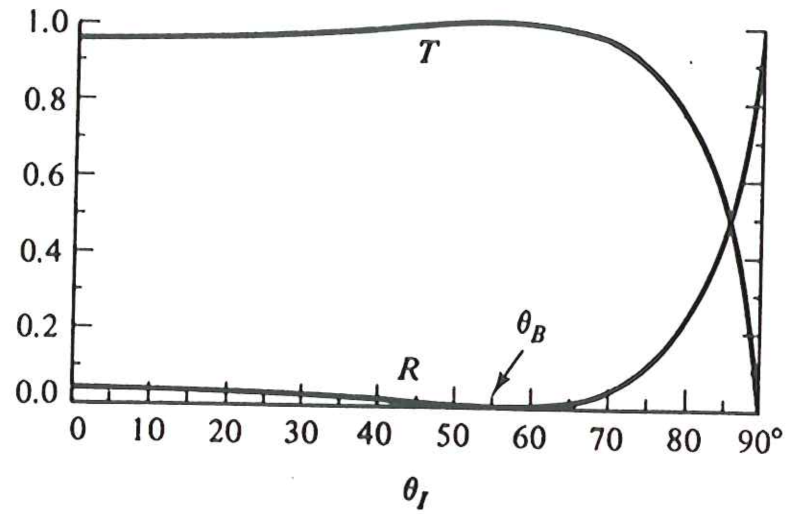
Transmission Coefficient $T = \dfrac{I_2}{I_0}$

$$= \frac{E_{20}^2 \, n_2 \cos\theta_2}{E_{00}^2 \, n_1 \cos\theta_1}$$

$$\boxed{T = \frac{4\alpha\beta}{(\alpha + \beta)^2}} \qquad \beta = \frac{n_2}{n_1}, \quad \alpha = \frac{\cos\theta_2}{\cos\theta_0}$$

Exercise: Check that $R + T = 1$

i.e. power reaching a given area on boundary equals power leaving area

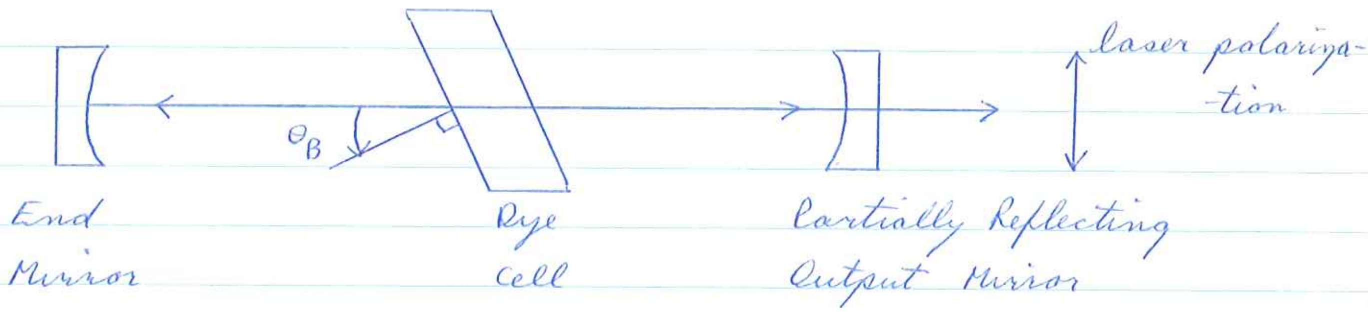


Brewsters Angle

The reflection coefficient is zero when $\alpha = \beta$. The angle of incidence then is called Brewsters angle θ_B.

Exercise: Show $\tan \theta_B = \frac{n_2}{n_1}$

Therefore a p polarized wave incident at Brewster's angle is completely transmitted. This is useful for designing lasers where losses are to be minimized.



Spontaneously emitted light from the dye molecules is unpolarized. Unpolarized light consists of equal amounts of s and p polarized light. The p polarized fraction is completely transmitted by the dye cell while the s polarized light is partly transmitted and partly reflected. Hence lasing threshold power is first reached by the p polarized wave, and the laser is polarized along the vertical direction.

Example

$n_1 = n_{air} = 1$

$n_2 = n_{glass} = 1.5$

$\tan\theta_B = 1.5 \implies \theta_B = 56.3^\circ$

Assignment 3

1. Consider the wave $\psi = 5\cos(10y + 6t)$. Distances are measured in cm. and time in seconds. Find the following.
   1) amplitude
   2) wavevector $|\vec{k}|$
   3) wavelength λ
   4) angular frequency ω
   5) frequency ν
   6) period T
   7) speed of wave
   8) propagation direction $\hat{k}$

2. Plane Waves in Nonconducting Media

   a) Derive $\nabla^2 \vec{B} - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0$ from Maxwell eqns.

   b) Show $\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ is a solution.

   c) What is the relation between $|\vec{E}|$ + $|\vec{B}|$?
   What is relation between direction of $\vec{E}$, $\vec{B}$ + $\vec{k}$?

3. For complex fields show that the energy density averaged over many periods can be written as:

$$\langle u \rangle = \frac{1}{16\pi}\left(\vec{E}\cdot\vec{D}^* + \vec{B}\cdot\vec{H}^*\right)$$

4. Plane Waves in Conducting Medium

Consider the case when the conducting current is much smaller than the displacement current.
i.e. $\frac{4\pi\sigma}{\omega\epsilon} << 1$

Show $\alpha \simeq \frac{\omega}{c}\sqrt{\mu\epsilon}\left[1 + \frac{1}{2}\left(\frac{2\pi\sigma}{\omega\epsilon}\right)^2\right]$

$$\beta \simeq \frac{2\pi\sigma}{c}\sqrt{\frac{\mu}{\epsilon}}$$

The attenuation factor β is then independent of frequency.

5. For copper $\sigma \simeq 5 \times 10^{17}$ ~~cm~~ sec^{-1}
Assuming $\mu = 1$, complete the following table.

Radiation	Frequency	δ - skin depth
UV		
visible		
infrared		
microwave		
TV, FM		
radiofrequency		

6) When going from a more dense to less dense medium (i.e. $n_1 > n_2$) a light beam bends away from the normal. At a so called critical angle θ_c of incidence, the refracted angle is 90°. If $\theta_I > \theta_c$

the wave is totally reflected. Evaluate θ_c for a beam reflecting at a glass ($n=1.5$) air interface.

7. Show that for a s polarized wave, the reflection and transmission coefficients are given by:

$$R = \left(\frac{1-\beta\alpha}{1+\beta\alpha}\right)^2 \qquad \text{where } \beta = \frac{n_2}{n_1}$$

$$\alpha = \frac{\cos\theta_2}{\cos\theta_0}$$

$$T = \frac{4\alpha\beta}{(1+\beta\alpha)^2}$$

8. Explain how polaroid sunglasses cut down on glare. (Hint: Think of Brewster's angle)

9. Circular Polarization
Instead of taking $\hat{x}$ + $\hat{y}$ as the 2 orthogonal polarization vectors for a wave propagating in the $\hat{z}$ direction, one can use the following:

$$\vec{\epsilon}_\pm = \hat{x}\cos\omega t \pm \hat{y}\sin\omega t$$

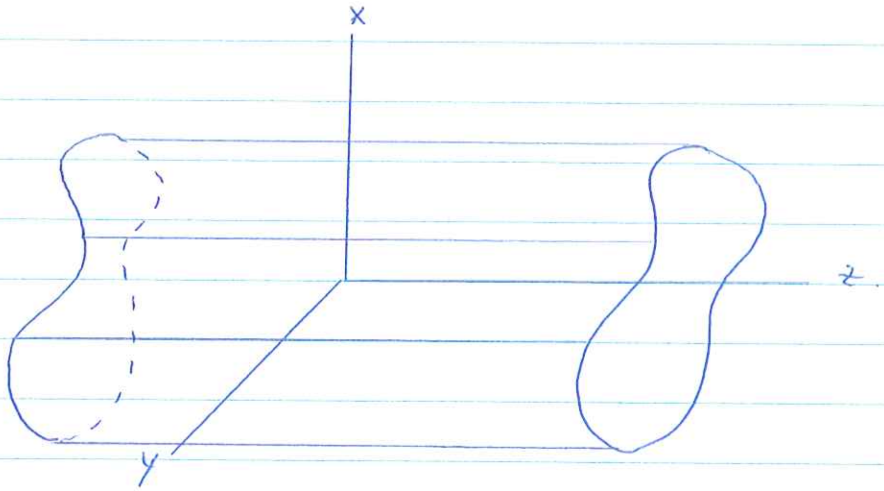
a) Show $\langle \vec{\epsilon}_a \cdot \vec{\epsilon}_b \rangle = \delta_{ab}$. $\quad a, b = \pm$

b) Show $\vec{\epsilon}_+$ is a vector rotating counterclockwise in xy plane. Which direction does $\vec{\epsilon}_-$ rotate?

c) Suppose $\vec{E} = \hat{x}E_0\cos\omega t$. What fraction of $\vec{E}$ is polarized "along $\vec{\epsilon}_+$"?

IV Waveguides

We shall investigate the propagation of waves in a hollow conducting pipe having a uniform cross section.



Hollow means there is a vacuum inside the pipe. Hence, inside the pipe the following wave equation must be solved for the fields $\vec{E}$ & $\vec{B}$.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0 \qquad (1)$$

We seek solutions of the form:

$$\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = \begin{pmatrix}\vec{E}_0(x,y)\\ \vec{B}_0(x,y)\end{pmatrix} e^{i(kz-\omega t)} \qquad (2)$$

The relation between ω & k is found by substituting (2) in (1) and making use of the geometry of a particular problem.

The fields also satisfy Maxwell's equations. (from which (1) is derived)

$$\nabla\cdot\vec{E}=0 \Longrightarrow \frac{\partial E_{ox}}{\partial x}+\frac{\partial E_{oy}}{\partial y}+ikE_{oz}=0 \qquad (3)$$

$$\nabla\cdot\vec{B}=0 \Longrightarrow \frac{\partial B_{ox}}{\partial x}+\frac{\partial B_{oy}}{\partial y}+ikB_{oz}=0 \qquad (4)$$

$$\underline{\nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}}$$

$$\nabla\times\vec{E}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix}$$

$$=\left(\frac{\partial E_{oz}}{\partial y}-ikE_{oy},\ -\frac{\partial E_{oz}}{\partial x}+ikE_{ox},\ \frac{\partial E_{oy}}{\partial x}-\frac{\partial E_{ox}}{\partial y}\right)e^{i(kz-\omega t)}$$

$$-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=i\frac{\omega}{c}\left(B_{ox},B_{oy},B_{oz}\right)e^{i(kz-\omega t)}$$

$$\therefore\quad \frac{\partial E_{oz}}{\partial y}-ikE_{oy}=i\frac{\omega}{c}B_{ox} \qquad (5a)$$

$$-\frac{\partial E_{oz}}{\partial x}+ikE_{ox}=i\frac{\omega}{c}B_{oy} \qquad (5b)$$

$$\frac{\partial E_{oy}}{\partial x}-\frac{\partial E_{ox}}{\partial y}=i\frac{\omega}{c}B_{oz} \qquad (5c)$$

Similarly $\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ implies:

$$\frac{\partial B_{oz}}{\partial y} - ikB_{oy} = -i\frac{\omega}{c}E_{ox} \qquad (6a)$$

$$-\frac{\partial B_{oz}}{\partial x} + ikB_{ox} = -i\frac{\omega}{c}E_{oy} \qquad (6b)$$

$$\frac{\partial B_{oy}}{\partial x} - \frac{\partial B_{ox}}{\partial y} = -i\frac{\omega}{c}E_{oz} \qquad (6c)$$

<u>Exercise:</u> Derive (6).

We next solve for E_{ox} using (5b) + (6a).

k (5b) $- \frac{\omega}{c}$ (6a) gives:

$$-k\frac{\partial E_{oz}}{\partial x} + ik^2 E_{ox} - \frac{\omega}{c}\frac{\partial B_{oz}}{\partial y} + ik\frac{\omega}{c}B_{oy} = i\frac{\omega}{c}kB_{oy} + i\left(\frac{\omega}{c}\right)^2 E_{ox}$$

$$iE_{ox}\left(-k^2 + \left(\frac{\omega}{c}\right)^2\right) = -k\frac{\partial E_{oz}}{\partial x} - \frac{\omega}{c}\frac{\partial B_{oz}}{\partial y}$$

$$\boxed{E_{ox} = \frac{i}{k_c^2}\left\{k\frac{\partial E_{oz}}{\partial x} + \frac{\omega}{c}\frac{\partial B_{oz}}{\partial y}\right\} \qquad (7a)}$$

where $k_c^2 \equiv \left(\frac{\omega}{c}\right)^2 - k^2$

Similarly we can use (5) & (6) to get the following.

$$\boxed{\begin{aligned} E_{0y} &= \frac{i}{k_c^2}\left\{ k\,\frac{\partial E_{0z}}{\partial y} - \frac{\omega}{c}\,\frac{\partial B_{0z}}{\partial x} \right\} && (7b) \\ B_{0x} &= \frac{-i}{k_c^2}\left\{ \frac{\omega}{c}\,\frac{\partial E_{0z}}{\partial y} - k\,\frac{\partial B_{0z}}{\partial x} \right\} && (7c) \\ B_{0y} &= \frac{i}{k_c^2}\left\{ \frac{\omega}{c}\,\frac{\partial E_{0z}}{\partial x} + k\,\frac{\partial B_{0z}}{\partial y} \right\} && (7d) \end{aligned}}$$

Exercise: Prove (7b - d).

(7) shows that the transverse field components are specified entirely by the longitudinal field components E_{0z} & B_{0z}. If $E_{0z} = B_{0z} = 0$ then $\vec{E}_0 = \vec{B}_0 = 0$ everywhere.

$\therefore$ TEM waves cannot propagate in a hollow conducting pipe.

TE Waves

These are waves where the electric field only has a transverse component. i.e. $E_{0z} = 0$.

TM Waves

These are waves where the magnetic field only has a transverse component. i.e. $B_{0z} = 0$.

From (7) we find that for a TE mode, B_{oz} completely specifies the wave while for a TM mode, E_{oz} completely specifies the wave.

Determination of $\left\{ \begin{matrix} E_{oz} \\ B_{oz} \end{matrix} \right\}$

Substituting (2) in (1) we get:

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\begin{pmatrix} \vec{E}_o \\ \vec{B}_o \end{pmatrix} + \left(\left(\frac{\omega}{c}\right)^2 - k^2\right)\begin{pmatrix} \vec{E}_o \\ \vec{B}_o \end{pmatrix} = 0 \qquad (8)}$$

This is called the Helmholtz equation. To solve it we need boundary conditions.

Boundary Conditions on E_{oz} & B_{oz}.

In a perfect conductor there can be no electric fields tangent to the surface or magnetic fields perpendicular to the surface S.

i.e. $\vec{E}_{tang.}\Big|_S = 0 \implies \boxed{E_{oz}\Big|_S = 0}$ (9a).

and $\vec{B}_{normal}\Big|_S = 0 \implies \hat{n}\cdot\vec{B}_o\Big|_S = 0$ $\hat{n}$ = unit vector normal to S

$$\hat{n}\cdot\left(B_{ox}\hat{x} + B_{oy}\hat{y} + B_{oz}\hat{z}\right)\Big|_S = 0$$

$$\hat{n}\cdot\left(B_{ox}\hat{x} + B_{oy}\hat{y}\right)\Big|_S = 0.$$

For TE waves, (7) gives the following expressions for B_{ox} + B_{oy}.

$$B_{ox} = \frac{ik}{k_c^2} \frac{\partial B_{oz}}{\partial x}$$

$$B_{oy} = \frac{ik}{k_c^2} \frac{\partial B_{oz}}{\partial y}$$

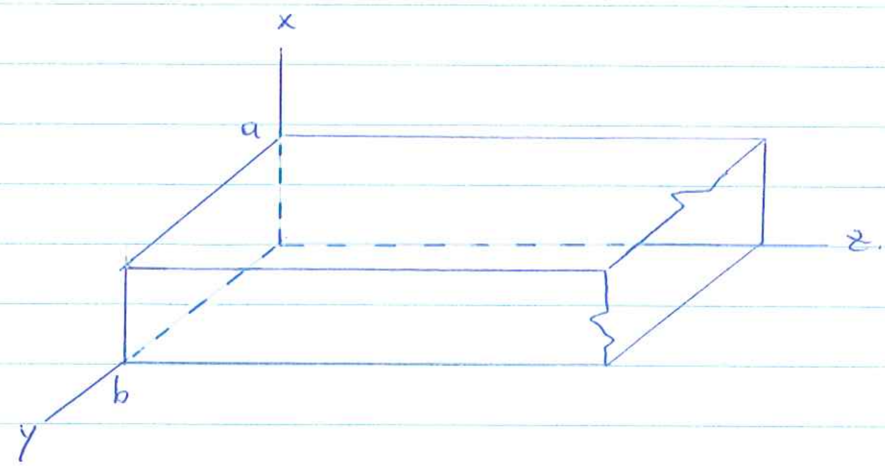
Bound. Cond. $\Rightarrow$ $\hat{n} \cdot \left(\frac{\partial B_{oz}}{\partial x} \hat{x} + \frac{\partial B_{oz}}{\partial y} \hat{y} \right) \Big|_S = 0.$

$$\hat{n} \cdot \nabla B_{oz} \Big|_S = 0.$$

or $\boxed{\left. \frac{\partial B_{oz}}{\partial n} \right|_S = 0}$ (9b).

Rectangular Waveguide

Consider the following rectangular waveguide.



We shall find the TE waves propagating in this waveguide. We therefore solve (8) subject to boundary condition (9b).

i.e. $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) B_{oz} + k_c^2 B_{oz} = 0.$$

$$\left.\frac{\partial B_{oz}}{\partial n}\right|_S = 0 \implies \left.\frac{\partial B_{oz}}{\partial x}\right|_{x=0,a} = 0 \quad \leftarrow \text{Top + bottom}$$

$$\left.\frac{\partial B_{oz}}{\partial y}\right|_{y=0,b} = 0 \quad \leftarrow \text{2 sides}$$

The equation for B_{oz} is the 2 dimensional analog of $\frac{\partial^2 B}{\partial x^2} + k^2 B = 0$ which has periodic solutions. We therefore let $B_{oz} = B_o \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$.

Boundary conditions are satisfied if $m, n \in \mathbb{Z}$.

Substituting B_{oz} into the wave equation, we get:

$$-\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 + k_c^2 = 0.$$

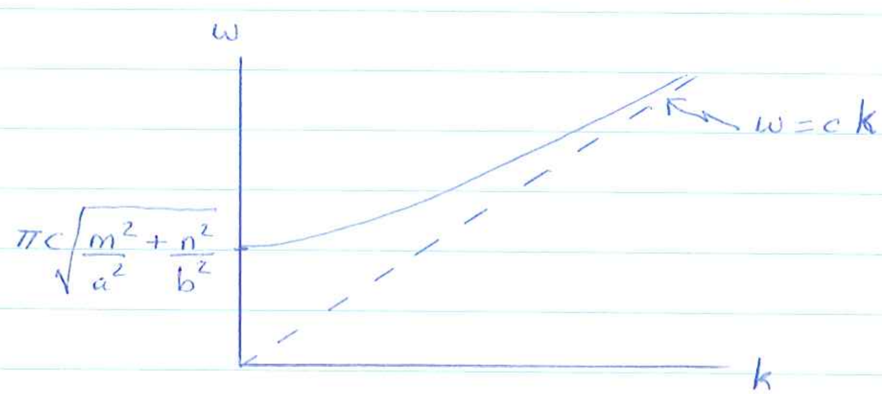
$$k_c = \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$ $m, n \in$ positive integers not both zero

Note that B_{oz} remains the same if m is replaced by $-m$ or n by $-n$. Also if $m = n = 0$, B_{oz} = constant and all other field components vanish. Hence no wave propagates if $m = n = 0$. We therefore restrict m & n to be positive integers with both not equal to zero at the same time.

Each possible solution is referred to as a TE_{mn} mode. The frequency and wavevector of this mode is related by:

$$k_c^2 = \frac{\omega^2}{c^2} - k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$$\omega^2 = \pi^2 c^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) + c^2 k^2.$$



The lowest frequency the TE_{mn} mode can propagate at is the so called cut off frequency given by:

$$\omega_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

If $a < b$, then the TE_{01} mode has the lowest cut off frequency $\omega_{01} = \frac{\pi c}{b}$.

If $\omega < \omega_{mn}$, k is imaginary and the wave is attenuated. For homework, you will show that the TE_{01} mode has the lowest cutoff frequency. It is therefore called the dominant mode.

Field Components of TE_{mn} Wave

$$B_{0z} = B_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

Other components are found using (7).

$$E_{0x} = \frac{i\omega}{k_c^2 c} \frac{\partial B_{0z}}{\partial y}$$

$$= -i \frac{n\pi}{b} \frac{\omega}{k_c^2 c} B_0 \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$E_{0y} = \frac{-i\omega}{k_c^2 c} \frac{\partial B_{0z}}{\partial x}$$

$$= -i \frac{m\pi}{a} \frac{\omega}{k_c^2 c} B_0 \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}.$$

$$B_{ox} = i \frac{k}{k_c^2} \frac{\partial B_{oz}}{\partial x}$$

$$= -i \frac{m\pi}{a} \frac{k}{k_c^2} B_o \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$B_{oy} = i \frac{k}{k_c^2} \frac{\partial B_{oz}}{\partial y}$$

$$= -i \frac{n\pi}{b} \frac{k}{k_c^2} B_o \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.$$

where $k_c^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right).$

TE_{01} Mode.

$$k_c = \frac{\pi}{b}.$$

$$B_{oz} = B_o \cos \frac{\pi y}{b}.$$

$$E_{ox} = -i \frac{b}{\pi} \frac{\omega}{c} B_o \sin \frac{\pi y}{b}.$$

$$E_{oy} = 0$$

$$B_{ox} = 0$$

$$B_{oy} = -i \frac{b}{\pi} k B_o \sin \frac{\pi y}{b}.$$

A sketch of the fields when $x = \frac{a}{2}$ is shown in the next figure. Recall that actual fields are:

$$\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} \vec{E}_o \\ \vec{B}_o \end{pmatrix} e^{i(kz - \omega t)}.$$

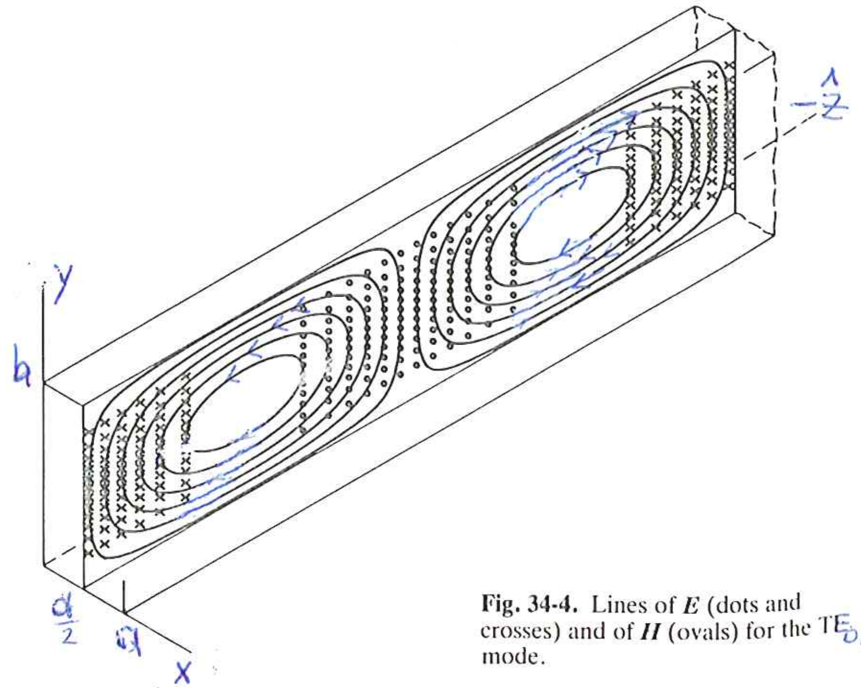


Fig. 34-4. Lines of E (dots and crosses) and of H (ovals) for the TE_{01} mode.

Energy Flow in Waveguide

Poynting Vector ≡ energy going through unit area per unit time.

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \, Re \, (\vec{E} \times \vec{B}^*).$$

We shall evaluate $\langle \vec{S} \rangle$ for the TE_{01} mode.

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \, Re \left(\vec{E}_0 \, e^{i(kz-\omega t)} \times \vec{B}_0^* \, e^{-i(kz-\omega t)} \right)$$

$$= \frac{c}{8\pi} \, Re \, (\vec{E}_0 \times \vec{B}_0^*)$$

$$= \frac{c}{8\pi} \, Re \left(E_{0x} \, \hat{x} \times (B_{0y}^* \, \hat{y} + B_{0z}^* \, \hat{z}) \right)$$

$$\langle \vec{S} \rangle = \hat{z}\,\frac{c}{8\pi}\,E_{0x}\,B_{0y}^{*}$$

$$= \hat{z}\,\frac{c}{8\pi}\,(-i)\,\frac{b}{\pi}\,\frac{\omega}{c}\,B_0 \sin\frac{\pi y}{b}\;(+i)\,\frac{b}{\pi}\,k\,B_0 \sin\frac{\pi y}{b}$$

$$\langle \vec{S} \rangle = \hat{z}\,\frac{1}{8\pi}\left(\frac{b}{\pi}\right)^2 \omega k\, B_0^2 \sin^2\frac{\pi y}{b}$$

To find the <u>total power</u> transmitted down the waveguide, we integrate $\langle \vec{S} \rangle$ over the waveguide cross section.

$$P = \int\limits_{\substack{\text{cross sectional}\\ \text{area of waveguide}}} \langle \vec{S} \rangle \cdot d\vec{a}$$

$$P = \int \langle S \rangle \, dx\, dy$$

$$= \frac{1}{8\pi}\left(\frac{b}{\pi}\right)^2 \omega k\, B_0^2 \int_0^b \int_0^a \sin^2\frac{\pi y}{b}\, dx\, dy$$

$$= \frac{1}{8\pi}\left(\frac{b}{\pi}\right)^2 \omega k\, B_0^2\; a \int_0^b \frac{1}{2}\left(1-\cos\frac{2\pi y}{b}\right) dy$$

$$= \frac{1}{8\pi}\left(\frac{b}{\pi}\right)^2 \omega k\, B_0^2\; a\,\frac{b}{2}$$

$$\boxed{P = \frac{1}{16\pi^3}\, a b^3\, \omega k\, B_0^2 . \quad \text{is power transmitted down waveguide}}$$

Average Field Energy

Next we compute the average field energy per unit length of waveguide.

Energy Density $u = \frac{1}{16\pi}(\vec{E}\cdot\vec{E}^* + \vec{B}\cdot\vec{B}^*)$

$$u = \frac{1}{16\pi}\left\{ E_{ox}^2 + B_{oy}^2 + B_{oz}^2 \right\}$$

$$= \frac{1}{16\pi}\left\{ \left(\frac{b}{\pi}\right)^2 \left(\frac{\omega}{c}\right)^2 B_o^2 \sin^2\frac{\pi y}{b}\right.$$

$$\left. + \left(\frac{b}{\pi}\right)^2 k^2 B_o^2 \sin^2\frac{\pi y}{b} + B_o^2 \cos^2\frac{\pi y}{b}\right\}.$$

If this is integrated over a volume of waveguide of unit length, we get:

$$\langle \mathcal{E} \rangle = \int_{\mathcal{V}} u \, dV$$

$$= \int_0^b \int_0^a u \; dx\, dy$$

$$= \frac{1}{16\pi}\left\{ \left(\frac{b}{\pi}\right)^2 \left(\frac{\omega}{c}\right)^2 B_o^2 \frac{ab}{2} + \left(\frac{b}{\pi}\right)^2 k^2 B_o^2 \frac{ab}{2}\right.$$

$$\left. + B_o^2 \frac{ab}{2}\right\}$$

$$= \frac{1}{32\pi} ab\, B_o^2 \left\{ \left(\frac{b}{\pi}\right)^2 \left(\left(\frac{\omega}{c}\right)^2 + k^2 \right) + 1 \right\}$$

$$\langle \mathcal{E} \rangle = \frac{1}{32\pi^3} \, a b^3 \, B_0^2 \left\{ \left(\frac{\omega}{c}\right)^2 + k^2 + \frac{\pi^2}{b^2} \right\}$$

For the TE_{01} mode $\quad \dfrac{\omega^2}{c^2} = k^2 + \dfrac{\pi^2}{b^2}$

$$\boxed{\therefore \langle \mathcal{E} \rangle = \frac{1}{16\pi^3} \, a b^3 \, B_0^2 \, \frac{\omega^2}{c^2} \text{ is average field energy ~~density~~ per unit length}}$$

Speed of Energy propagation v_g

$$v_g = \frac{P}{\langle \mathcal{E} \rangle}$$

$$= \frac{\omega k}{\omega^2/c^2}$$

$$= \frac{c^2}{\omega/k}.$$

Recall $v_p \equiv \dfrac{\omega}{k}$ is phase velocity.

$$\therefore \quad \boxed{v_g = \frac{c^2}{v_p}}$$

This result is true for all modes.

v_g + v_p for TE_{01} Mode

Phase Velocity $\quad v_p = \dfrac{\omega}{k}$

$$= \frac{1}{k} \, c \sqrt{k^2 + \frac{\pi^2}{b^2}}$$

$$v_p = c \sqrt{1 + \frac{\pi^2}{k^2 b^2}}$$

$\therefore\ v_p > c \quad \forall\, k.$

Group velocity $v_g = \dfrac{c^2}{v_p}$

$$= \frac{c}{\sqrt{1 + \frac{\pi^2}{k^2 b^2}}}$$

$$\boxed{v_g = \frac{ck}{\sqrt{k^2 + \frac{\pi^2}{b^2}}}}$$

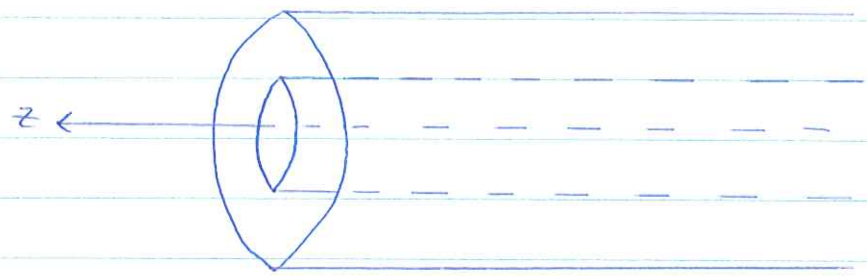
When $\omega = \omega_{01} = \dfrac{c\pi}{b}$, $k = 0$ and $v_g = 0$. Hence no power propagates down waveguide.

Comment

Waveguides are used to transport energy at frequencies c / dimension of waveguide $\sim 10^{10}$ Hz. The cutoff wavelength can be chosen to select the wavelength to be transported. i.e. waveguides can be used as filters

Coaxial Waveguide

Coax Cable



We shall use cylindrical coordinates (ρ, ϕ, z).

$\rho < a$ inner conductor
$a < \rho < b$ vacuum
$\rho > b$ outer conductor

We shall now show that there is a TEM wave with the electric field given by:

$$\vec{E} = \hat{\rho}\,\frac{E_0}{\rho}\,e^{i(kz-\omega t)}$$

(Note: For static case, this reduces to $\vec{E} = \hat{\rho}\,\frac{E_0}{\rho}$.)

We therefore need to show that this field satisfies Maxwell's equations and the boundary conditions.

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{E} = \hat{\phi}\,\frac{\partial E_\rho}{\partial z} - \hat{z}\,\frac{\partial E_\rho}{\partial \phi}$$

$$= \hat{\phi}\; ik\,\frac{E_0}{\rho}\,e^{i(kz-\omega t)}$$

Assuming $\vec{B} = \vec{B}_0\, e^{i(kz-\omega t)}$ we find

$$-\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \frac{i\omega}{c}\,\vec{B}.$$

$$\therefore\ \vec{B} = \hat{\phi}\,\frac{ck}{\omega}\,\frac{E_0}{\rho}\,e^{i(kz-\omega t)}$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{B} = -\hat{\rho}\,\frac{\partial B_\phi}{\partial z} + \hat{z}\,\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\phi)$$

$$= -\hat{\rho}\,\frac{ck}{\omega}\,\frac{E_0}{\rho}\,ik\,e^{i(kz-\omega t)}$$

$$= -\hat{\rho}\; i\,\frac{ck^2}{\omega}\,\frac{E_0}{\rho}\,e^{i(kz-\omega t)}$$

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \hat{\rho}\,\frac{1}{c}\,\frac{E_0}{\rho}\,(-i\omega)\,e^{i(kz-\omega t)}$$

$$= -\hat{\rho}\; i\,\frac{\omega}{c}\,\frac{E_0}{\rho}\,e^{i(kz-\omega t)}$$

Equating the expressions for $\nabla \times \vec{B}$ and $\frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ gives:

$$\frac{ck^2}{\omega} = \frac{\omega}{c}$$

$$\therefore \boxed{\omega = ck.}$$

Note that all frequencies down to DC are allowed.

Exercise: Show $\nabla \cdot \vec{E} = 0$ and $\nabla \cdot \vec{B} = 0$ are satisfied.

Results

TEM mode is given by:

$$\vec{E} = \hat{\rho}\, \frac{E_0}{\rho}\, e^{i(kz-\omega t)}$$

$$\vec{B} = \hat{\phi}\, \frac{E_0}{\rho}\, e^{i(kz-\omega t)}$$

where $\omega = ck$.

These fields also satisfy the boundary conditions at the conductors since nowhere is there a tangential $\vec{E}$ or normal $\vec{B}$ field at these surfaces.

Energy Flow in Waveguide

Poynting Vector $\langle \vec{S} \rangle = \frac{c}{8\pi} \, Re(\vec{E} \times \vec{B}^*)$

$$\langle \vec{S} \rangle = \frac{c}{8\pi}\, Re\left\{ \hat{\rho}\, \frac{E_0}{\rho}\, e^{i(kz-\omega t)} \times \hat{\phi}\, \frac{E_0}{\rho}\, e^{-i(kz-\omega t)} \right\}$$

$$= \hat{z}\, \frac{c}{8\pi}\, \frac{E_0^2}{\rho^2}$$

Power transported in waveguide

$$P = \int_{\substack{\text{Waveguide} \\ \text{Cross section}}} \vec{S} \cdot d\vec{a}$$

$$= \int_a^b S(\rho)\, 2\pi \rho\, d\rho$$

$$= \frac{c}{8\pi} E_0^2 \int_a^b 2\pi \frac{\rho\, d\rho}{\rho^2}$$

$$\therefore P = \frac{c E_0^2}{4} \ln\left(\frac{b}{a}\right)$$

Average Field Energy Density in Waveguide

$$u = \frac{1}{16\pi} \left\{ \vec{E} \cdot \vec{E}^* + \vec{B} \cdot \vec{B}^* \right\}$$

$$= \frac{1}{16\pi} \left\{ \frac{E_0^2}{\rho^2} + \frac{E_0^2}{\rho^2} \right\}$$

$$u = \frac{1}{8\pi} \frac{E_0^2}{\rho^2}$$

Average Field Energy per unit length of waveguide is

$$\langle E \rangle = \int_{\text{volume.}} u\, dV$$

$$\langle \mathcal{E} \rangle = \int_a^b \frac{1}{8\pi} \frac{E_0^2}{\rho^2} 2\pi \rho \, d\rho$$

$$= \frac{E_0^2}{4} \int_a^b \frac{d\rho}{\rho}$$

$$\langle \mathcal{E} \rangle = \frac{E_0^2}{4} \ln\left(\frac{b}{a}\right)$$

$\therefore$ speed of energy propagation $v_g = \dfrac{P}{\langle \mathcal{E} \rangle} = c$

Comment

One can also show that TE & TM modes exist. These will have cut off frequencies $\sim \dfrac{c}{a} \simeq 10^{10}$ Hz.

Below this frequency, all frequencies are transmitted at the same speed c. Therefore coax cables are used in labs to transport signals without distortion.

Assignment 4

1. For the rectangular waveguide, find the following.

   a) Fields of TM_{mn} modes.

   b) Cutoff frequencies of TM modes,
   What is lowest cutoff frequency?

2. For the TM_{11} mode, find:

   a) Poynting vector $\langle \vec{S} \rangle$

   b) Power propagating in waveguide

   c) Average energy density

   d) Average energy per unit length

   e) Speed of energy propagation
   How is this related to the phase velocity?

V Radiation

We shall now discuss how light waves are generated. Let's begin by writing Maxwell's equations in vacuum.

$$\nabla \cdot \vec{E} = 4\pi\rho \quad (1) \qquad\qquad \nabla \cdot \vec{B} = 0 \quad (3)$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad (2) \qquad\qquad \nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \quad (4)$$

Equation (3) allows us to write

$$\vec{B} = \nabla \times \vec{A} \qquad (5)$$

where $\vec{A}$ is the vector potential. Substituting this expression in Faraday's law (2) yields:

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \vec{A})$$

$$= \nabla \times \left(-\frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)$$

$$\nabla \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0$$

This implies $\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\nabla \Phi$

$$\text{or } \vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad (6)$$

Φ is the electric potential and for time independent fields is identical to the potential discussed earlier. Therefore to find fields $\vec{E}$ + $\vec{B}$, we must find $\vec{A}$ + Φ.

Wave Equations for Φ and $\vec{A}$

Taking the divergence of (6) gives:

$$-\nabla^2 \Phi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla . \vec{A}) = \nabla . \vec{E}$$

$$\nabla^2 \Phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla . \vec{A}) = -4\pi\rho \quad \text{using (1)}$$

Recall that we have the gauge freedom to assign a value to $\nabla . \vec{A}$. We shall use the Lorentz or radiation gauge where:

$$\nabla . \vec{A} = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$$

$$\Rightarrow \boxed{\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -4\pi\rho \qquad (7) \quad \text{Wave Eqn. for } \Phi}$$

Similarly one can show the wave equation for $\vec{A}$ is given by:

$$\boxed{\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J} \qquad (8).}$$

Exercise: Derive (8).

Solution of Wave Equations

For the static case, the wave equations reduce to:

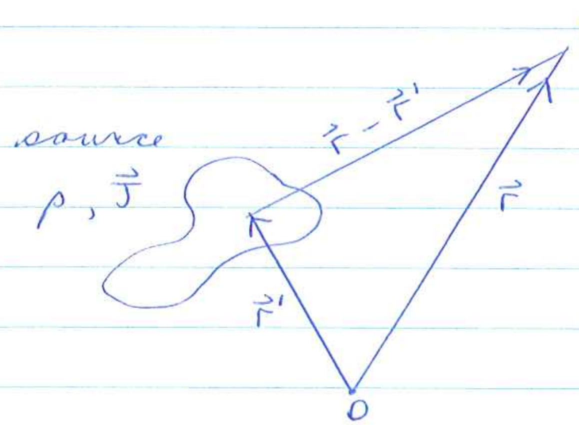
$$\nabla^2 \Phi = -4\pi\rho$$

$$\nabla^2 \vec{A} = -\frac{4\pi}{c}\vec{J}$$

which have the solutions

$$\Phi(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{r}\, d^3r'$$

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{\vec{J}(\vec{r}\,')}{r}\, d^3r'$$



$r \equiv |\vec{r}-\vec{r}\,'|$ is distance from source of ρ or $\vec{J}$ to observer at point P

Now suppose the value of ρ + $\vec{J}$ changes. This produces a new field which travels to the observer at P at the speed of light. Therefore the field observed at P at time t depends on ρ or $\vec{J}$ evaluated at an earlier or retarded time

$$t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}.$$

Therefore we guess the solution of the wave equations to be: $$\Phi(\vec{r},t) = \int_V \frac{\rho(\vec{r}\,',t_r)}{r}\, d^3r' \qquad (9)$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int \frac{\vec{J}(\vec{r}\,',t_r)}{r}\, d^3r' \qquad (10)$$

Rigorous Proof that (9) + (10) are Correct

For (9) + (10) to be correct, they must

1) satisfy the Lorentz gauge (homework)

2) satisfy the wave equations (7) + (8).

We shall show (9) is a solution of (7). We begin by dividing volume V into a volume V_1 surrounding point P and a volume $V_2 = V - V_1$.

$$\therefore \Phi(\vec{r},t) = \underbrace{\int_{V_1} \frac{\rho(\vec{r}\,',t_r)}{r}\, d^3r'}_{\equiv \Phi_1} + \underbrace{\int_{V_2} \frac{\rho(\vec{r}\,',t_r)}{r}\, d^3r'}_{\equiv \Phi_2}$$

If volume V_1 is small, the points in V_1 are close to P. The retardation r/c is then negligible and $t_r = t$.

$$\therefore \Phi_1 \simeq \int_{V_1} \frac{\rho(\vec{r}\,',t)}{r}\, d^3r'$$

This statement is exact in the limit $V_1 \to 0$.

The right hand side of the preceding equation is the solution of Poisson's equation for static fields.

$$\therefore \nabla^2 \Phi_1 = -4\pi \rho \quad \text{in the limit } V_1 \to 0.$$

Next consider $\Phi_2(\vec{r}, t) = \int_{V_2} \frac{\rho(\vec{r}', t - \mathscr{r}/c)}{\mathscr{r}} \, d^3 r'$

We must evaluate $\nabla^2 \Phi_2$ where ∇^2 means differentiating with respect to $\vec{r}$, i.e. $\nabla^2 = \nabla^2_{\vec{r}}$. Alternatively we can differentiate with respect to $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$, if $\vec{r}'$ is held fixed.

i.e. $\nabla^2_{\vec{r}} = \nabla^2_{\vec{\mathscr{r}}}$ if $\vec{r}'$ is fixed

$$\therefore \nabla^2 \Phi_2 = \nabla^2_{\vec{\mathscr{r}}} \int_{V_2} \frac{\rho(\vec{r}', t - \mathscr{r}/c)}{\mathscr{r}} \, d^3 r'$$

The integrand $\frac{\rho(\vec{r}', t - \mathscr{r}/c)}{\mathscr{r}}$ only depends on the magnitude of $\vec{\mathscr{r}}$. Therefore when taking the Laplacian in spherical coordinates there will be no angular terms.

$$\therefore \nabla^2_{\mathscr{r}} \left(\frac{\rho}{\mathscr{r}} \right) = \frac{1}{\mathscr{r}^2} \frac{\partial}{\partial \mathscr{r}} \left(\mathscr{r}^2 \frac{\partial}{\partial \mathscr{r}} \left(\frac{\rho}{\mathscr{r}} \right) \right)$$

$$= \frac{1}{\mathscr{r}^2} \frac{\partial}{\partial \mathscr{r}} \left(-\rho + \mathscr{r} \frac{\partial \rho}{\partial \mathscr{r}} \right)$$

$$= \frac{1}{\mathscr{r}} \frac{\partial^2 \rho}{\partial \mathscr{r}^2}$$

$$\therefore \quad \nabla^2 \Phi_2 = \int_{V_2} \frac{1}{r} \frac{\partial^2 \rho}{\partial r^2} \, d^3 r'$$

Next we show that $\dfrac{\partial^2 \rho}{\partial r^2} = \dfrac{1}{c^2} \dfrac{\partial^2 \rho}{\partial t^2}$

$\rho = \rho(\vec{r}\,', w)$ where $w = t - r/c$

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial w} \frac{\partial w}{\partial t} = \frac{\partial \rho}{\partial w}$$

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 \rho}{\partial w^2}$$

$$\frac{\partial \rho}{\partial r} = \frac{\partial \rho}{\partial w} \frac{\partial w}{\partial r} = -\frac{1}{c} \frac{\partial \rho}{\partial w}$$

$$\frac{\partial^2 \rho}{\partial r^2} = \frac{1}{c^2} \frac{\partial^2 \rho}{\partial w^2}$$

$$\Rightarrow \quad \frac{\partial^2 \rho}{\partial r^2} = \frac{1}{c^2} \frac{\partial^2 \rho}{\partial t^2}$$

$$\therefore \quad \nabla^2 \Phi_2 = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_{V_2} \frac{\rho(\vec{r}\,', t_r)}{r} \, d^3 r'$$

In the limit. $V_1 \to 0$, $V_2 \to V$, and we see the integral is just Φ.

$$\therefore \quad \nabla^2 \Phi = \nabla^2 \Phi_1 + \nabla^2 \Phi_2$$

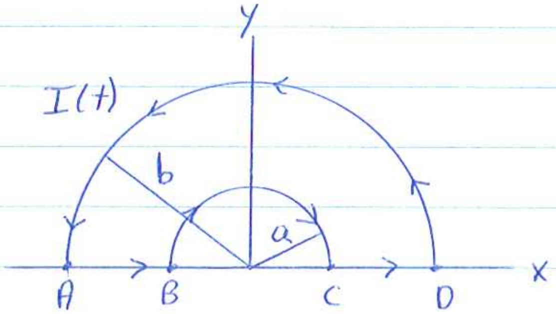
$$= -4\pi \rho + \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$

or $\nabla^2 \Phi - \dfrac{1}{c^2} \dfrac{\partial^2 \Phi}{\partial t^2} = -4\pi \rho.$ $\quad \therefore$ (9) is solution of (7).

Similarly since $\vec{A} = (A_x, A_y, A_z)$ where A_x, A_y, A_z are scalar functions like Φ, it follows that (10) is a solution of (8).

Example

A wire is bent into a loop as shown below. It carries a time dependent current. Find the vector potential at the origin.



$$\vec{A}(\vec{r}, t) = \frac{1}{c} \oint \frac{I(\vec{r}\,', t_r)\, d\vec{l}}{r}$$

$$\vec{A}(0,t) = \frac{1}{c} \int_{A\to B} \frac{I\, d\vec{l}}{r} + \frac{1}{c} \int_{B\to C} \frac{I\, d\vec{l}}{r} + \frac{1}{c} \int_{C\to D} \frac{I\, d\vec{l}}{r} + \frac{1}{c} \int_{D\to A} \frac{I\, d\vec{l}}{r}$$

$A \to B$: $\quad r = -x$.

$$t_r = t - \frac{(-x)}{c} = t + \frac{x}{c}$$

$$\frac{1}{c} \int_{A\to B} \frac{I(\vec{r}\,', t_r)\, d\vec{l}}{r} = \hat{x} \frac{1}{c} \int_b^a \frac{I(t + \frac{x}{c})}{-x} dx.$$

$$= \hat{x} \frac{1}{c} \int_{-a}^{-b} \frac{I(t - x/c)}{x} dx.$$

$B \to C$: $\quad r = a$

$$t_r = t - \frac{a}{c}$$

$$\frac{1}{c}\int_{B\to C} \frac{I(\vec{r}\,', t_r)}{r}\, d\vec{l} = \frac{1}{c}\int_{B\to C} \frac{I(t-a/c)}{a}\, d\vec{l}$$

$$= \frac{1}{c}\,\frac{I(t-a/c)}{a}\int_{B\to C} d\vec{l}$$

$$= \frac{1}{c}\,\frac{I(t-a/c)}{a}\; 2a\,\hat{x}$$

$$= \hat{x}\;\frac{2I(t-a/c)}{c}$$

Similarly one finds $\displaystyle \frac{1}{c}\int_{C\to D} \frac{I(\vec{r}\,', t_r)\,d\vec{l}}{r} = \hat{x}\,\frac{1}{c}\int_a^b \frac{I(t-x/c)\,dx}{x}$

$$\frac{1}{c}\int_{D\to A} \frac{I(\vec{r}\,', t_r)\,d\vec{l}}{r} = -\hat{x}\;\frac{2I(t-b/c)}{c}$$

$$\therefore\; \vec{A}(0,t) = \hat{x}\,\frac{1}{c}\left\{2I(t-a/c) - 2I(t-b/c) + \int_{-a}^{-b} \frac{I(t-x/c)}{x}\,dx + \int_a^b \frac{I(t-x/c)}{x}\,dx\right\}$$

Suppose $I = kt$, k = constant

$$\vec{A}(0,t) = \frac{\hat{x}}{c}\left\{2k\left(t-\frac{a}{c}\right) - 2k\left(t-\frac{b}{c}\right) + \int_{-a}^{-b}\frac{k(t-x/c)}{x}\,dx + \int_a^b \frac{k(t-x/c)}{x}\,dx\right\}$$

$$\vec{A}(0,t) = \frac{\hat{x}}{c}\left\{ \frac{2k}{c}(b-a) + kt\ln\left(\frac{b}{a}\right) - \frac{k}{c}(-b+a) \right.$$

$$\left. + kt\ln\left(\frac{b}{a}\right) - \frac{k}{c}(b-a) \right\}$$

$$\therefore \vec{A}(0,t) = \frac{\hat{x}}{c}\left\{ \frac{2k}{c}(b-a) + 2kt\ln\left(\frac{b}{a}\right) \right\}$$

The electric potential $\Phi(\vec{r},t) = 0$ since $\rho = 0$.

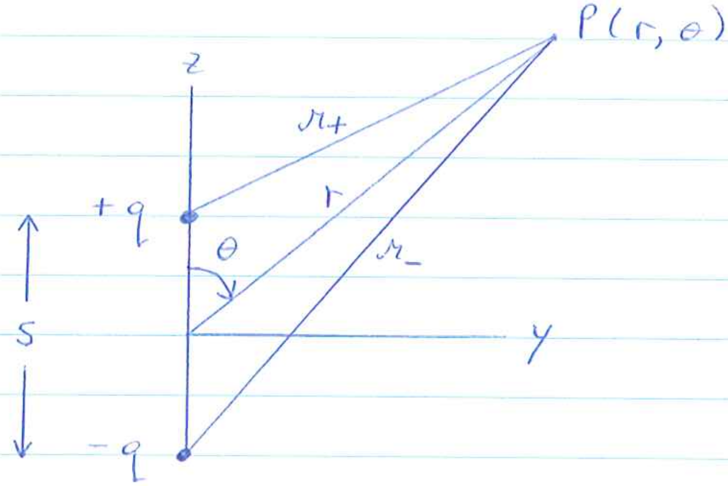
Electric field $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\Phi$

$$\vec{E}(0,t) = -\hat{x}\frac{2k}{c^2}\ln\left(\frac{b}{a}\right)$$

To find the magnetic field, $\vec{B} = \nabla \times \vec{A}$, we must know the potential $\vec{A}(\vec{r},t)$ not just $\vec{A}(0,t)$.

Electric Dipole Radiation

Consider two small metal spheres separated by a distance s and connected by a small wire. The spheres have charges $q(t)$ and $-q(t)$ where $q(t) = q_0 \cos \omega t$.



The two charges act as an oscillating electric dipole of moment

$$\vec{p}(t) = q(t)\, s\, \hat{z}$$

$$= p_0 \cos \omega t \;\hat{z} \quad \text{where } p_0 = q_0 s.$$

We shall find the electric and magnetic fields radiated by the oscillating dipole. To do this we need the potentials Φ & $\vec{A}$.

$$\Phi(r,\theta,t) = \frac{q(t - r_+/c)}{r_+} - \frac{q(t - r_-/c)}{r_-}$$

$$= \frac{q_0 \cos \omega(t - r_+/c)}{r_+} - \frac{q_0 \cos \omega(t - r_-/c)}{r_-}$$

where $r_{\pm} = \sqrt{r^2 \mp r s \cos\theta + \left(\frac{s}{2}\right)^2}$

We are interested in finding the fields at positions far from the dipole.

i.e. $\boxed{r >> s}$

An atomic dipole has a size $s \simeq 1\,\mathring{A}$. $\therefore r >> 1\,\mathring{A}$

An ideal dipole is formed in the limit $q_0 \to \infty$, $s \to 0$ such that $p_0 = q_0 s$ is finite. We therefore expand $r_{\pm}$ in terms of s keeping only the first order term.

$$r_{\pm} = r\left[1 \mp \frac{s}{r}\cos\theta + \left(\frac{s}{2r}\right)^2\right]^{1/2}$$

$$\simeq r\left(1 \mp \frac{s}{2r}\cos\theta\right)$$

$$\frac{1}{r_{\pm}} \simeq \frac{1}{r}\left(1 \pm \frac{s}{2r}\cos\theta\right)$$

$$\cos w(t - r_{\pm}/c) = \cos w\left[t - \frac{r}{c}\left(1 \mp \frac{s}{2r}\cos\theta\right)\right]$$

$$= \cos\left\{w\left(t - \frac{r}{c}\right) \pm \frac{ws}{2c}\cos\theta\right\}$$

$$= \cos w\left(t - \frac{r}{c}\right)\cos\left(\frac{ws}{2c}\cos\theta\right)$$

$$\mp \sin w\left(t - \frac{r}{c}\right)\sin\left(\frac{ws}{2c}\cos\theta\right)$$

Next we assume the dipole size $\boxed{s << \frac{c}{w} \simeq \lambda}$ where λ is the wavelength of radiated light. For visible light $\lambda \simeq 5000\,\mathring{A} >> s \simeq 1\,\mathring{A}$.

In the limit $\frac{\omega s}{c} \ll 1$ we get:

$$\cos\omega\left(t - r_\pm/c\right) = \cos\omega\left(t-\frac{r}{c}\right) \mp \frac{\omega s}{2c}\cos\theta\,\sin\omega\left(t-\frac{r}{c}\right)$$

$$\therefore\ \Phi(r,\theta,t) = q_0\left[\cos\omega\left(t-\frac{r}{c}\right) - \frac{\omega s}{2c}\cos\theta\,\sin\omega\left(t-\frac{r}{c}\right)\right]$$

$$\cdot\frac{1}{r}\left(1+\frac{s}{2r}\cos\theta\right)$$

$$-q_0\left[\cos\omega\left(t-\frac{r}{c}\right) + \frac{\omega s}{2c}\cos\theta\,\sin\omega\left(t-\frac{r}{c}\right)\right]$$

$$\cdot\frac{1}{r}\left(1-\frac{s}{2r}\cos\theta\right)$$

Now we take the dipole limit $\begin{cases} s \to 0 \\ q_0 \to \infty \end{cases}$ such that

$q_0 s = p_0$ is finite.

$$\Phi(r,\theta,t) = q_0\left\{\cos\omega\left(t-\frac{r}{c}\right)\left[\frac{s}{r^2}\cos\theta\right]\right.$$

$$\left.+\sin\omega\left(t-\frac{r}{c}\right)\left[-\frac{\omega s}{rc}\cos\theta\right]\right\}$$

$$= \frac{p_0\cos\theta}{r}\left\{-\frac{\omega}{c}\sin\omega\left(t-\frac{r}{c}\right) + \frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right)\right\}$$

Exercise: Check that in the static field limit $(\omega \to 0)$ $\Phi = \frac{p_0\cos\theta}{r^2}$ as expected for a classical dipole.

We are interested in fields at a large distance from the source in the so called radiation zone. This is defined as the region where:

$$\boxed{r \gg \frac{c}{\omega} \simeq \lambda}$$

$$\Rightarrow \boxed{\Phi(r,\theta,t) = -\frac{p_0\,\omega}{c}\,\frac{\cos\theta}{r}\,\sin\omega\left(t-\frac{r}{c}\right)}$$

Next we find the vector potential.

Current flowing in wire $\vec{I}(t) = \hat{z}\,\frac{dq}{dt}$

$$= -\hat{z}\,q_0\,\omega\sin\omega t$$

Vector Potential $\vec{A}(\vec{r},t) = \frac{1}{c}\int_{-s/2}^{s/2} \frac{\vec{I}(t-r(z)/c)}{r(z)}\,dz$

To obtain $\vec{A}(\vec{r},t)$ to first order in s, the integrand is evaluated at $s=0$.

$$\vec{A}(r,\theta,t) = -\hat{z}\,\frac{q_0 s\,\omega}{rc}\,\sin\omega(t-r/c)$$

Taking the dipole limit $\begin{cases} q_0 \to \infty \\ s \to 0 \end{cases}$ such that $q_0 s = p$ is finite

we get:

$$\vec{A}(r,\theta,t) = -\hat{z}\,\frac{p_0\,\omega}{rc}\,\sin\omega(t-r/c)$$

$$\boxed{\vec{A}(r,\theta,t) = -\frac{p_0 \omega}{rc} \sin\omega(t - r/c)\ (\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})}$$

Now we are in position to find the electric and magnetic fields.

$$\nabla \Phi = \frac{\partial \Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\hat{\theta}$$

$$= -\frac{p_0 \omega}{c} \cos\theta \left[\frac{-1}{r^2} \sin\omega\left(t - \frac{r}{c}\right) - \frac{\omega}{rc} \cos\omega\left(t - \frac{r}{c}\right)\right]\hat{r}$$

$$-\frac{p_0 \omega}{c} \frac{(-\sin\theta)}{r^2} \sin\omega\left(t - \frac{r}{c}\right)\hat{\theta}$$

$$= p_0 \frac{\omega^2}{c^2} \frac{\cos\theta}{r} \cos\omega\left(t - \frac{r}{c}\right)\hat{r}$$

where the other terms have been dropped since $r \gg \lambda \simeq \frac{c}{\omega}$

$$\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\frac{p_0 \omega^2}{rc^2} \cos\omega\left(t - r/c\right) (\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}).$$

Electric field $\vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

$$\boxed{\vec{E}(r,\theta,t) = -p_0 \frac{\omega^2}{c^2} \frac{\sin\theta}{r} \cos\omega\left(t - \frac{r}{c}\right)\hat{\theta}}$$

The magnetic field $\vec{B} = \nabla \times \vec{A}$

$$\boxed{\vec{B}(r,\theta,t) = -p_0 \frac{\omega^2}{c^2} \frac{\sin\theta}{r} \cos\omega\left(t - \frac{r}{c}\right)\hat{\phi}}$$

Exercise: Derive the preceding result.

Comments

From the expressions for the electric and magnetic fields one observes the following.

1. $\vec{E}$ and $\vec{B}$ form a so called spherical wave propagating in the $\hat{\theta} \times \hat{\phi} = \hat{r}$ direction. The amplitudes decrease as $1/r$. At large r, the wave is approximately planar over small regions. This is analogous to the Earth's surface appearing flat over small regions. It is also a transverse wave since $\vec{E}$, $\vec{B}$ + $\hat{r}$ are all perpendicular.
2. $\vec{E}$ + $\vec{B}$ are in phase.
3. $\vec{E}$ + $\vec{B}$ have equal amplitudes.
4. $\vec{E}$ + $\vec{B}$ form a monochromatic wave of frequency ω.

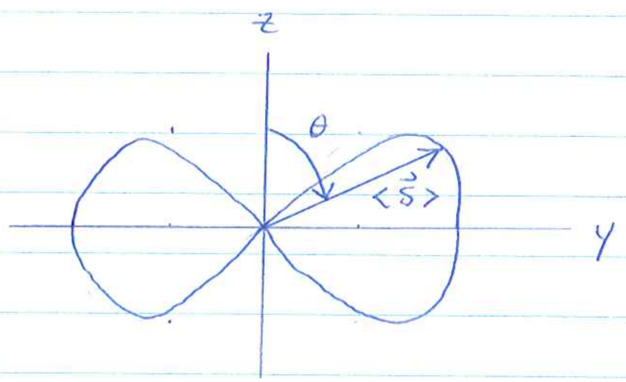
Flow of Energy

Poynting vector $\vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{B})$ Formula for real fields is used here.

$$\vec{S} = \frac{c}{4\pi} p_0^2 \frac{\omega^4}{c^4} \frac{\sin^2\theta}{r^2} \cos^2\omega\left(t - \frac{r}{c}\right) \hat{r}$$

Averaging over many optical periods we get:

$$\boxed{\langle \vec{S} \rangle = \frac{1}{8\pi} p_0^2 \frac{\omega^4}{c^3} \frac{\sin^2\theta}{r^2} \hat{r}}$$



Notice that no energy is radiated along the direction of the dipole axis.

Power radiated through sphere of radius r is

$$P = \int_{\substack{\text{surface of sphere} \\ \text{of radius } r}} \langle \vec{S} \rangle \cdot d\vec{a}$$

$$= \frac{p_0^2}{8\pi} \frac{\omega^4}{c^3} \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi .$$

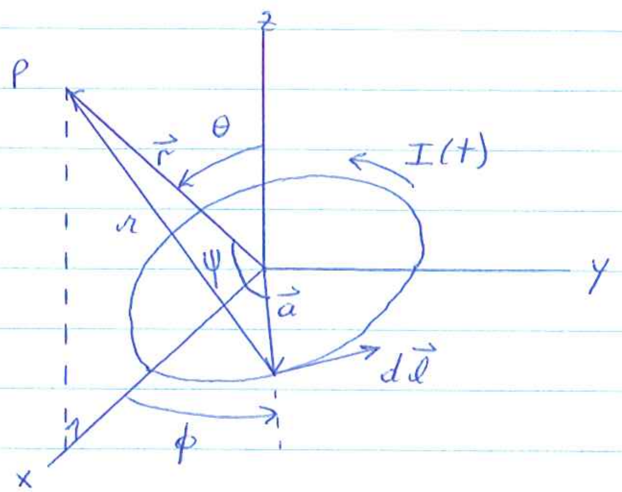
$$P = \frac{p_0^2}{8\pi} \frac{\omega^4}{c^3} \; 2\pi \int_0^\pi \sin^3\theta \, d\theta \quad \text{independent of } r$$

$$\boxed{P = \frac{p_0^2}{3} \frac{\omega^4}{c^3}}$$

This result explains why the sky is blue. Sunlight consists of a broad band of frequencies - white light. It is absorbed and reradiated by molecules in the atmosphere that act as oscillating dipoles. Since $P \propto \omega^4$, light at higher frequencies (blue) is scattered more than light at lower frequencies (red). ∴ the sky appears blue and the sun red.

Magnetic Dipole Radiation

Consider a loop of wire of radius a having a current
$I(t) = I_0 \cos \omega t$



$$\vec{a} = a(\cos\phi, \sin\phi, 0)$$

$$d\vec{l} = a(-\sin\phi, \cos\phi, 0)\, d\phi.$$

The changing current produces an oscillating magnetic dipole moment

$$\vec{m}(t) = \pi a^2 I(t)\, \hat{z}$$

$$= m_0 \cos \omega t\, \hat{z} \quad \text{where } m_0 = \pi a^2 I_0$$

We shall find the electric and magnetic fields radiated by $m(t)$. To do this we need Φ and $\vec{A}$. Since the loop is uncharged $\Phi = 0$.

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \oint \frac{I_0 \cos \omega(t - r/c)}{r}\, d\vec{l}$$

By symmetry $\vec{A}$ points in the $\hat{\phi}$ direction. For simplicity we shall evaluate $\vec{A}$ at P, a point directly above the x axis. $\vec{A}(P)$ points in $\hat{y}$ direction.

$$\vec{A}(P,t) = \hat{y}\ \frac{I_0}{c} \int_0^{2\pi} \frac{\cos\omega(t - \imath/c)}{\imath}\ a\cos\phi\, d\phi$$

where $\imath = \sqrt{r^2 - 2ra\cos\psi + a^2}$ and ψ is the angle between vectors $\vec{r} = r(\sin\theta, 0, \cos\theta)$ and $\vec{a} = a(\cos\phi, \sin\phi, 0)$.

$$\vec{r} \cdot \vec{a} = ra\cos\psi$$

$$\vec{r} \cdot \vec{a} = r(\sin\theta, 0, \cos\theta) \cdot a(\cos\phi, \sin\phi, 0)$$
$$= ra\sin\theta\cos\phi$$

$$\Rightarrow \cos\psi = \sin\theta\cos\phi.$$

$$\therefore \imath = \sqrt{r^2 - 2ra\sin\theta\cos\phi + a^2}$$

We are interested in finding the fields at positions far from the dipole. i.e. $r \gg a$. We therefore expand $\imath$ keeping only the terms that are first order in a.

$$\imath = r\left[1 - \frac{2a}{r}\sin\theta\cos\phi + \frac{a^2}{r^2}\right]^{1/2}$$

$$\simeq r\left(1 - \frac{a}{r}\sin\theta\cos\phi\right)$$

$$\frac{1}{\imath} \simeq \frac{1}{r}\left(1 + \frac{a}{r}\sin\theta\cos\phi\right)$$

$$\cos\omega(t - \imath/c) = \cos\omega\left[t - \frac{r}{c}\left(1 - \frac{a}{r}\sin\theta\cos\phi\right)\right]$$

$$= \cos\omega\left(t - \frac{r}{c}\right)\cos\left(\frac{\omega}{c}a\sin\theta\cos\phi\right) - \sin\omega\left(t - \frac{r}{c}\right)\sin\left(\frac{\omega}{c}a\sin\theta\cos\phi\right)$$

Next we assume $a << \frac{c}{\omega} \simeq \lambda$.

$$\therefore \cos\omega\left(t - \frac{r}{c}\right) = \cos\omega\left(t - \frac{r}{c}\right) - \frac{\omega a}{c}\sin\theta\cos\phi\sin\omega\left(t - \frac{r}{c}\right)$$

Then $$\vec{A}(P,t) = \hat{y}\,\frac{I_0}{c}\int_0^{2\pi}\left[\cos\omega\left(t-\frac{r}{c}\right) - \frac{\omega a}{c}\sin\theta\cos\phi\sin\omega\left(t-\frac{r}{c}\right)\right]$$

$$\cdot\frac{1}{r}\left(1 + \frac{a}{r}\sin\theta\cos\phi\right) a\cos\phi\, d\phi$$

$$\vec{A}(P,t) = \hat{y}\,\frac{I_0}{c}\,\frac{a}{r}\int_0^{2\pi}\left\{\cos\omega\left(t-\frac{r}{c}\right) - \frac{\omega a}{c}\sin\theta\cos\phi\sin\omega\left(t-\frac{r}{c}\right)\right.$$

$$\left. + \frac{a}{r}\sin\theta\cos\phi\cos\omega\left(t-\frac{r}{c}\right)\right\}\cos\phi\, d\phi$$

where the 2nd order term $O\left(\frac{a}{r}\cdot\frac{\omega a}{c}\right)$ was dropped.

The first term integrates to zero since $\int_0^{2\pi}\cos\phi\, d\phi = 0$

The next terms contain $\int_0^{2\pi}\cos^2\phi\, d\phi = \pi$

$$\therefore \vec{A}(P,t) = \hat{y}\,\frac{I_0}{c}\,\frac{a}{r}\,\pi\left\{-\frac{\omega a}{c}\sin\theta\sin\omega\left(t-\frac{r}{c}\right)\right.$$

$$\left. + \frac{a}{r}\sin\theta\cos\omega\left(t-\frac{r}{c}\right)\right\}$$

$$\vec{A}(P,t) = \hat{y}\ \frac{I_0 \pi a^2}{c}\ \frac{\sin\theta}{r} \left\{ -\frac{\omega}{c} \sin\omega\left(t-\frac{r}{c}\right) + \frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right) \right\}$$

Since there is rotational symmetry about the z axis, we can write:

$$\vec{A}(r,\theta,t) = \hat{\phi}\ \frac{m_0}{c}\ \frac{\sin\theta}{r} \left\{ -\frac{\omega}{c} \sin\omega\left(t-\frac{r}{c}\right) + \frac{1}{r}\cos\omega\left(t-\frac{r}{c}\right) \right\}$$

In the static limit ($\omega = 0$) $\vec{A} = \hat{\phi}\ \dfrac{m_0 \sin\theta}{c r^2}$ as expected.

We are interested in fields in the radiation zone for which $r \gg \frac{c}{\omega} \simeq \lambda$.

$$\boxed{\vec{A}(r,\theta,t) = -\, m_0 \frac{\omega}{c^2}\ \frac{\sin\theta}{r}\ \sin\omega\left(t-\frac{r}{c}\right)\ \hat{\phi}}$$

Electric field $\vec{E} = -\dfrac{1}{c}\dfrac{\partial \vec{A}}{\partial t} - \nabla \Phi$

$$\boxed{\vec{E}(r,\theta,t) = m_0 \frac{\omega^2}{c^3}\ \frac{\sin\theta}{r}\ \cos\omega\left(t-\frac{r}{c}\right)\ \hat{\phi}}$$

One can show the magnetic field $\vec{B} = \nabla\times\vec{A}$ is given by:

$$\boxed{\vec{B}(r,\theta,t) = -\, m_0 \frac{\omega^2}{c^3}\ \frac{\sin\theta}{r}\ \cos\omega\left(t-\frac{r}{c}\right)\ \hat{\theta}}$$

Exercise: Derive the expression for $\vec{B}$.

These expressions for $\vec{E}$ & $\vec{B}$ are very similar to those produced by an oscillating electric dipole. We get a spherical wave propagating in the $\hat{\phi} \times (-\hat{\theta}) = \hat{r}$ direction.

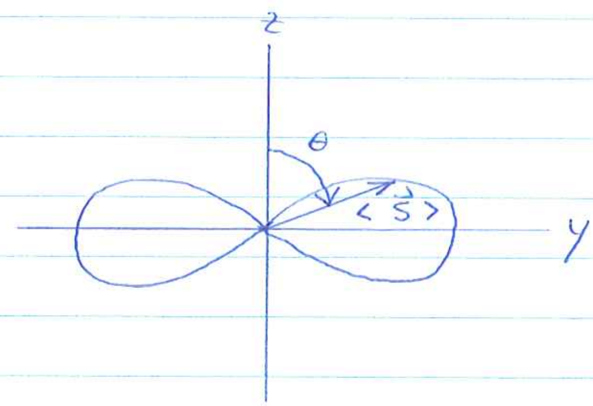
Flow of Energy

Poynting Vector $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$

$$\vec{S} = \frac{c}{4\pi} m_0^2 \frac{\omega^4}{c^6} \frac{\sin^2\theta}{r^2} \cos^2\omega\left(t - \frac{r}{c}\right) \hat{r}$$

Averaging over many optical periods we get:

$$\boxed{\langle \vec{S} \rangle = \frac{1}{8\pi} m_0^2 \frac{\omega^4}{c^5} \frac{\sin^2\theta}{r^2} \hat{r}}$$



Note that no energy is radiated along the magnetic dipole axis.

Power radiated through spherical surface of radius r is

$$P = \int_{\text{spherical surface}} \langle \vec{S} \rangle \cdot d\vec{a}$$

$$P = \frac{1}{8\pi} \; m_0^2 \; \frac{\omega^4}{c^5} \int \frac{\sin^2\theta}{r^2} \; r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4} \; m_0^2 \; \frac{\omega^4}{c^5} \int_0^{\pi} \sin^3\theta \, d\theta.$$

$$\boxed{P = \frac{m_0^2 \; \omega^4}{3 \, c^5}}$$

Note that $P \propto \omega^4$ for both electric & magnetic dipole radiation.

Next we consider an electric & magnetic dipole of similar dimensions and currents.

$m_0 = \pi a^2 I_0$ where $I_0 \simeq q_0 \omega$

$p_0 = q_0 a$

$$\frac{P_{mag.\, dipole}}{P_{elect.\, dipole}} = \frac{m_0^2}{p_0^2 c^2}$$

$$= \left(\frac{\pi a^2 q_0 \omega}{q_0 a c} \right)^2$$

$$= \left(\frac{a\omega}{c} \right)$$

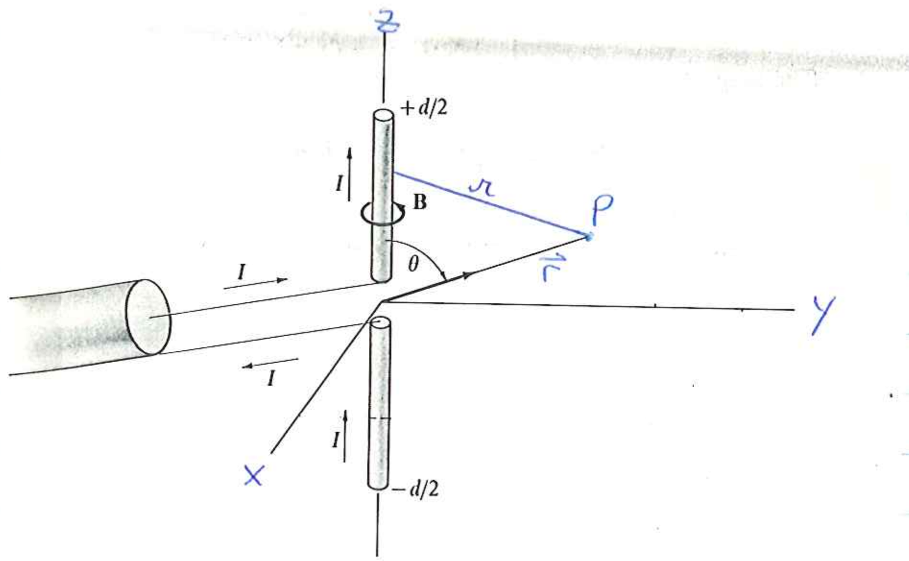
$$\simeq \left(\frac{a}{\lambda} \right)^2$$

$\ll 1$ since $a \ll \lambda$

$\therefore$ normally electric dipole radiation predominates.

Linear Antenna

Consider a linear antenna driven at its center by a coax cable. The antenna has length d as shown below.



The input current varies harmonically in time. It is zero at the ends of the antenna and is the same in the top and lower antenna halfs. We assume the current to be given by the following:

$$\vec{I}(z,t) = \hat{z}\; I_0 \cos \omega t \;\sin k\left(\frac{d}{2} - |z|\right), \quad k = \frac{\omega}{c}$$

To simplify the algebra we shall use a complex current:

$$\vec{I}(z,t) = \hat{z}\; I_0\, e^{i\omega t} \sin k\left(\frac{d}{2} - |z|\right).$$

We shall find the electric and magnetic fields radiated by $\vec{I}(z,t)$. First we find $\vec{A}$.

$$\vec{A}(r,\theta,t) = \frac{1}{c}\int_{-d/2}^{d/2} \frac{\vec{I}\,(z', t - \mathscr{r}/c)}{\mathscr{r}}\, dz'$$

$$= \hat{z}\,\frac{I_0}{c}\int_{-d/2}^{d/2} \frac{e^{i\omega(t-\mathscr{r}/c)} \sin k\left(\frac{d}{2} - |z'|\right)}{\mathscr{r}}\, dz'$$

where $\mathscr{r} = \sqrt{r^2 + z'^2 - 2rz'\cos\theta}$

$$\simeq r\left[1 - 2\frac{z'}{r}\cos\theta + \left(\frac{z'}{r}\right)^2\right]^{1/2}$$

We are interested in finding the fields at positions far from the antenna. i.e. $r \gg d$.

$$\therefore\ \mathscr{r} \simeq r\left(1 - \frac{z'}{r}\cos\theta\right)$$

To lowest order in $\frac{d}{r}$, $\vec{A}$ is found by substituting the above formula for $\mathscr{r}$ in $e^{i\omega(t-\mathscr{r}/c)}$ and $\mathscr{r} \cong r$ in the denominator.

$$e^{i\omega(t-\mathscr{r}/c)} = \exp i\omega\left[t - \frac{r}{c}\left(1 - \frac{z'}{r}\cos\theta\right)\right]$$

$$= \exp i\omega\left(t - \frac{r}{c}\right)\ \exp\left(i\omega\frac{z'}{c}\cos\theta\right)$$

$$\therefore\ \vec{A}(r,\theta,t) = \hat{z}\,\frac{I_0}{rc}\, e^{i\omega(t-r/c)} \underbrace{\int_{-d/2}^{d/2} e^{i\omega\frac{z'}{c}\cos\theta} \sin k\left(\frac{d}{2} - |z'|\right) dz'}_{\equiv \mathcal{I}}$$

Evaluation of Integral I

$$I \equiv \int_{-d/2}^{d/2} e^{i\alpha z'} \sin k\left(\frac{d}{2} - |z'|\right) dz' \qquad \text{where } \alpha \equiv \frac{\omega}{c}\cos\theta$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = k\cos\theta$$

$$= \int_{0}^{d/2} e^{i\alpha z'} \sin k\left(\frac{d}{2} - z'\right) dz' + \int_{-d/2}^{0} e^{i\alpha z'} \sin k\left(\frac{d}{2} + z'\right) dz'$$

$$= \int_{0}^{d/2} e^{i\alpha z'} \sin k\left(\frac{d}{2} - z'\right) dz' + \int_{0}^{d/2} e^{-i\alpha z'} \sin k\left(\frac{d}{2} - z'\right) dz'$$

$$= 2\int_{0}^{d/2} \cos\alpha z' \sin k\left(\frac{d}{2} - z'\right) dz'$$

$$= \int_{0}^{d/2} \left\{ \sin\left[(\alpha - k)z' + \frac{kd}{2}\right] - \sin\left[(\alpha + k)z' - \frac{kd}{2}\right] \right\} dz'$$

$$= \left\{ \frac{-\cos\left[(\alpha - k)z' + \frac{kd}{2}\right]}{\alpha - k} + \frac{\cos\left[(\alpha + k)z' - \frac{kd}{2}\right]}{\alpha + k} \right\}_{0}^{d/2}$$

$$= \frac{-\cos\left[(\alpha - k)\frac{d}{2} + \frac{kd}{2}\right] + \cos\frac{kd}{2}}{\alpha - k}$$

$$+ \frac{\cos\left[(\alpha + k)\frac{d}{2} - \frac{kd}{2}\right] - \cos\frac{kd}{2}}{\alpha + k}$$

$$I = \frac{-\cos\frac{\alpha d}{2} + \cos\frac{kd}{2}}{\alpha - k} + \frac{\cos\frac{\alpha d}{2} - \cos\frac{kd}{2}}{\alpha + k}$$

$$= \left(\cos\frac{\alpha d}{2} - \cos\frac{kd}{2}\right)\left\{\frac{-1}{\alpha - k} + \frac{1}{\alpha + k}\right\}$$

$$= \left(\cos\frac{\alpha d}{2} - \cos\frac{kd}{2}\right)\left\{\frac{-\alpha - k + \alpha - k}{\alpha^2 - k^2}\right\}$$

$$= \frac{-2k}{\alpha^2 - k^2}\left(\cos\frac{\alpha d}{2} - \cos\frac{kd}{2}\right)$$

$$= \frac{-2k}{k^2(\cos^2\theta - 1)}\left(\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}\right)$$

$$I = \frac{2}{k}\,\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin^2\theta}$$

$$\therefore \vec{A}(r,\theta,t) = \hat{z}\,\frac{2I_0}{rck}\,e^{i\omega(t - r/c)}\,\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin^2\theta}$$

$$\boxed{\vec{A}(r,\theta,t) = \frac{2I_0}{krc}\,e^{i\omega(t - r/c)}\,\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin^2\theta}\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)}$$

For convenience define $F(\theta) \equiv \dfrac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin^2\theta}$

Electric field $\vec{E} = -\dfrac{1}{c}\dfrac{\partial \vec{A}}{\partial t} - \nabla \Phi$

$$\boxed{\vec{E}(r,\theta,t) = -\frac{2i\,\omega I_0}{krc^2} e^{i\omega(t - r/c)} F(\theta)\,(\hat{r}\cos\theta - \hat{\theta}\sin\theta)}$$

Magnetic field $\vec{B} = \nabla \times \vec{A}$

$$\vec{B} = \hat{\phi}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right\}$$

$$rA_\theta = -\frac{2I_0}{kc} e^{i\omega(t - r/c)} F(\theta)\sin\theta$$

$$\frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} = i\,\frac{2I_0\omega}{rkc^2} e^{i\omega(t-r/c)} F(\theta)\sin\theta$$

We discard the remaining term since it contributes a term $\propto 1/r^2$ which is much smaller than the first term.

$$\therefore \boxed{\vec{B}(r,\theta,t) = 2i\,\frac{I_0\,\omega}{rkc^2} e^{i\omega(t-r/c)} F(\theta)\sin\theta\,\hat{\phi}.}$$

Flow of Energy.

Time averaged Poynting vector

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \, Re \, (\vec{E} \times \vec{B}^*)$$

We are interested in energy flowing radially outwards away from antenna. This is given by $E_\theta \, \hat{\theta} \times B_\phi \, \hat{\phi}$.

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \, Re \left(E_\theta \, \hat{\theta} \times B_\phi^* \, \hat{\phi} \right)$$

$$= \hat{r} \frac{c}{8\pi} \frac{2\omega}{krc^2} I_0 F(\theta) \sin\theta \; \frac{2 I_0 \omega}{r k c^2} F(\theta) \sin\theta$$

$$= \hat{r} \frac{1}{2\pi} \frac{\omega^2 I_0^2}{k^2 c^3} \frac{F^2(\theta) \sin^2\theta}{r^2}.$$

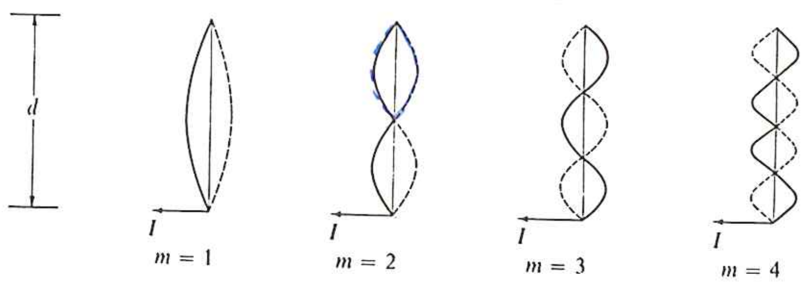
$$\boxed{\langle \vec{S} \rangle = \frac{\hat{r}}{r^2} \frac{I_0^2}{2\pi c} \left(\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin\theta} \right)^2}$$

This is the energy radiated/sec into a unit area. If we multiply $\langle \vec{S} \rangle$ by $r^2 \hat{r}$, we obtain the power radiated into a unit solid angle $\frac{dP}{d\Lambda}$.

$$\therefore \frac{dP}{d\Lambda} = \frac{I_0^2}{2\pi c} \left(\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin\theta} \right)^2$$

The angular distribution of power radiation therefore strongly depends on value of $\frac{kd}{2}$. We consider situations where $\frac{kd}{2} = \frac{m\pi}{2}$ $m \in N$.

The resulting current distributions in the antenna are shown below for $m=1,2,3,4$.



Solid and dashed lines are currents in different halfs of current cycle.

$$\frac{dP}{d\Omega} = \frac{I_0^2}{2\pi c} \left(\frac{\cos\left(\frac{m\pi}{2}\cos\theta\right) - \cos\left(\frac{m\pi}{2}\right)}{\sin\theta} \right)^2$$

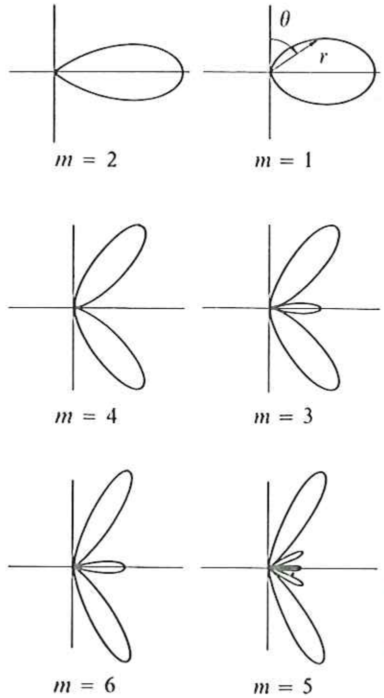
For the so called half-wave case ($m=1$)

$$\frac{dP}{d\Omega} = \frac{I_0^2}{2\pi c} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

For the so called full-wave case ($m=2$)

$$\frac{dP}{d\Omega} = \frac{2 I_0^2}{\pi c} \frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

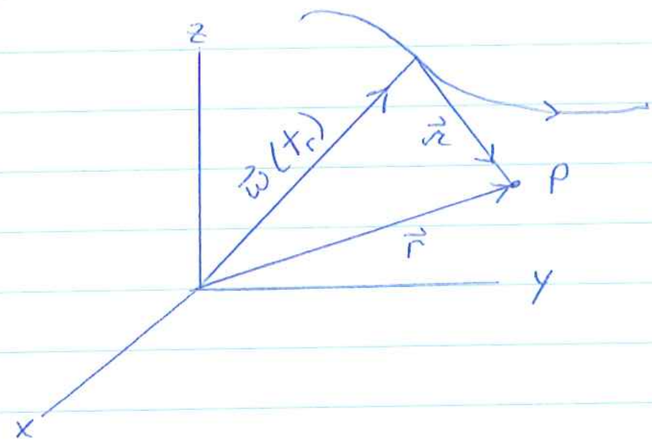
$\frac{dP}{d\Omega}$ for $m=1,2 \ldots 6$ are shown in the next figure.



The most intense radiation lobe is found nearest the antenna. As $m \to \infty$, this lobe points along the antenna direction.

Lienard - Wiechert Potentials

We wish to calculate the fields of a moving point charge. Consider a point charge q moving along a path $\vec{w}(t)$.



$$\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$$

We are interested in the field at point P. The potentials $\Phi(\vec{r},t)$ & $\vec{A}(\vec{r},t)$ depend on the arrangement of the charge or current distribution at the earlier time $t_r = t - \frac{\mathscr{r}}{c}$.

i.e. $$\Phi(\vec{r},t) = \int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}\, d^3r'$$

where $\mathscr{r} = |\vec{r}-\vec{r}'|$, $t_r = t - |\vec{r}-\vec{r}'|/c$. For a point charge $\vec{r}'$ is replaced by $\vec{w}(t_r)$.

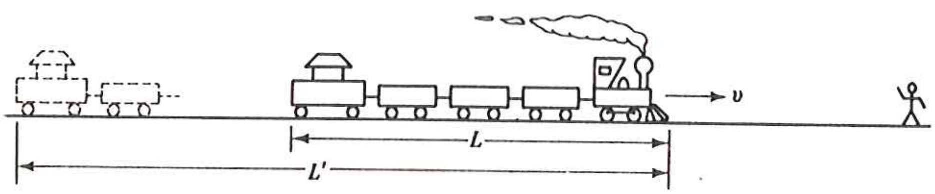
The denominator $\mathscr{r} = |\vec{r}-\vec{r}'| = |\vec{r}-\vec{w}|$ can be taken outside the integral.

$$\Phi(\vec{r},t) = \frac{1}{\mathscr{r}} \int \rho(\vec{r}',t_r)\, d^3r'$$

For a particle at rest, the integral of charge density is simply the total charge q. However, when the charge moves, the integral is greater than q because moving objects appear to be larger than when they are at rest.

Moving Objects

Consider a train of length L moving towards an observer. The observer sees the whole train at once. The light he sees from the caboose has travelled farther to reach him than light coming from the engine.



The ~~extra~~ distance L' is travelled by light in a time $\frac{L'}{c}$. During this time interval, the train moves a distance $L'-L$ at speed V.

$$\therefore \quad \frac{L'}{c} = \frac{L'-L}{V}$$

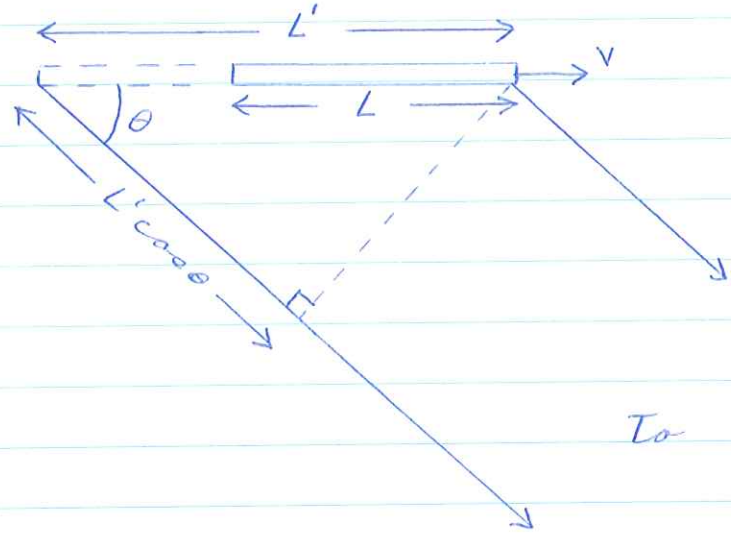
$$L'\left(\frac{1}{c} - \frac{1}{V}\right) = -\frac{L}{V}$$

$$L' = \frac{-L/V}{\frac{1}{c} - \frac{1}{V}}$$

$$\boxed{L' = \frac{L}{1 - \frac{V}{c}}}$$

An observer not knowing the train is moving thinks the train has length L' which is larger than its actual length L by $\left(1 - \frac{v}{c}\right)^{-1}$.

Next suppose $\vec{v}$ makes an angle θ with line of sight of a distant observer. (By distant we mean θ is constant)



To reach observer light from caboose travels an extra distance $L' \cos\theta$ in a time $\frac{L'\cos\theta}{c}$. During this time the train moves distance $L' - L$ at speed v.

$$\frac{L'\cos\theta}{c} = \frac{L' - L}{v}$$

$$L' = \frac{L}{1 - \frac{v}{c}\cos\theta}$$

$$\boxed{L' = \frac{L}{1 - \frac{\vec{v}}{c}\cdot\hat{n}}}$$

where $\hat{n}$ is a unit vector from the train to the observer.

If $\theta = \frac{\pi}{2}$, $L' = L$. Therefore dimensions perpendicular to motion are not distorted.

When measuring volumes, 3 lengths are measured, but only one is distorted.

$\therefore$ apparent volume $V' = \dfrac{V}{1 - \frac{\hat{r}\cdot\vec{v}}{c}}$ V = actual volume

Evaluation of $\int \rho(\vec{r}', t_r)\, d^3r'$

To evaluate this integral for a point charge we let the point charge have volume V and later take the limit $V \to 0$.

$\therefore \rho = \dfrac{q}{V}$ for "point charge".

$$\int \rho(\vec{r}', t_r)\, d^3r' = \int \frac{q}{V} (d^3r')_{\text{moving charge}}$$

$$= \int \frac{q}{V} \frac{(d^3r')_{\text{charge at rest}}}{1 - \frac{\hat{r}\cdot\vec{v}}{c}}$$

$$= \frac{q}{1 - \frac{\hat{r}\cdot\vec{v}}{c}} \frac{1}{V} \int d^3r'$$

$$\int \rho(\vec{r}', t_r)\, d^3r' = \frac{q}{1 - \frac{\hat{r}\cdot\vec{v}}{c}}$$ which is independent of V.

$\therefore$ potential of moving point charge is

$$\boxed{\Phi(\vec{r}, t) = \frac{q}{r\left(1 - \frac{\hat{r}\cdot\vec{v}}{c}\right)}}$$ $\vec{r} = \vec{r} - \vec{w}(t_r)$ and $\vec{v} = \vec{v}(t_r)$

Vector Potential

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int \frac{\vec{J}(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}\,d^3r' \qquad t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$$

$$= \frac{1}{c}\int \frac{\rho(\vec{r}\,',t_r)\,\vec{v}(t_r)}{|\vec{r}-\vec{r}\,'|}\,d^3r'$$

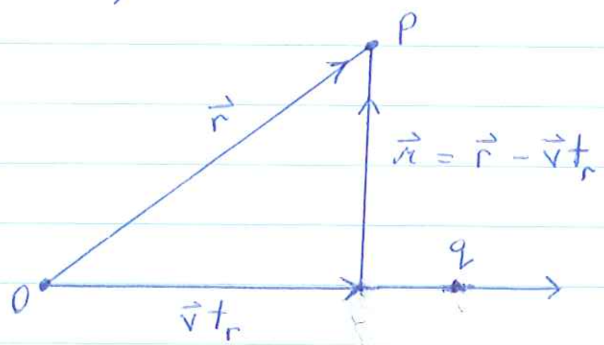
$$= \frac{1}{c}\,\frac{1}{\imath}\,\vec{v}(t_r)\int \rho(\vec{r}\,',t_r)\,d^3r'$$

$$\boxed{\vec{A}(\vec{r},t) = \frac{\vec{v}(t_r)}{c}\,\Phi(\vec{r},t)}$$

The expressions for $\Phi(\vec{r},t)$ & $\vec{A}(\vec{r},t)$ are called the Liénard - Wiechert potentials for a moving point charge.

Example

Find the potential of a point charge moving with constant velocity. For convenience we let particle pass through origin at $t=0$. $\therefore\ \vec{w}(t) = \vec{v}t$



Potential $\quad \Phi(\vec{r},t) = \dfrac{q}{\imath\left(1 - \dfrac{\hat{\imath}\cdot\vec{v}}{c}\right)}$ i.e. We aim to express the right side as a function of $\vec{r}$ & t.

First we compute the retarded time t_r.

$t - t_r$ = time for light to travel from particle to reach observer at time t & location P.

$\therefore$ distance light travels $|\vec{\mathscr{r}}| = c(t - t_r) = |\vec{r} - \vec{v}t_r|$

Squaring this result gives $c^2(t-t_r)^2 = (\vec{r} - \vec{v}t_r)\cdot(\vec{r} - \vec{v}t_r)$

$$c^2(t^2 - 2tt_r + t_r^2) = r^2 - 2\vec{r}\cdot\vec{v}\,t_r + v^2 t_r^2$$

$$t_r^2(c^2 - v^2) + t_r(2\vec{r}\cdot\vec{v} - 2tc^2) + (c^2t^2 - r^2) = 0.$$

$$t_r = \frac{-(2\vec{r}\cdot\vec{v} - 2tc^2) \pm \sqrt{(2\vec{r}\cdot\vec{v} - 2tc^2)^2 - 4(c^2 - v^2)(c^2t^2 - r^2)}}{2(c^2 - v^2)}$$

$$t_r = \frac{c^2t - \vec{r}\cdot\vec{v} \pm \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

To check which sign should be used, consider the case $\vec{v} = 0$.

$$\Rightarrow t_r = \frac{c^2t \pm \sqrt{c^4t^2 + c^2r^2 - c^4t^2}}{c^2}$$

$$= t \pm \frac{r}{c}$$

Since $t_r < t$, we want the negative sign.

$$\therefore t_r = \frac{c^2t - \vec{r}\cdot\vec{v} - \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

Now $r = c(t - t_r)$ and $\hat{r} = \dfrac{\vec{r} - \vec{v}\,t_r}{c(t-t_r)}$

$$\Rightarrow r\left(1 - \frac{\hat{r}\cdot\vec{v}}{c}\right) = c(t-t_r)\left[1 - \frac{\vec{v}}{c}\cdot\frac{(\vec{r} - \vec{v}\,t_r)}{c(t-t_r)}\right]$$

$$= c(t-t_r) - \frac{\vec{v}}{c}\cdot\vec{r} + \frac{v^2}{c}t_r$$

$$= \frac{1}{c}\left[(c^2 t - \vec{r}\cdot\vec{v}) - t_r(c^2 - v^2)\right]$$

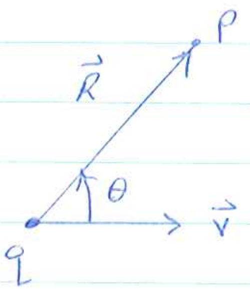
$$= \frac{1}{c}\sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}$$

using the expression for t_r

$$\therefore \boxed{\Phi(\vec{r},t) = \frac{q\,c}{\sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}}$$

Exercise: Show $\Phi(\vec{r},t) = \dfrac{q}{R\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}}$

where $\vec{R} \equiv \vec{r} - \vec{v}\,t$ is vector from present position of particle to observer and θ is angle between $\vec{R}$ & $\vec{v}$.



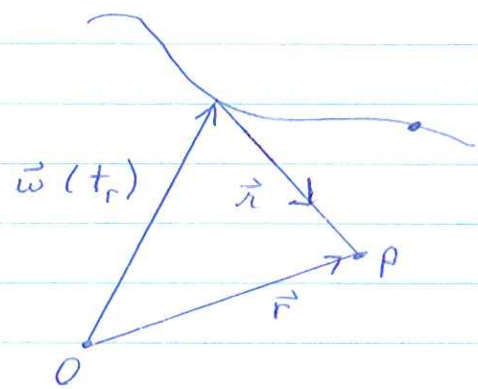
Point Charges in Motion

Potentials of a point charge in motion along path $\vec{w}(t)$ are:

$$\Phi(\vec{r},t) = \frac{q}{\imath\left(1 - \frac{\hat{\imath}\cdot\vec{v}}{c}\right)}$$

$$\vec{A}(\vec{r},t) = \frac{\vec{v}}{c}\,\Phi(\vec{r},t)$$

where $\vec{v} = \vec{v}(t_r)$, $t_r \equiv t - \frac{\imath}{c}$ and $\vec{\imath} \equiv \vec{r} - \vec{w}(t_r)$.



The fields are found using:

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

$$\vec{B} = \nabla\times\vec{A}$$

After tons of algebra (see Griffith pg. 370-372) one gets the following.

$$\vec{E}(\vec{r},t) = q\,\frac{\imath}{(\vec{\imath}\cdot\vec{u})^3}\left[\vec{u}(c^2 - v^2) + \vec{\imath}\times(\vec{u}\times\vec{a})\right]$$

$\vec{B}(\vec{r},t) = \hat{\imath}\times\vec{E}$ where $\vec{u} = c\hat{\imath} - \vec{v}(t_r)$ and $\vec{a} = \vec{a}(t_r)$ is the acceleration. Note the magnetic field is always perpendicular to the electric field and to vector $\vec{r}$.

Let's consider the electric field for a charge at rest.

$\vec{v} = 0 \implies \vec{u} = c\hat{r}$ & $\vec{a} = 0$

Then $$\vec{E}(\vec{r},t) = q \frac{r}{(\vec{r}\cdot c\hat{r})^3} c\hat{r}\, c^2$$

$$= q\frac{r}{r^3}\hat{r}$$

$$= \frac{q}{r^2}\hat{r}$$

Therefore the first term of the electric field is called the generalized Coulomb field. It is also called sometimes the velocity field since it is independent of the acceleration.

The second term which depends on the acceleration is called the acceleration field. It falls off as $1/r$ while the first term falls off as $1/r^2$. It therefore predominates at large distances and is called the radiation field.

Example

Find electric and magnetic fields of a point charge moving with constant velocity.

i.e. charge follows path $\vec{w} = \vec{v}t$ where $\vec{a} = 0$.

$$\therefore \vec{E}(\vec{r},t) = q\frac{(c^2 - v^2)}{(\vec{r}\cdot\vec{u})^3} r\,\vec{u}$$

$$\mathscr{r}\vec{u} = \mathscr{r}(c\hat{\mathscr{r}} - \vec{v})$$
$$= c\vec{\mathscr{r}} - \mathscr{r}\vec{v}$$
$$= c(\vec{r} - \vec{w}(t_r)) - c(t - t_r)\vec{v}$$ using definitions of $\vec{\mathscr{r}}$ and t_r.
$$= c(\vec{r} - \vec{v}t_r) - c(t - t_r)\vec{v}$$
$$= c(\vec{r} - \vec{v}t)$$
$$= c\vec{R} \quad \text{where } \vec{R} \equiv \vec{r} - \vec{v}t$$

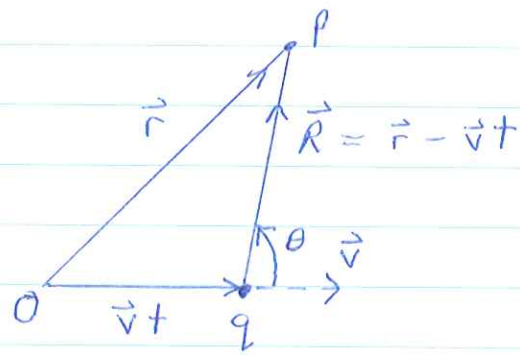
$\vec{R}$ is vector from the present charge position at time t to the observer.

$$\vec{\mathscr{r}} \cdot \vec{u} = \vec{\mathscr{r}} \cdot (c\hat{\mathscr{r}} - \vec{v})$$

$$= \mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{v}$$

$$= \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}$$

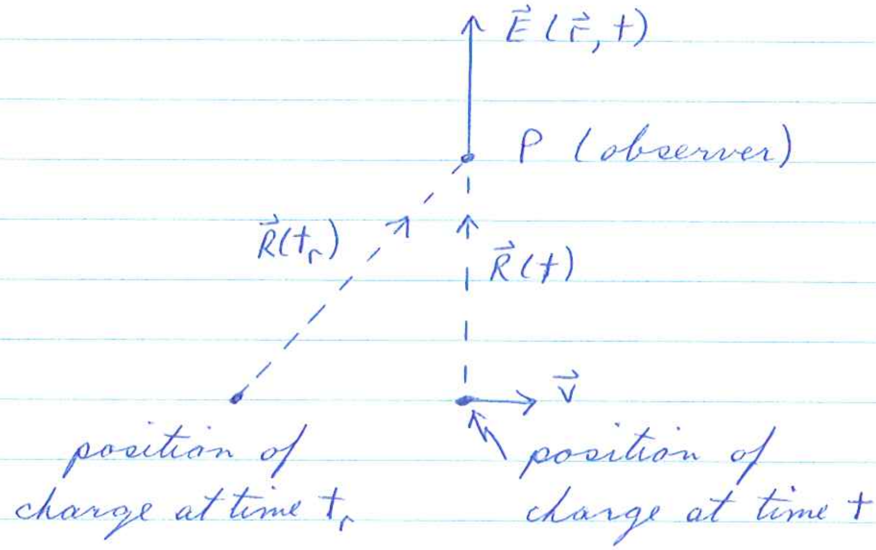
$$= cR\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}$$ as was shown in the previous lecture



$$\therefore \vec{E}(\vec{r},t) = q\,\frac{(c^2 - v^2)}{c^3R^3\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\,c\vec{R}$$

$$\boxed{\vec{E}(\vec{r},t) = q\,\frac{\hat{R}}{R^2}\,\frac{(1 - v^2/c^2)}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}}$$

$\vec{E}$ points in $\hat{R}$ direction, the vector to P from its present position, even though the message originated from the retarded position!



Furthermore $\vec{E}$ is not spherically symmetric as is the case for a particle at rest. It is reduced along the axis of motion as shown in figure on next page.

Magnetic Field $\vec{B} = \hat{\mathscr{r}} \times \vec{E}$

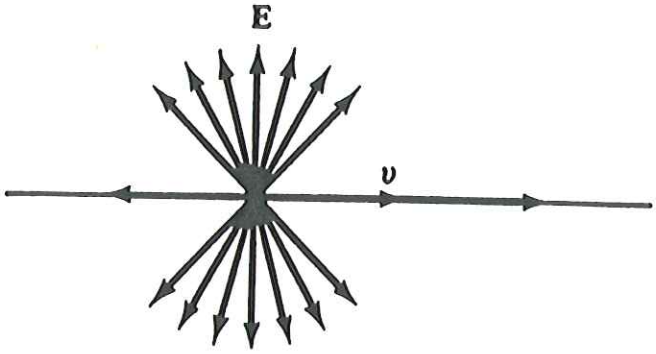
$$\hat{\mathscr{r}} = \frac{\vec{r} - \vec{v}t_r}{\mathscr{r}}$$

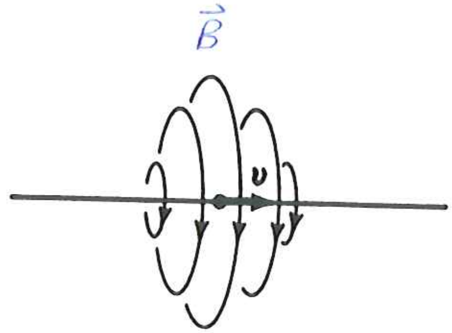
$$= \frac{\vec{r} - \vec{v}t}{\mathscr{r}} + \frac{\vec{v}(t-t_r)}{\mathscr{r}}$$

$$= \frac{\vec{R}}{\mathscr{r}} + \frac{\vec{v}(t-t_r)}{c(t-t_r)}$$

$$= \frac{\vec{R}}{\mathscr{r}} + \frac{\vec{v}}{c}$$

$$\therefore \boxed{\vec{B} = \frac{\vec{v}}{c} \times \vec{E}}$$





Flow of Energy

Energy passing through unit area per second is given by Poynting vector.

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

$$= \frac{c}{4\pi} \vec{E} \times (\hat{r} \times \vec{E})$$

$$= \frac{c}{4\pi} \left[\hat{r} E^2 - \vec{E} (\hat{r} \cdot \vec{E}) \right]$$

Electric field $\vec{E} = q \frac{r}{(\vec{r} \cdot \vec{u})^3} \left[\vec{u} (c^2 - v^2) + \vec{r} \times (\vec{u} \times \vec{a}) \right]$

where $\vec{u} \equiv c\hat{r} - \vec{v}$.

Only the second term contributes to radiation since first term $\propto \frac{1}{r^2}$ while second $\propto \frac{1}{r}$.

$$\therefore \vec{E}_{rad} = q \frac{r}{(\vec{r} \cdot \vec{u})^3} (\vec{r} \times (\vec{u} \times \vec{a}))$$

$$\hat{r} \cdot \vec{E}_{rad} = 0$$

$$\therefore \vec{S}_{rad} = \frac{c}{4\pi} \hat{r} \; E^2_{rad}$$

We consider the case where the charge is instantaneously at rest at time t_r. i.e. $\vec{v}(t_r)=0$. $\Rightarrow \vec{u} = c\,\hat{\mathscr{r}}$

$$\vec{E}_{rad} = q\,\frac{\mathscr{r}}{c^3\mathscr{r}^3}\,\vec{\mathscr{r}}\times(c\,\hat{\mathscr{r}}\times\vec{a})$$

$$= \frac{q}{c^2\mathscr{r}}\,\hat{\mathscr{r}}\times(\hat{\mathscr{r}}\times\vec{a})$$

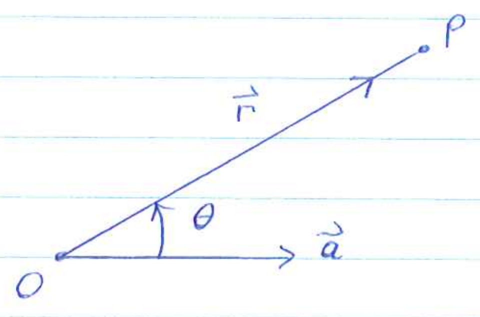
$$= \frac{q}{c^2\mathscr{r}}\left[\hat{\mathscr{r}}\,(\hat{\mathscr{r}}\cdot\vec{a})-\vec{a}\right]$$

$$\therefore\ \vec{S}_{rad} = \hat{\mathscr{r}}\,\frac{c}{4\pi}\,\frac{q^2}{c^4\mathscr{r}^2}\left[\hat{\mathscr{r}}\,(\hat{\mathscr{r}}\cdot\vec{a})-\vec{a}\right]\cdot\left[\hat{\mathscr{r}}\,(\hat{\mathscr{r}}\cdot\vec{a})-\vec{a}\right]$$

$$= \hat{\mathscr{r}}\,\frac{q^2}{4\pi c^3\mathscr{r}^2}\left(a^2-(\hat{\mathscr{r}}\cdot\vec{a})^2\right).$$

Exercise: Show $\vec{r} = \vec{\mathscr{r}}$ if $\vec{v}(t_r)=0$.

$$\therefore\ \vec{S}_{rad} = \frac{q^2}{4\pi c^3}\,\frac{\hat{r}}{r^2}\left(a^2-(\hat{r}\cdot\vec{a})^2\right)$$



Defining θ to be the angle between $\vec{r}$ + $\vec{a}$, we can write:

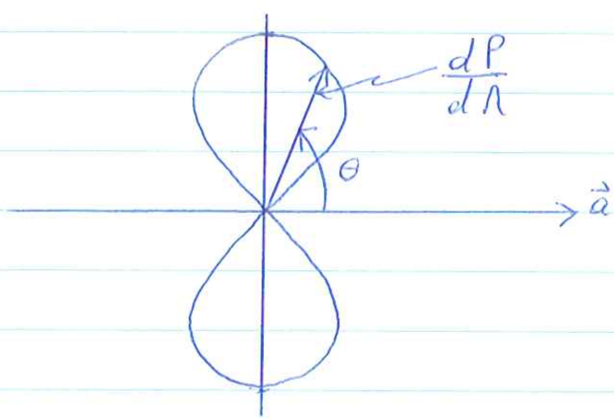
$$\vec{S}_{rad} = \frac{q^2}{4\pi c^3} \frac{\hat{r}}{r^2} \left(a^2 - a^2 \cos^2\theta \right)$$

$$= \frac{q^2 a^2}{4\pi c^3} \frac{\sin^2\theta}{r^2} \hat{r}$$

Power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = r^2 \hat{r} \cdot \vec{S}_{rad}$$

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \sin^2\theta}$$



∴ no energy is radiated along the acceleration axis.

Total power radiated is $P = \int \frac{dP}{d\Omega} \, d\Omega$

$$P = \frac{q^2 a^2}{4\pi c^3} \int \sin^2\theta \sin\theta \, d\theta \, d\phi$$

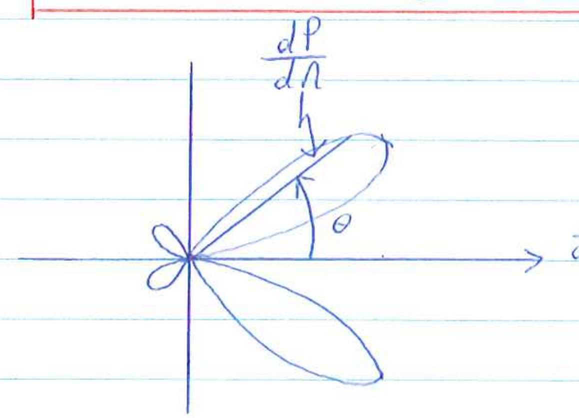
$$P = \frac{q^2 a^2}{4\pi c^3} \, 2\pi \int_0^{\pi} \sin^3\theta \, d\theta$$

$$\boxed{P = \frac{2}{3} \frac{q^2 a^2}{c^3}}$$

This is known as the Larmor radiation formula.

For the general case where the particle velocity is not zero, one can show the following.

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \frac{a^2 \sin^2\theta}{\left(1 - \frac{v}{c}\cos\theta\right)^5}}$$



As $v \to c$, the radiation becomes increasingly concentrated in the forward direction $\vec{a}$.

Total power $P = \frac{2}{3} \frac{q^2 a^2}{c^3} \gamma^6$ where $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

In the low velocity limit $v \to c$, this reduces to Larmor formula.

Therefore radiation is emitted when charges accelerate or decelerate. eg. When fast electrons hit a metal target, X-rays are emitted. This "braking radiation" is better known by the German name bremsstrahlung.

Assignment 5

15 1. Show that $\Phi(\vec{r},t) = \int_V \frac{\rho(\vec{r}', t_r)}{r} d^3r'$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int \frac{\vec{J}(\vec{r}', t_r)}{r} d^3r'$$

satisfy the Lorentz condition. Hint: Proceed as follows.

a) Show $\nabla \cdot \left(\frac{\vec{J}}{r}\right) = \frac{1}{r}(\nabla \cdot \vec{J}) + \frac{1}{r}(\nabla' \cdot \vec{J}) - \nabla' \cdot \left(\frac{\vec{J}}{r}\right)$

where ∇ denotes differentiation w.r.t. $\vec{r}$ & ∇' w.r.t. $\vec{r}'$.

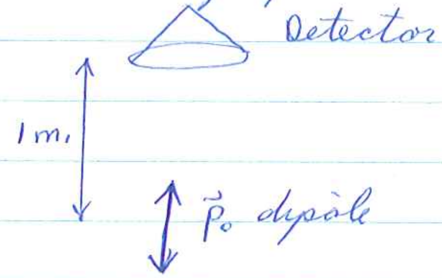
b) Show $\nabla \cdot \vec{J} = -\frac{1}{c}\frac{\partial \vec{J}}{\partial t_r} \cdot (\nabla r)$

c) Show $\nabla' \cdot \vec{J} = -\frac{\partial \rho}{\partial t} - \frac{1}{c}\frac{\partial \vec{J}}{\partial t_r} \cdot (\nabla' r)$

d) Show $\nabla \cdot \vec{A} = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$

10 2a) Discuss the assumptions made when discussing electric dipole radiation.

b) A detector having a radius of 10 cm. is placed 1 meter away from an electric dipole



What fraction of total solid angle does the detector cover and what fraction of total dipole power does it detect (assuming 100% detection efficiency)?

25 3. Consider two half wave antennas each having a current

$$\vec{I}(z,t) = \hat{z}\ I_0 \cos\omega t \sin k\left(\frac{d}{2} - |z|\right) \quad , \ k = \frac{\omega}{c}$$

Each antenna has length d and points in direction $\hat{z}$. Antenna one is at position $\left(\frac{\Delta}{2}, 0, 0\right)$ and the other is at $\left(-\frac{\Delta}{2}, 0, 0\right)$

a) Find the vector potential. $\vec{A}(\vec{r},t)$.

b) Find the electric and magnetic fields.

c) Find $\frac{dP}{d\Omega}$.

d) Evaluate $\frac{dP}{d\Omega}$ in the xy plane when antennae are separated by a distance $\frac{\lambda}{2}$. Along what direction is radiation preferentially emitted?

10 4. A simple model of H is to picture the electron orbiting the proton.

a) How much energy should the electron emit per second if $P = \frac{2}{3}\frac{e^2 a^2}{c^3}$ holds?

b) What is the kinetic energy of H in the ground state?

c) Crudely estimate how long it takes for the electron to radiate away this kinetic energy.

10 5. An accelerator has a current of 1 μamp. of electrons at a speed of .9999 c. The accelerator radius is 1 km.

a) How many electrons are in the accelerator beam?

b) What is the power radiated by the e^- beam?

70

VI. Relativity and Electromagnetism

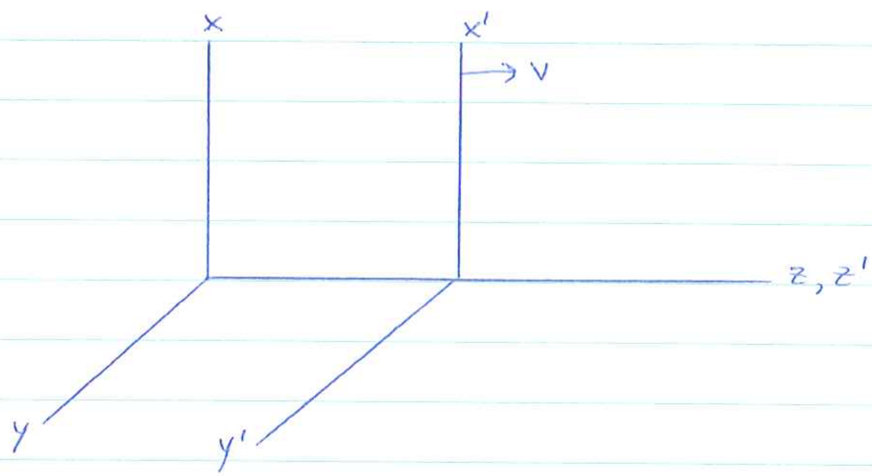
Maxwell's Equations

$$\nabla \cdot \vec{E} = 4\pi\rho \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

What is c?

c is the speed of light. We now consider a stationary observer O and a moving observer O'.



Let x, y, z, t be coordinates at rest for observer O.
" x', y', z', t' " " " O'.

According to Newtonian physics, coordinates in the rest and moving frame are related as follows.

$$x = x'$$
$$y = y'$$
$$z = z' + vt'$$
$$t = t'$$

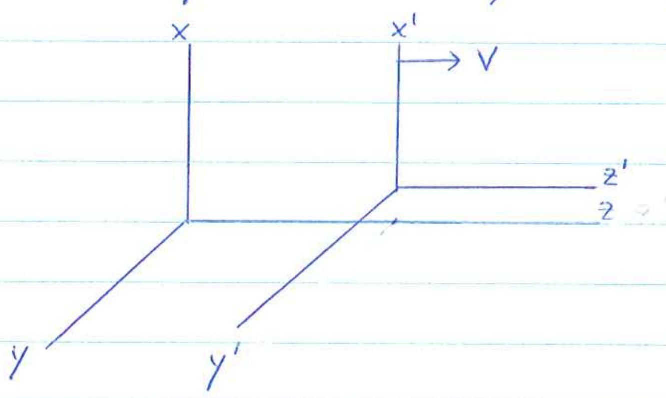
Therefore speeds $\vec{u}$ measured by O and $\vec{u}'$ measured by O' are related as follows.

$$u_x = u'_x$$
$$u_y = u'_y$$
$$u_z = u'_z + v$$

Therefore if O' measures $u'_z = c$ = speed of light, then O should measure $u_z = c + v$ = speed of light.
This last prediction disagrees totally with experiment. Michelson and Morley carefully measured c parallel and perpendicular to the Earths motion (i.e. along E-W and N-S axes) and found the same result.
We therefore conclude that Newtonian physics needs modification.

Relativity

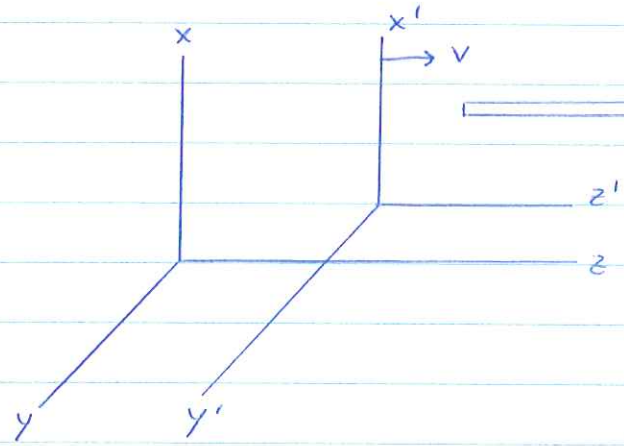
Einstein proposed mechanics such that the speed of light c is the same for all inertial observers. The coordinates in the rest and moving frames are related by the Lorentz transformations.



$$x = x'$$
$$y = y'$$
$$z = \gamma(z' + vt')$$
$$t = \gamma\left(t' + \frac{vz'}{c^2}\right)$$
$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Lorentz Contraction

Consider a rod of length L_0 moving with speed v. An observer O at rest measures the rod length by measuring the rod endpoints at the same time.



Observer O' moving with the rod, measures rod length to be
$$\begin{aligned} L_0 &= z_2' - z_1' \\ &= \gamma(z_2 \mp vt) - \gamma(z_1 \mp vt) \\ &= \gamma(z_2 - z_1) \\ &= \gamma L \end{aligned}$$

Observer O measures rod length to be:
$$L = \frac{L_0}{\gamma} < L_0$$

$\therefore$ moving objects are measured to have a length shorter than their rest length. This is called Lorentz contraction.

Time Dilation

An observer O watches a moving clock. Let O' be an observer in frame of reference where clock is at rest.

clock interval measured by O is: $\Delta t = t_2 - t_1$

$$= \gamma\left(t_2' + \frac{vz'}{c^2}\right) - \gamma\left(t_1' + \frac{vz'}{c^2}\right)$$

$$= \gamma(t_2' - t_1')$$

$$\therefore \Delta t = \gamma \Delta t' > \Delta t'$$

$\therefore$ moving clock ticks slower than clock at rest.

Velocities

Velocities $\vec{u}$ measured by O and $\vec{u}'$ measured by O' are related as follows.

$$u_x = \frac{u_x'}{\gamma\left(1 + \frac{vu_z'}{c^2}\right)}$$

$$u_y = \frac{u_y'}{\gamma\left(1 + \frac{vu_z'}{c^2}\right)}$$

$$u_z = \frac{u_z' + v}{1 + \frac{vu_z'}{c^2}}$$

Exercise: Show $u_z = c$ if $u_z' = c$.

Force

Force $\vec{F} \equiv \frac{d\vec{p}}{dt}$ where momentum $\vec{p} \equiv \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$.

Relation between force $\vec{F}$ measured by O and $\vec{F}'$ measured by O' is:

$$F_x = \frac{F_x'}{\gamma\left(1 + \frac{vu_z'}{c^2}\right)}$$

$$F_y = \frac{F_y'}{\gamma\left(1 + \frac{vu_z'}{c^2}\right)}$$

$$F_z = \frac{F_z' + \frac{v}{c^2}(\vec{u}'.\vec{F}')}{1 + \frac{vu_z'}{c^2}}$$

If the particle is at rest in the moving frame $\vec{u}'=0$ and we get the following.

$$\vec{F}_\perp = \frac{\vec{F}_\perp'}{\gamma}$$

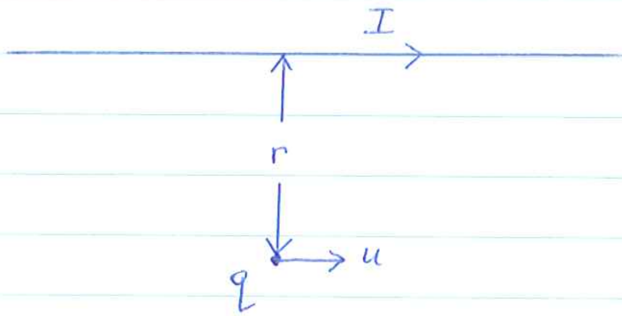
$$F_{\parallel} = F_{\parallel}'$$

$\perp$ is component of $\vec{F}$ perpendicular to $\vec{v}$.
$\parallel$ " " parallel to $\vec{v}$.

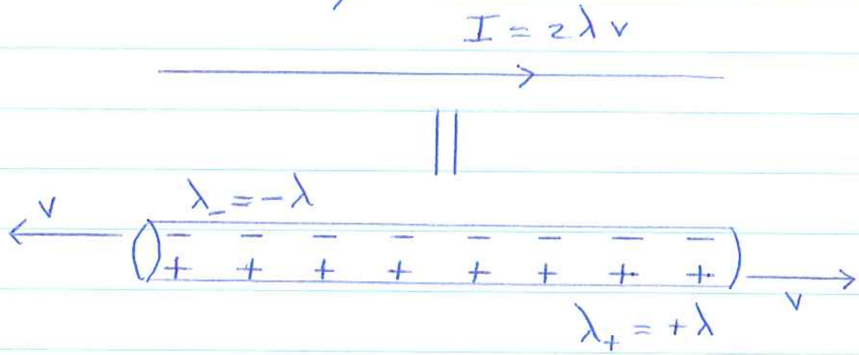
Magnetism

We shall now show that magnetism is a consequence of relativity and electrostatics.

Consider a charge q moving at speed u parallel to a wire carrying a current I.



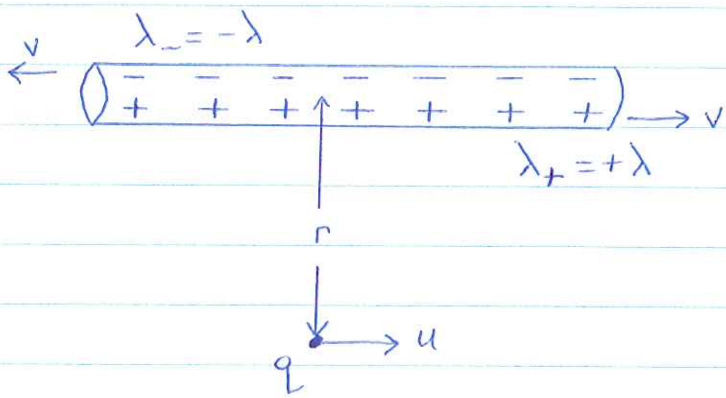
We shall calculate the force on q by first computing force seen in moving frame where q is at rest. To do this we picture the current as consisting of a row of + charges and − charges moving in opposite directions at speed v



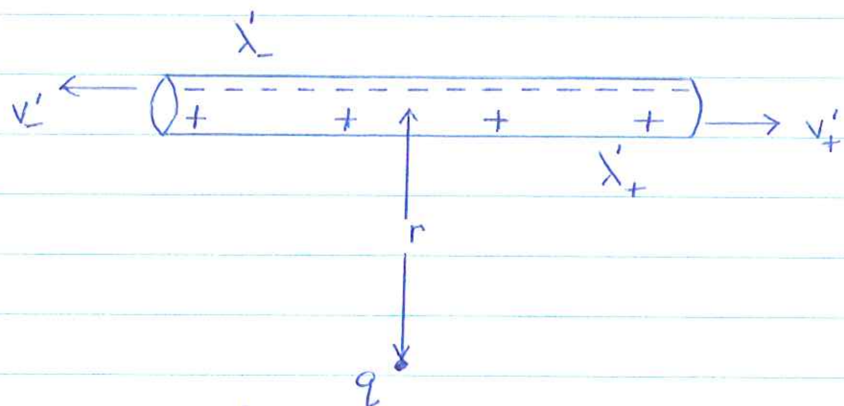
Charge densities $\lambda_\pm$ are equal and opposite since wire has no net charge.

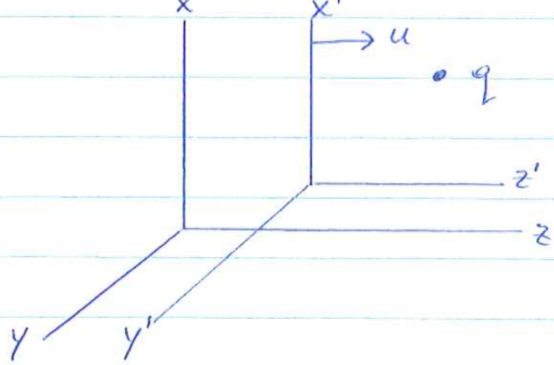
Consider the situation as seen in the following two frames of reference.

Frame O: − & + charges move with equal and opposite speeds



Frame O': Charge q is at rest. + charge moves slower in O' than in O and hence has smaller Lorentz contraction. The reverse holds for the − charges.





$$\therefore\ v'_\pm = \frac{\pm v - u}{1 - \frac{u(\pm v)}{c^2}}$$

$$v'_\pm = \frac{-u \pm v}{1 \mp \frac{uv}{c^2}}$$

Charge Density

Let $+\lambda_0$ be charge density in frame of reference where + charge is at rest. By symmetry $-\lambda_0$ is − charge density in a different frame of reference where − charge is at rest.

The distance between moving charges is measured to be smaller than when charges are at rest. Hence the charge density increases by factor γ.

In frame O $\quad \lambda_{\pm} = \pm\gamma(v)\lambda_0 \equiv \pm\lambda \quad$ where $I = 2\lambda v$

In frame O' $\quad \lambda'_{\pm} = \pm\gamma(v'_{\pm})\lambda_0$

Force on q in Ref. Frame O'

q is at rest $\Rightarrow$ no magnetic force

Net charge density on wire $\lambda'_{TOT} = \lambda'_+ + \lambda'_-$

$$= \gamma(v'_+)\lambda_0 - \gamma(v'_-)\lambda_0$$

$$= \frac{\lambda}{\gamma(v)}\left(\gamma(v'_+) - \gamma(v'_-)\right)$$

Electric field using Gauss law for infinite line of charge density λ'_{TOT} is:

$$E' = \frac{2\lambda'_{TOT}}{r}$$

$$= \frac{2}{r}\frac{\lambda}{\gamma(v)}\left(\gamma(v'_+) - \gamma(v'_-)\right)$$

Force on q in O' frame $F' = qE'$

$$= \frac{2q}{r} \frac{\lambda}{\gamma(v)} \left(\gamma(v'_+) - \gamma(v'_-) \right)$$

($\vec{F}'$ points away from wire if $E' > 0$).

Aside: Here we assume that charge q is constant i.e. independent of the particles speed. This was already implicitly assumed earlier when evaluating charge densities.

Since F' is $\perp$ to motion, force F in O is related to F by:

$$F = \frac{1}{\gamma(u)} F'$$

$$F = \frac{2q\lambda}{r} \frac{1}{\gamma(u)\gamma(v)} \left(\gamma(v'_+) - \gamma(v'_-) \right)$$

Exercise: $\gamma(v'_\pm) = \gamma(v)\gamma(u)\left(1 \mp uv/c^2\right)$

$$\therefore\ F = \frac{2\lambda q}{r} \left[1 - \frac{uv}{c^2} - \left(1 + \frac{uv}{c^2} \right) \right]$$

$$= -\frac{4\lambda q}{r} \frac{uv}{c^2}$$

$$F = -q\left(\frac{u}{c}\right)\left(\frac{2I}{rc}\right)$$

$\vec{F}$ because of $-$ sign pts. toward wire.

This is simply the magnetic force $q\frac{u}{c}B$ where $B = -\frac{2I}{rc}$.

$\therefore$ magnetic fields are result of electrostatics + relativity.

Field Transformations

In the previous lecture we saw that an electric field in one frame of reference appears to be a magnetic field in another reference frame. Today we will learn how to transform fields between reference frames.

Assumptions

1. Charge of a particle is independent of its motion. We also say Charge is invariant.

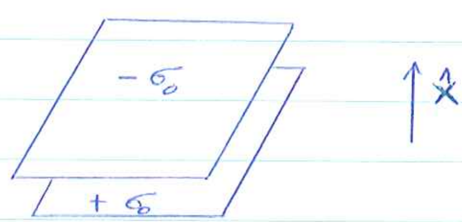
2. The field transformation rules are independent of the way in which the fields were produced. i.e. We are only interested in $\vec{E}$ + $\vec{B}$ at some point, not on cost, machinery, labour etc. needed to create them.

This last assumption allows us to choose convenient charge configurations for which the fields are easily found.

Transformation of $\vec{E}$

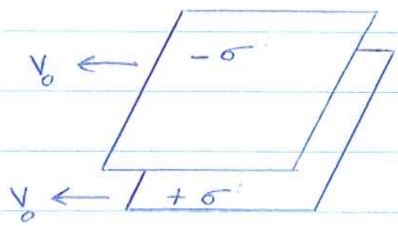
Consider a parallel plate capacitor.

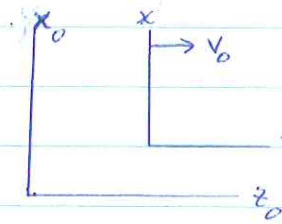
Frame S_0: Capacitor is at rest.



$$\vec{E}_0 = 4\pi \sigma_0 \hat{x}$$

Frame S: Observer is moving with speed $v_0 \hat{z}$.





$$\vec{E} = 4\pi\sigma \, \hat{x}$$

Total charge on plates is same in reference frames S_0 and S. (assumption 1) The plate width is the same in S_0 and S but the length is shorter in S.

$$\therefore \sigma = \gamma(v_0)\sigma_0.$$

$$\vec{E} = 4\pi\sigma_0 \gamma(v_0) \, \hat{x}$$

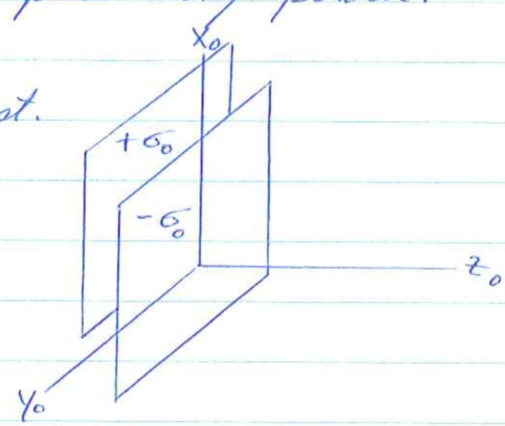
$$\boxed{\vec{E} = \gamma(v_0)\vec{E}_0}$$

Since $\hat{x} \perp$ direction of motion we conclude $\vec{E}_\perp = \gamma \vec{E}_{0\perp}$.

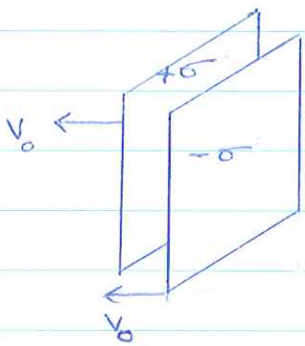
To see how fields parallel to the direction of motion transform, we consider the following plates.

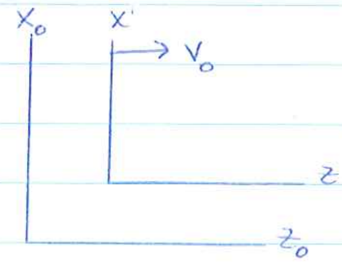
Frame S_0: Capacitor is at rest.

$$\vec{E}_0 = 4\pi\sigma_0 \, \hat{z}$$



Frame S : Observer is moving with speed $v_0\hat{z}$.





$$\vec{E}' = 4\pi\sigma' \hat{z}$$

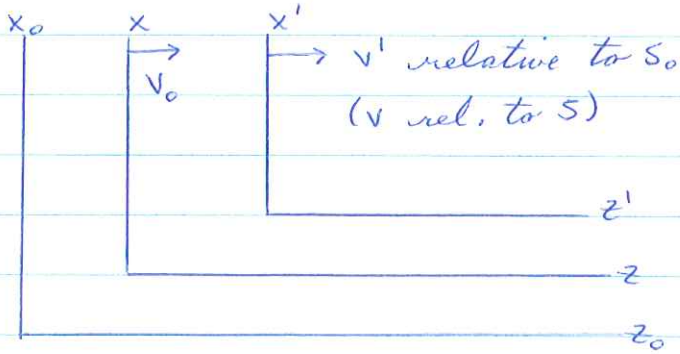
The plate dimensions measured by O_0 and O are the same. $\therefore \sigma' = \sigma_0$.
The plate separation distance is Lorentz contracted but electric field doesn't depend on separation distance.

$$\therefore \vec{E} = 4\pi\sigma_0\hat{z}$$
$$= \vec{E}_0$$

$$\Rightarrow \boxed{E_{\parallel} = E_{0\parallel}}$$

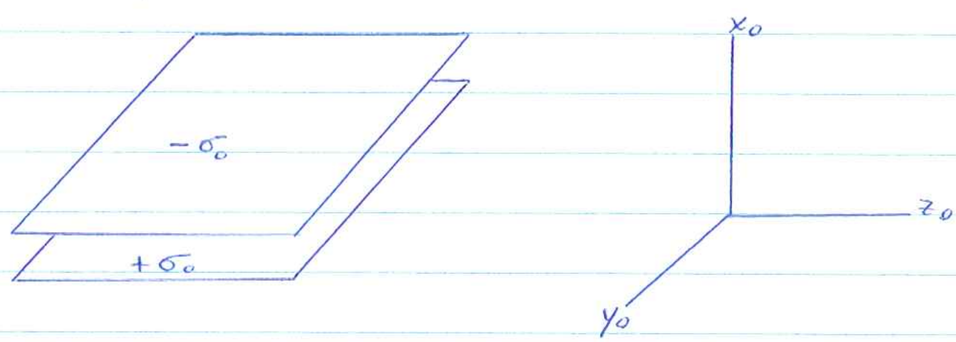
Transformation of Electric & Magnetic Fields

To see how $\vec{E}$ and $\vec{B}$ both transform, we consider a situation where both $\vec{E}$ and $\vec{B}$ are nonzero. We consider the parallel plate capacitor in the rest frame S_o and two moving frames S and S'.



First we consider capacitor with plates parallel to the yz plane.

Frame S_o : Capacitor is at rest.

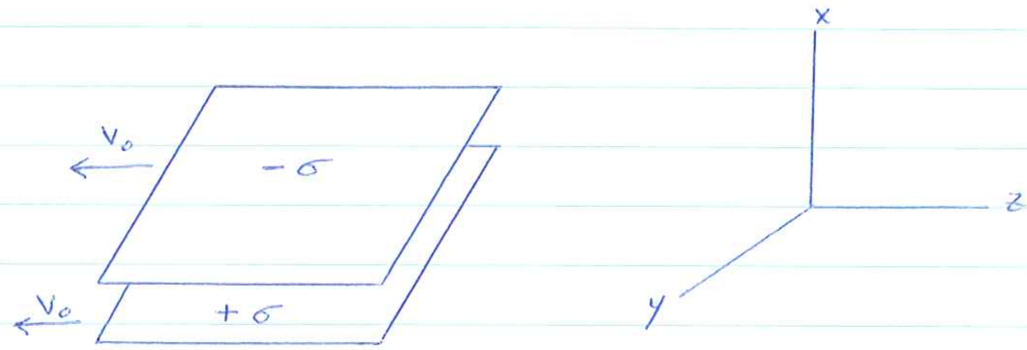


Electric field $\vec{E}_o = E_{ox}\ \hat{x}$

$$= 4\pi\sigma_o\ \hat{x}$$

Magnetic Field $\vec{B}_o = 0$.

Frame S: Observer moves with velocity $\vec{v}_0 = v_0 \hat{z}$ relative to S_0.



Charge Density $\sigma = \gamma(v_0)\,\sigma_0$

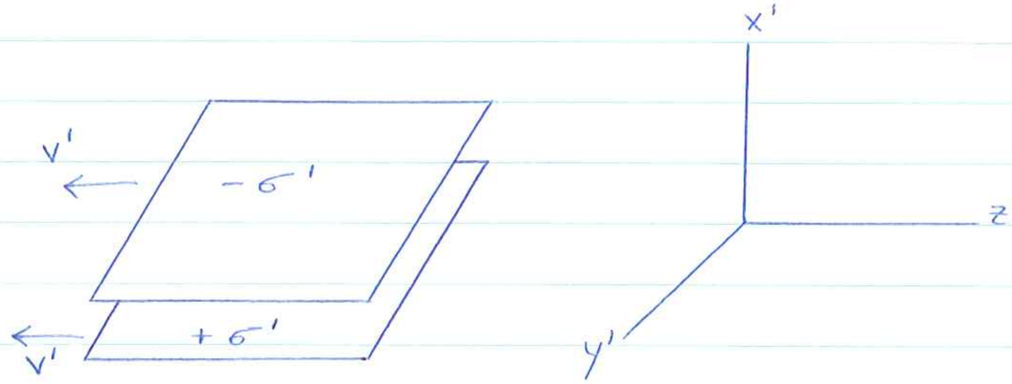
Electric field $\vec{E} = E_x \hat{x}$ where $E_x = 4\pi\sigma$.

Surface Current Density $\vec{K}_\pm = \mp\, \sigma\, v_0\, \hat{z}$

Using Ampere's Law one can show $\vec{K}_\pm$ produces a magnetic field

$$\vec{B} = B_y \hat{y} \quad \text{where} \quad B_y = -4\pi\sigma \frac{v_0}{c}$$

Frame S': Observer moves with velocity $\vec{v}\,\hat{z}$ relative to S or $\vec{v}'\,\hat{z}$ relative to S_0.



Relation Between v_0, v' and v is : $v' = \dfrac{v + v_0}{1 + \frac{v v_0}{c^2}}$

Charge Density $\sigma' = \gamma(v')\sigma_0$

Electric Field $\vec{E}' = E'_x \hat{x}$ where $E'_x = 4\pi\sigma'$

Surface Current Density $\vec{K}'_{\pm} = \mp \sigma' v' \hat{z}$

Magnetic Field $\vec{B}' = B'_y \hat{y}$ where $B'_y = -4\pi\sigma' \frac{v'}{c}$.

We seek to find an expression relating $\left\{\begin{matrix}\vec{E}' \\ \vec{B}'\end{matrix}\right\}$ and $\left\{\begin{matrix}\vec{E} \\ \vec{B}\end{matrix}\right\}$.

We shall need the following identity.

Exercise: Show $\dfrac{\gamma(v')}{\gamma(v_0)} = \gamma(v)\left(1 + \dfrac{v v_0}{c^2}\right)$

Electric Field
$$\begin{aligned} E'_x &= 4\pi\sigma' \\ &= 4\pi\sigma_0\gamma(v') \\ &= 4\pi\sigma\frac{\gamma(v')}{\gamma(v_0)} \\ &= 4\pi\sigma\gamma(v)\left(1 + \frac{v v_0}{c^2}\right) \\ &= \gamma(v)\left(4\pi\sigma + 4\pi\sigma\frac{v_0}{c}\frac{v}{c}\right) \end{aligned}$$

$$\boxed{E'_x = \gamma(v)\left(E_x - \frac{v}{c}B_y\right)}$$

Exercise: Show $\boxed{B'_y = \gamma(v)\left(B_y - \frac{v}{c}E_x\right)}$

Next we consider a capacitor whose plates are parallel to the xz plane. One can show the following:

$$\boxed{\begin{aligned} E'_y &= \gamma(v)\left(E_y + \frac{v}{c} B_x\right) \\ B'_x &= \gamma(v)\left(B_x + \frac{v}{c} E_y\right) \end{aligned}}$$

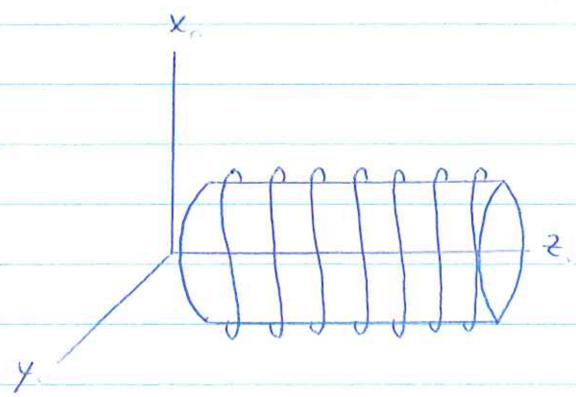
Transformation of Longitudinal Fields

To see how E_z & B_z transform, one could suggest orienting the capacitor plates parallel to xy plane. But as we showed previously the field E_z is unaffected.

i.e. $\boxed{E'_z = E_z}$

To see how B_z transforms consider a solenoid as shown below.

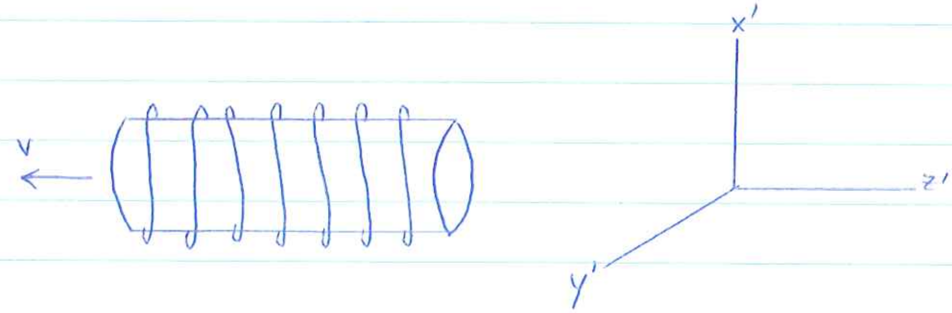
Frame S : Solenoid is stationary



Magnetic Field $\vec{B} = B_z \hat{z}$ where $B_z = 4\pi n \frac{I}{c}$

n = # turns per unit length

Frame S': Observer moves with velocity $\vec{v} = v\hat{z}$.



Distance between coils is Lorentz contracted.

$\therefore$ # turns per unit length $n' = \gamma(v)\, n$.

Clocks in moving frame are slower than in rest frame.

$\therefore$ current $I' = \dfrac{I}{\gamma(v)}$

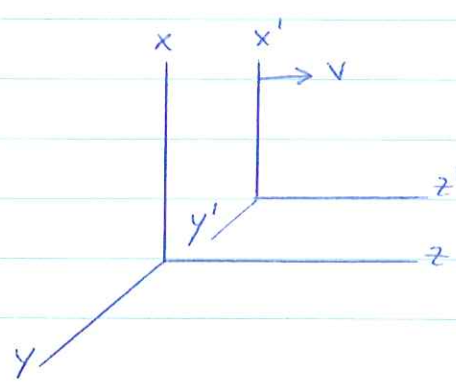
Magnetic field $\vec{B}' = B_z'\,\hat{z}$ where $B_z' = 4\pi n' \dfrac{I'}{c}$

$$= 4\pi n \gamma(v) \frac{1}{c} \frac{I}{\gamma(v)}$$

$$= 4\pi n \frac{I}{c}$$

$$\therefore \boxed{B_z' = B_z}$$

Field Transformation Summary.



$$\boxed{\begin{array}{lll} E_z' = E_z & E_y' = \gamma(v)\left(E_y + \frac{v}{c} B_x\right) & E_x' = \gamma(v)\left(E_x - \frac{v}{c} B_y\right) \\ B_z' = B_z & B_y' = \gamma(v)\left(B_y - \frac{v}{c} E_x\right) & B_x' = \gamma(v)\left(B_x + \frac{v}{c} E_y\right) \end{array}}$$

Assignment 6

1. Derive $E_y' = \gamma(v)\left(E_y + \frac{v}{c} B_x\right)$

$$B_x' = \gamma(v)\left(B_x + \frac{v}{c} E_y\right)$$

by considering a capacitor whose plates are parallel to the xz plane. The capacitor moves in the $\hat{z}$ direction.

2. If $\vec{B}' = 0$ in frame S', show $\vec{B} = \frac{\vec{v} \times \vec{E}}{c}$ in frame S.

Appendix

VECTOR IDENTITIES

TRIPLE PRODUCTS

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_a^b (\nabla f)\cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_{\text{volume}} (\nabla\cdot\mathbf{A})\, d\tau = \int_{\text{surface}} \mathbf{A}\cdot d\mathbf{a}$

Curl Theorem: $\int_{\text{surface}} (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint_{\text{line}} \mathbf{A}\cdot d\mathbf{l}$

VECTOR DERIVATIVES

CARTESIAN. $d\mathbf{l} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$; $d\tau = dx\,dy\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial x}\hat{\imath} + \frac{\partial t}{\partial y}\hat{\jmath} + \frac{\partial t}{\partial z}\hat{k}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$$

Laplacian. $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

SPHERICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl. $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) + \frac{\partial v_r}{\partial \theta}\right]\hat{\phi}$$

Laplacian. $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

CYLINDRICAL $d\mathbf{l} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z}$; $d\tau = r\,dr\,d\phi\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla \times \mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial \phi}\right]\hat{z}$$

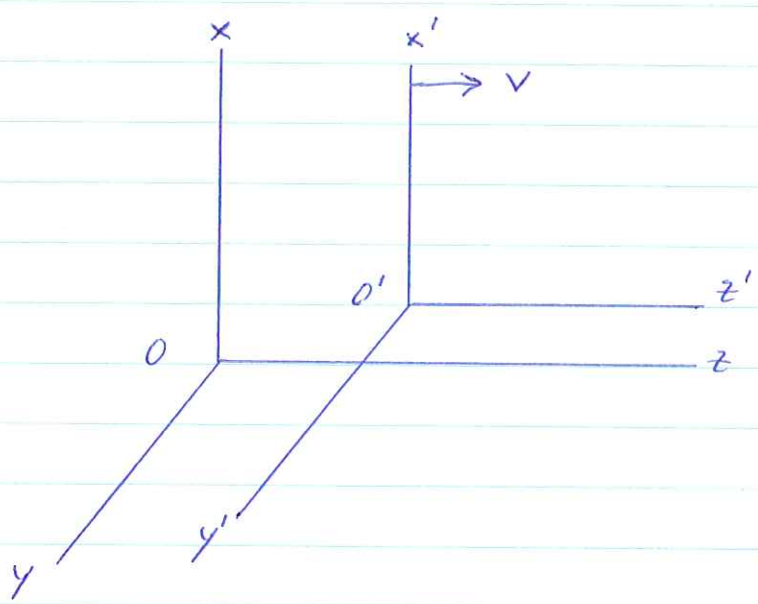
Laplacian. $$\nabla^2 t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Relativistic Electrodynamics

This section introduces the mathematical formalism convenient for special relativity and essential for future study of quantum field theory and general relativity.

Lorentz Transformation

Consider an observer O' moving with speed v relative to observer O.
Let x, y, z, t be the rest frame coordinates.
Let x', y', z', t' " " moving " ".



The coordinates of the rest frame and moving frame are related by the Lorentz transformation.

$$x' = x$$

$$y' = y$$

$$z' = \gamma(z - vt)$$

$$t' = \gamma\left(t - \frac{\beta}{c} z\right)$$

where $\beta \equiv \frac{v}{c}$ and $\gamma \equiv (1-\beta^2)^{-1/2}$

We now define

$$x_1 = x$$
$$x_2 = y$$
$$x_3 = z$$
$$x_4 = ict$$

The Lorentz transformation then becomes:

$$x_1' = x_1$$
$$x_2' = x_2$$
$$x_3' = \gamma(x_3 + i\beta x_4)$$

$$x_4' = \gamma(x_4 - i\beta x_3)$$

OR

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}}_{\equiv \lambda} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

We now define a 4 component vector

$X \equiv (\vec{x}, ict)$ which represents the space and time position of a point. This 4 dimensional space (3 spatial coords. + 1 time coord.) is called Minkowski Space.

The previous equation can then be written as follows.

$$X' = \lambda X$$

or $X'_\mu = \sum_\nu \lambda_{\mu\nu} X_\nu$ where $\mu, \nu = 1, 2, 3, 4$.

Einstein Summation Convention

In relativity theory, summation over repeated indices is assumed unless otherwise stated. Hence the summation sign in the previous equation may be omitted and we get:

$$X'_\mu = \lambda_{\mu\nu} X_\nu$$

4- Vector

In 4 dimensional space, a quantity M is called a 4-vector if its components M_μ transform according to the relation

$$M'_\mu = \lambda_{\mu\nu} M_\nu$$

Example:

The space time position $X = (\vec{x}, ict)$ is a 4-vector. Similarly $dX = (d\vec{x}, ic\,dt)$ is also a 4-vector.

Length of a 4-Vector

The length of a 4-Vector is defined to be $\sqrt{M_\mu M_\mu}$.

Lorentz Invariant

Quantities whose value is unchanged by a Lorentz transformation are called Lorentz invariants.

Theorem

The length of a 4-vector is a Lorentz invariant.

Proof

We need to show $M'_\mu M'_\mu = M_\mu M_\mu$.

$$
\begin{aligned}
M'_\mu M'_\mu &= M_1'^2 + M_2'^2 + M_3'^2 + M_4'^2 \\
&= M_1^2 + M_2^2 + \gamma^2 (M_3 + i\beta M_4)^2 + \gamma^2 (M_4 - i\beta M_3)^2 \\
&= M_1^2 + M_2^2 + \gamma^2 \big[M_3^2 + 2i\beta M_3 M_4 - \beta^2 M_4^2 \\
&\qquad + M_4^2 - 2i\beta M_3 M_4 - \beta^2 M_3^2 \big] \\
&= M_1^2 + M_2^2 + \gamma^2 \left[M_3^2 (1-\beta^2) + M_4^2 (1-\beta^2) \right] \\
&= M_1^2 + M_2^2 + M_3^2 + M_4^2
\end{aligned}
$$

$\therefore M'_\mu M'_\mu = M_\mu M_\mu$

Comment

Therefore we see that the Lorentz transformation does not change the length of a 4-vector. It is analogous to a rotation in 3 dimensional Cartesian space. Hence we say that a Lorentz transformation is a rotation in 4 dimensional Minkowski space. It changes the direction but not the length of a 4-vector.

Examples of Lorentz Invariants

The length of 4-vector dx is an invariant.

$$ds \equiv \sqrt{dx_\mu \, dx_\mu}$$

$$= \sqrt{dx_j \, dx_j - c^2 dt^2}$$

Here we have adopted the convention that Greek indices $\mu, \nu, \delta \ldots$ sum over 1, 2, 3, 4 while Roman indices $i, j, k \ldots$ sum only over the spatial coordinates 1, 2, 3.

Similarly $d\tau \equiv \frac{i}{c} ds$ is a Lorentz invariant.

$d\tau$ is called the element of proper time in Minkowski space.

$$d\tau = \frac{i}{c}\sqrt{dx_j \, dx_j - c^2 dt^2}$$

$$= dt \sqrt{1 - \frac{1}{c^2}\frac{dx_j}{dt}\frac{dx_j}{dt}}$$

$$= dt \sqrt{1 - \frac{u^2}{c^2}} \quad \text{where } \vec{u} = \text{velocity}$$

4-Vector Velocity

$U \equiv \frac{dx}{d\tau}$ is a 4-vector since dx is a 4-vector and $d\tau$ is a Lorentz invariant.

$$U = \left(\frac{d\vec{x}}{d\tau} , ic\frac{dt}{d\tau} \right)$$

$$= \left(\frac{d\vec{x}}{dt\sqrt{1-\frac{u^2}{c^2}}} , \frac{ic}{\sqrt{1-\frac{u^2}{c^2}}} \right)$$

$$= \left(\frac{\vec{u}}{\sqrt{1-u^2/c^2}} , \frac{ic}{\sqrt{1-u^2/c^2}} \right)$$

Exercise: By evaluating $U_\mu U_\mu$, show that it is a Lorentz invariant.

4-Vector Momentum

$P \equiv m_o U$ where m_o is the particles rest mass

$$= \left(\frac{m_o\vec{u}}{\sqrt{1-u^2/c^2}} , \frac{i m_o c}{\sqrt{1-u^2/c^2}} \right)$$

$$= (m\vec{u} , imc)$$

where $m \equiv \frac{m_o}{\sqrt{1-u^2/c^2}}$ is the mass of particle moving with speed u.

Hence we see that the space components of P are just the ordinary momentum $\vec{p} = m\vec{u}$.

Interpretation of P_4.

The force on the particle is given by:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$= \frac{d}{dt}\left(\frac{m_0 \vec{u}}{\sqrt{1-u^2/c^2}}\right)$$

Rate of Change of particle's kinetic energy T is

$$\frac{dT}{dt} = \vec{F} \cdot \vec{u}$$

$$= \vec{u} \cdot \frac{d}{dt}\left(\frac{m_0 \vec{u}}{\sqrt{1-u^2/c^2}}\right)$$

Exercise: a) Evaluate the Lorentz invariant P^2.

b) Differentiate $P^2 = P_\mu P_\mu$ and show that

$$\vec{u} \cdot \frac{d}{dt}\left(\frac{m_0 \vec{u}}{\sqrt{1-u^2/c^2}}\right) = m_0 c^2 \frac{d}{dt}\left(\frac{1}{\sqrt{1-u^2/c^2}}\right)$$

$$\therefore \frac{dT}{dt} = m_0 c^2 \frac{d}{dt}\left(\frac{1}{\sqrt{1-u^2/c^2}}\right)$$

Integrating from time when particle is at rest until particle has velocity $\vec{u}$ we get:

$$T = m_0 c^2 \left. \frac{1}{\sqrt{1-u^2/c^2}} \right|_0^u$$

$$= m_0 c^2 \left(\frac{1}{\sqrt{1-u^2/c^2}} - 1 \right)$$

$\therefore$ Kinetic Energy $T = mc^2 - m_0 c^2$

mc^2 is an energy associated with a moving particle.
$m_0 c^2$ " " particle at rest.

$\Rightarrow$ we interpret Total Energy $E \equiv mc^2$

$$= \underset{\text{Kinetic Energy}}{T} + \underset{\text{rest energy}}{m_0 c^2}$$

Therefore $p_4 = \frac{iE}{c}$ and the 4-vector momentum can be written as: $p = \left(m\vec{u}, \frac{iE}{c} \right)$.

4-Vectors In Electrodynamics

Electric Charge

The charge of a particle is experimentally found to be Lorentz invariant.

eg. Charge of electron at rest = Charge of moving electron in atom

Charge Density

We next consider the volume and charge density of a particle as measured by an observer:

1) at rest with respect to the charge

2) moving with respect to the charge

	Rest Frame	Moving Frame
Charge	q	q
Volume	V_0	$V = \frac{V_0}{\gamma}$
Charge Density	$\rho_0 = \frac{q}{V_0}$	$\rho = \frac{q}{V} = \gamma \rho_0$

The volume is smaller in the moving frame by a factor γ due to Lorentz contraction along the direction of motion. Hence a moving charge has a larger charge density than when the charge is at rest.

4- Vector Charge Current Density

Current Density $\vec{J} = \rho \vec{u}$

We now define $J \equiv (\vec{J}, ic\rho)$.

$$= (\rho \vec{u}, ic\rho)$$

$$= \rho_0 \left(\frac{\vec{u}}{\sqrt{1-u^2/c^2}}, \frac{ic}{\sqrt{1-u^2/c^2}} \right)$$

$$\therefore J = \rho_0 U$$

$\therefore J$ is a 4- vector since ρ_0 is Lorentz invariant and U is a 4- vector.

Continuity Equation

We shall now use the 4-vector notation to express the continuity equation.

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0.$$

$$\frac{\partial J_j}{\partial x_j} + \frac{\partial (ic\rho)}{\partial (ict)} = 0$$

$$\frac{\partial J_j}{\partial x_j} + \frac{\partial J_4}{\partial x_4} = 0$$

$$\boxed{\frac{\partial J_\mu}{\partial x_\mu} = 0}$$

This is also written as:

$$\Box \cdot J = 0$$

where $\Box \equiv \left(\nabla , \frac{\partial}{\partial x_4} \right)$ is a 4 dimensional operator.

Exercise: 1) Show that $\Box$ is a 4-vector.

2) $\Box^2 \equiv \frac{\partial^2}{\partial x_\mu \partial x_\mu}$ is called the D'Alembertian

a) Show that $\Box^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$.

b) Prove $\Box^2$ is Lorentz invariant by showing that $\frac{\partial^2}{\partial x_\mu \partial x_\mu} = \frac{\partial^2}{\partial x'_\mu \partial x'_\mu}$.

Lorentz Gauge

$$\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0$$

$\vec{A}$ = vector potential
Φ = scalar potential

Defining $A \equiv (\vec{A}, i\Phi)$ we can rewrite this equation as:

$$\boxed{\Box \cdot A = 0}$$

Exercise: Prove this.

We shall next show that A is a 4-vector. Recall that $\vec{A}$ and Φ satisfy the following inhomogeneous wave equations.

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}$$

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -4\pi\rho$$

Using the d'Alembertian we get:

$$\Box^2 \vec{A} = -\frac{4\pi}{c}\vec{J} \qquad (1)$$

$$\Box^2 \Phi = -4\pi\rho.$$

Using the definitions of A_4 and J_4, the last equation becomes:

$$\Box^2 \frac{A_4}{i} = -4\pi \frac{J_4}{ic}$$

or $\Box^2 A_4 = -\frac{4\pi}{c} J_4$ (2)

Combining (1) and (2) we obtain:

$$\boxed{\Box^2 A = -\frac{4\pi}{c} J}$$

$\therefore$ A is a 4-vector since $\Box^2$ is a Lorentz invariant and J is a 4-vector.

Electromagnetic Field Tensor

$$F_{\mu\nu} \equiv \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

We shall state later why $F_{\mu\nu}$ is a tensor or what a tensor actually is.

Exercise: Prove that $F_{\mu\nu}$ is antisymmetric.

i.e. $F_{\mu\nu} = -F_{\nu\mu}$

Hence diagonal elements $F_{\mu\mu} = 0$.

Evaluation of $F_{\mu\nu}$

Recall the relations between the electric and magnetic fields and the scalar and vector potentials.

$$\vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$F_{12} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$$
$$= (\nabla \times \vec{A})_3$$
$$= B_3$$

$$F_{13} = \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3}$$
$$= -(\nabla \times \vec{A})_2$$
$$= -B_2$$

$$F_{14} = \frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4}$$
$$= i\frac{\partial \Phi}{\partial x_1} - \frac{1}{ic}\frac{\partial A_1}{\partial t}$$

$$= -i\left[-\frac{\partial \Phi}{\partial x_1} - \frac{1}{c}\frac{\partial A_1}{\partial t}\right]$$
$$= -iE_1$$

Exercise: Show $F_{23} = B_1$, $F_{24} = -iE_2$, $F_{34} = -iE_3$.

$$\therefore \quad F = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}$$

Maxwell Equations in Terms of EM Field Tensor

Theorem 1

$$\frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0$$

Proof:

$$L.S. = \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\mu}{\partial x_\lambda} - \frac{\partial A_\lambda}{\partial x_\mu}\right) + \frac{\partial}{\partial x_\lambda}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) + \frac{\partial}{\partial x_\mu}\left(\frac{\partial A_\lambda}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\lambda}\right)$$

$$= \frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\lambda} - \frac{\partial^2 A_\lambda}{\partial x_\nu \partial x_\mu} + \frac{\partial^2 A_\nu}{\partial x_\lambda \partial x_\mu} - \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} + \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu} - \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\lambda}$$

$$= 0.$$

Theorem 2

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} J_\mu$$

Proof:

$$L.S. = \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right)$$

$$= \frac{\partial}{\partial x_\mu}\left(\frac{\partial A_\nu}{\partial x_\nu}\right) - \frac{\partial^2}{\partial x_\nu^2} A_\mu$$

Now $\frac{\partial A_\nu}{\partial x_\nu} = 0$ Lorentz Gauge

$$\Box^2 A_\mu = -\frac{4\pi}{c} J_\mu \quad \text{Wave Equation}$$

$$\therefore \ L.S. = \frac{\partial}{\partial x_\mu} 0 + \frac{4\pi}{c} J_\mu$$

$$= \frac{4\pi}{c} J_\mu .$$

We shall now show that the equations of these two theorems are merely a statement of Maxwell's equations as is illustrated below.

1) Let $\lambda \mu \nu$ be any combination of 1, 2, 3 in Theorem 1.

$$\frac{\partial F_{12}}{\partial x_3} + \frac{\partial F_{23}}{\partial x_1} + \frac{\partial F_{31}}{\partial x_2} = 0.$$

$$\frac{\partial B_3}{\partial x_3} + \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} = 0.$$

$$\nabla \cdot \vec{B} = 0.$$

2) Let $\lambda = 1, \mu = 2, \nu = 4$ in Theorem 1.

$$\frac{\partial F_{12}}{\partial x_4} + \frac{\partial F_{24}}{\partial x_1} + \frac{\partial F_{41}}{\partial x_2} = 0.$$

$$\frac{\partial B_3}{\partial x_4} - i\frac{\partial E_2}{\partial x_1} + i\frac{\partial E_1}{\partial x_2} = 0.$$

$$\frac{1}{ic}\frac{\partial B_3}{\partial t} - i\left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\right) = 0.$$

$$\frac{1}{c}\frac{\partial B_3}{\partial t} + (\nabla \times \vec{E})_3 = 0.$$

$$(\nabla \times \vec{E})_3 = \frac{-1}{c}\frac{\partial B_3}{\partial t}.$$

Exercise: Show the remaining two Maxwell equations are expressed by $\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} J_\mu$.

Covariance

The fundamental postulate of relativity is that the laws of physics are the same in all inertial reference frames. Hence an observer O at rest discovers the same Maxwell Equations as does a moving observer O'.

If O experiences EM tensor $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$

O' must experience EM tensor $F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu}$

The property that the equations have the same form in different inertial frames is called covariance.

Transformation of Field Tensor

We shall now study the relation between $F'_{\mu\nu}$ + $F_{\mu\nu}$. i.e. How does $F_{\mu\nu}$ transform under a Lorentz transformation?

Since x and A are 4-vectors we have:

$$x'_\mu = \lambda_{\mu\sigma} x_\sigma$$

$$A'_\nu = \lambda_{\nu\rho} A_\rho$$

Exercise: In the next assignment, you will show that $\lambda_{\mu\sigma}\lambda_{\mu\nu} = \delta_{\sigma\nu}$. Using this property show that $x_\sigma = \lambda_{\mu\sigma} x'_\mu$.

$$\therefore \frac{\partial x_\sigma}{\partial x'_\mu} = \lambda_{\mu\sigma}.$$

Now $$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu}$$

$$= \lambda_{\nu\rho} \frac{\partial A_\rho}{\partial x'_\mu} - \lambda_{\mu\sigma} \frac{\partial A_\sigma}{\partial x'_\nu}$$

$$= \lambda_{\nu\rho} \frac{\partial A_\rho}{\partial x_\sigma} \frac{\partial x_\sigma}{\partial x'_\mu} - \lambda_{\mu\sigma} \frac{\partial A_\sigma}{\partial x_\rho} \frac{\partial x_\rho}{\partial x'_\nu}$$

$$= \lambda_{\nu\rho} \lambda_{\mu\sigma} \frac{\partial A_\rho}{\partial x_\sigma} - \lambda_{\mu\sigma} \lambda_{\nu\rho} \frac{\partial A_\sigma}{\partial x_\rho}$$

$$= \lambda_{\mu\sigma} \lambda_{\nu\rho} \left(\frac{\partial A_\rho}{\partial x_\sigma} - \frac{\partial A_\sigma}{\partial x_\rho} \right)$$

$$\therefore \boxed{F'_{\mu\nu} = \lambda_{\mu\sigma} \lambda_{\nu\rho} F_{\sigma\rho}}$$

This transformation must be satisfied in order to satisfy the requirement of covariance.

Definition of a Tensor

In 4 dimensional space, a tensor T of rank 2 is a set of 4^2 quantities $T_{\mu\nu}$ that transform under a Lorentz transformation in the following way.

$$T'_{\mu\nu} = \lambda_{\mu\alpha} \lambda_{\nu\beta} T_{\alpha\beta}$$

$\therefore$ a 4-vector is a tensor of rank 1.
$F_{\mu\nu}$ " " 2.

Transformation of Fields

$$F'_{\mu\nu} = \lambda_{\mu\sigma}\lambda_{\nu\rho}F_{\sigma\rho}$$

$$= \lambda_{\mu\sigma}F_{\sigma\rho}\lambda^{T}_{\rho\nu} \quad \text{where } \lambda^T \text{ means transpose}$$

or $F' = \lambda F \lambda^T$

$$F' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} 0 & B_3 & -\gamma(B_2-\beta E_1) & -i\gamma(E_1-\beta B_2) \\ -B_3 & 0 & \gamma(B_1+\beta E_2) & -i\gamma(E_2+\beta B_1) \\ \gamma(B_2-\beta E_1) & -\gamma(B_1+\beta E_2) & 0 & -iE_3 \\ i\gamma(E_1-\beta B_2) & i\gamma(E_2+\beta B_1) & iE_3 & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} E_1' &= \gamma(E_1-\beta B_2) & B_1' &= \gamma(B_1+\beta E_2) \\ E_2' &= \gamma(E_2+\beta B_1) & B_2' &= \gamma(B_2-\beta E_1) \\ E_3' &= E_3 & B_3' &= B_3 \end{aligned}}$$

Lorentz Force Density

$$K_\mu \equiv \frac{1}{c} F_{\mu\nu} J_\nu$$

Exercise: Prove K_μ is a 4-vector.

$$K_1 = \frac{1}{c} F_{1\nu} J_\nu$$

$$= \frac{1}{c} \left[F_{12} J_2 + F_{13} J_3 + F_{14} J_4 \right]$$

$$= \frac{1}{c} \left[B_3 \rho u_2 + (-B_2) \rho u_3 + (-iE_1) ic\rho \right]$$

$$= \frac{\rho}{c} (B_3 u_2 - B_2 u_3) + \rho E_1$$

$$= \frac{\rho}{c} (\vec{u} \times \vec{B})_1 + \rho E_1$$

$$\therefore \vec{K} = \frac{\rho}{c} \vec{u} \times \vec{B} + \rho \vec{E}.$$

Hence the spatial part of K is simply a Lorentz force on a charge density ρ.

$$K_4 = \frac{1}{c} F_{4\nu} J_\nu$$

$$= \frac{1}{c} \left[F_{41} J_1 + F_{42} J_2 + F_{43} J_3 \right]$$

$$= \frac{1}{c} \left[iE_1 J_1 + iE_2 J_2 + iE_3 J_3 \right]$$

$$= \frac{i}{c} \vec{E} \cdot \vec{J}$$

$\vec{J} \cdot \vec{E}$ is the rate at which the Lorentz force does work on charge in a unit volume.

Energy Momentum Tensor

$$T_{\mu\nu} \equiv \frac{1}{4\pi}\left[F_{\mu\sigma} F_{\sigma\nu} + \frac{1}{4}\delta_{\mu\nu} F_{\lambda\rho} F_{\lambda\rho} \right]$$

Exercise: a) Prove $T_{\mu\nu}$ is a tensor.

b) Prove T is a symmetrical tensor. i.e. $T_{\mu\nu} = T_{\nu\mu}$.

c) Prove $Tr\, T = 0$

One can evaluate $T_{\mu\nu}$ and find the following.

$$T_{jk} = \frac{1}{4\pi}\left[E_j E_k + B_j B_k - \frac{\delta_{jk}}{2}(E^2 + B^2) \right]$$

Maxwell Stress Tensor

$$T_{4k} = \frac{-i}{4\pi}(\vec{E} \times \vec{B})_k$$

$$= \frac{-i S_k}{c}$$ where $\vec{S}$ is the Poynting vector

$$T_{44} = \frac{1}{8\pi}(E^2 + B^2)$$ is energy density of electric & magnetic fields.

$$\equiv \mathcal{E}$$

Relation Between Lorentz Force Density and Energy Momentum Tensor

$$\boxed{\frac{\partial T_{\mu\nu}}{\partial x_\nu} = K_\mu}$$

Proof:

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = \frac{1}{4\pi}\frac{\partial}{\partial x_\nu}\left[F_{\mu\sigma}F_{\sigma\nu} + \frac{1}{4}\delta_{\mu\nu}F_{\lambda\rho}F_{\lambda\rho}\right]$$

$$= \frac{1}{4\pi}\left[\frac{\partial F_{\mu\sigma}}{\partial x_\nu}F_{\sigma\nu} + F_{\mu\sigma}\frac{\partial F_{\sigma\nu}}{\partial x_\nu} + \frac{1}{4}\frac{\partial (F_{\lambda\rho}F_{\lambda\rho})}{\partial x_\mu}\right]$$

We shall now show that $\frac{\partial F_{\mu\sigma}}{\partial x_\nu}F_{\sigma\nu} = \frac{-1}{4}\frac{\partial (F_{\lambda\rho}F_{\lambda\rho})}{\partial x_\mu}$.

$$\frac{\partial F_{\mu\sigma}}{\partial x_\nu}F_{\sigma\nu} = \frac{\partial F_{\sigma\mu}}{\partial x_\nu}F_{\nu\sigma} \quad \text{since } F_{\mu\nu} = -F_{\nu\mu}$$

$$= \frac{\partial F_{\nu\mu}}{\partial x_\sigma}F_{\sigma\nu} \quad (1)$$ interchanging dummy indices ν & σ.

Using (1) we may write:

$$\frac{\partial F_{\mu\sigma}}{\partial x_\nu}F_{\sigma\nu} = \frac{1}{2}\left(\frac{\partial F_{\mu\sigma}}{\partial x_\nu} + \frac{\partial F_{\nu\mu}}{\partial x_\sigma}\right)F_{\sigma\nu}$$

But $\frac{\partial F_{\mu\sigma}}{\partial x_\nu} + \frac{\partial F_{\nu\mu}}{\partial x_\sigma} + \frac{\partial F_{\sigma\nu}}{\partial x_\mu} = 0$ (Maxwell's Equation)

$$\therefore \frac{\partial F_{\mu\sigma}}{\partial x_\nu} F_{\sigma\nu} = -\frac{1}{2} \frac{\partial F_{\sigma\nu}}{\partial x_\mu} F_{\sigma\nu}$$

$$= -\frac{1}{4} \frac{\partial (F_{\sigma\nu} F_{\sigma\nu})}{\partial x_\mu}$$ where σ & ν are dummy indices

$$\Rightarrow \frac{\partial T_{\mu\nu}}{\partial x_\nu} = \frac{1}{4\pi} F_{\mu\sigma} \frac{\partial F_{\sigma\nu}}{\partial x_\nu}$$

$$= \frac{1}{c} F_{\mu\sigma} J_\sigma$$ since $\frac{\partial F_{\sigma\nu}}{\partial x_\nu} = \frac{4\pi}{c} J_\sigma$

$$\therefore \frac{\partial T_{\mu\nu}}{\partial x_\nu} = K_\mu$$

Significance of Above Equation

First we consider the 4 th component.

$$K_4 = \frac{\partial T_{4\nu}}{\partial x_\nu}$$

$$= \frac{\partial T_{4k}}{\partial x_k} + \frac{\partial T_{44}}{\partial x_4}.$$

$$\frac{i}{c} \vec{E} \cdot \vec{J} = -\frac{i}{c} \frac{\partial S_k}{\partial x_k} + \frac{\partial \mathcal{E}}{\partial (ict)}$$

$$-\vec{E} \cdot \vec{J} = \nabla \cdot \vec{S} + \frac{\partial \mathcal{E}}{\partial t}$$

$$\frac{\partial \mathcal{E}}{\partial t} = -\vec{J}\cdot\vec{E} - \nabla\cdot\vec{S}.$$

This is merely a statement of energy conservation.

$\frac{\partial \mathcal{E}}{\partial t}$ = rate of change of energy stored in electric and magnetic field in unit volume.

$+\vec{J}\cdot\vec{E}$ = rate at which Lorentz force does work in unit volume.

$\nabla\cdot\vec{S}$ = energy flux coming out of unit volume.

Next the remaining components of $K_\mu = \frac{\partial T_{\mu\nu}}{\partial x_\nu}$ are studied.

$$K_j = \frac{\partial T_{j\nu}}{\partial x_\nu}$$

$$= \frac{\partial T_{jk}}{\partial x_k} + \frac{\partial T_{j4}}{\partial (ict)}$$

$$K_j = \frac{\partial T_{jk}}{\partial x_k} - \frac{1}{c^2}\frac{\partial S_j}{\partial t}$$

$\frac{\partial T_{jk}}{\partial x_k}$ is the divergence of the Maxwell stress tensor $\{T^M\}$

$$\therefore \vec{K} = \nabla\{T^M\} - \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t}$$

$$\vec{K} + \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t} = \nabla\{T^M\}$$

Integrating over volume V, we get:

$$\int_V \left(\vec{K} + \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t} \right) dV = \int_V \nabla\{T^M\}\, dV$$

Since $\vec{K} \equiv$ Lorentz force density

$$\int_V \vec{K}\, dV = \text{Lorentz force}$$

$$= \frac{d\vec{p}}{dt}$$ where $\vec{p}$ is the momentum due to motion of charged particles

We define $\vec{g} \equiv \frac{\vec{S}}{c^2}$

$$\vec{G} \equiv \int_V \vec{g}\, dV$$

$$\therefore \frac{d}{dt}(\vec{p} + \vec{G}) = \int_V \nabla\{T^M\}\, dV$$

$$\boxed{\frac{d}{dt}(\vec{p} + \vec{G}) = \int_S \{T^M\}\cdot \hat{n}\, da.}$$

We shall show that this is a statement of momentum conservation.

If the electric and magnetic fields vanish outside volume V, then

$$\frac{d}{dt}(\vec{p} + \vec{G}) = 0$$

$$\vec{p} + \vec{G} = \text{constant}$$

$\vec{p}$ is just the momentum of the moving charged particles. $\vec{G}$ is produced by the electromagnetic fields. It is called the field momentum.

$$\therefore \boxed{\vec{g} \equiv \frac{\vec{S}}{c^2} = \frac{1}{4\pi c}\vec{E} \times \vec{B} \quad \text{is Momentum Density of Electromagnetic Field}}$$

In general, the electric & magnetic fields don't vanish outside V and we see that:

$$-\{T^M\} \cdot \hat{n} = \text{flow of momentum per unit area out of volume V through surface S}$$

Example: Radiation Pressure

Consider a light wave propagating through space and completely absorbed by a black surface. Find the pressure exerted by the light on the black object.

Momentum Density of light wave is $\vec{g}$

Time averaged mom. density is $\langle \vec{g} \rangle$

Radiation Pressure $P_{rad} \equiv$ average force per unit area

$=$ time averaged mom. density $\times$ speed of wave

$$P_{rad} = \langle |\vec{g}| \rangle c$$

$$\boxed{P_{rad} = \frac{\langle |\vec{S}| \rangle}{c}}$$

eg. Energy flux of sunlight at Earth $= 1400$ watt/m^2.

$$\therefore P_{rad} = \frac{1400 \text{ watt/m}^2}{3 \times 10^8 \text{ m/sec}}$$

$$= 4.67 \times 10^{-6} \text{ Nt/m}^2$$

This is the force responsible for bending comet tails.

Assignment

10 1. The length of a 4-vector x_μ is a Lorentz invariant,
i.e. $x_\mu x_\mu = x'_\mu x'_\mu$

a) Use this property to show that $\lambda \lambda^T = 1$
i.e. $\lambda_{\mu\sigma} \lambda_{\mu\rho} = \delta_{\sigma\rho}$

b) Verify by multiplying matrices that $\lambda \lambda^T = 1$.

20 2 a) Show that $(\vec{x}, 0)$ is not a 4-vector.

b) Show that $\Box$ is a 4-vector.

c) Show that t is not a Lorentz invariant.

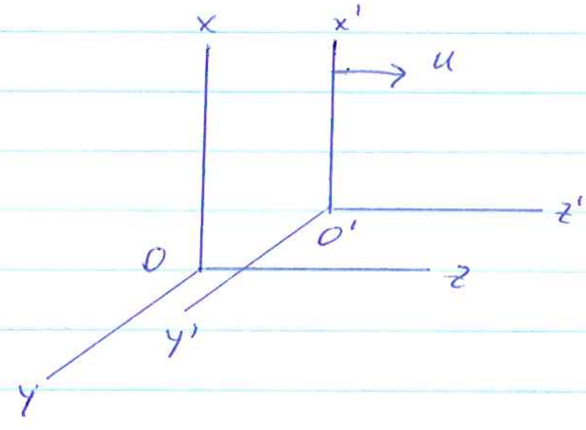
d) Prove $\delta_{\mu\rho}$ is a tensor.

5 3) Show that $\dfrac{\partial F_{\mu\nu}}{\partial x_\nu} = \dfrac{4\pi}{c} J_\mu$ is an expression of two of Maxwell's equations.

10 4) a) Prove that the energy momentum tensor $T_{\mu\nu}$ is indeed a tensor.

b) Evaluate T_{11}, T_{41}, T_{44} using the definition.

30 5) A particle is at rest in frame S'.
It is seen to have a speed u in frame S.



Observer O' sees: electric field $\vec{E}'$
magnetic field $\vec{B}'$
particle charge density ρ_0.

Observer O sees: electric field $\vec{E}$
magnetic field $\vec{B}$
particle charge density ρ

a) Write down the Lorentz force density K'.

b) Find K by applying a Lorentz transformation to K'.

c) What are the relations between $\vec{E}'$, $\vec{B}'$ and $\vec{E}$, $\vec{B}$?

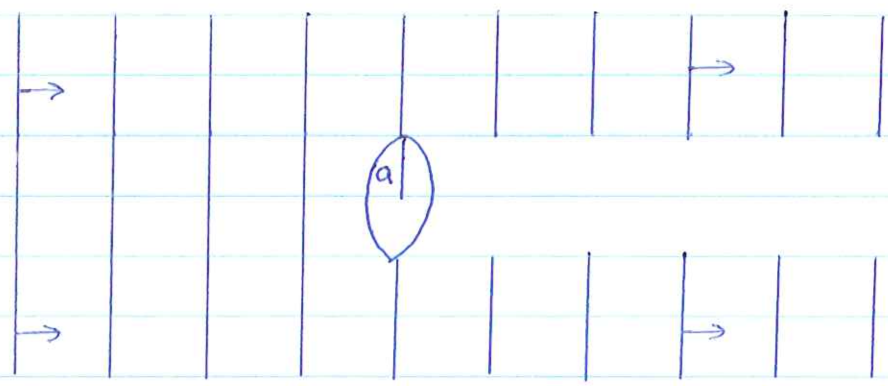
d) How is ρ related to ρ_0?

e) Derive the expression for the Lorentz force density.
i.e. show $\vec{K} = \rho(\vec{E} + \vec{\beta} \times \vec{B})$.

Tot = 75

Diffraction

Diffraction refers to the bending of light when it passes an object. Consider a plane wave incident on a circular disk. If light travelled in truly straight lines, the disk would produce a shadow extending to infinity as shown below.



Instead at a distance behind the disk about equal to the disk radius, one observes a bright spot. Hence light behaves similarly to water waves which bend around objects. Observations made in the early 1800's of diffraction, was key evidence confirming the wave nature of light.

Explanation

The intensity of diffracted light can be computed by solving Maxwells equations for the electric and magnetic vector fields using appropriate boundary conditions at the surface of the diffracting object. The mathematics for this vector diffraction theory is formidable and has only been done for rather simple problems. A simpler although approximate scalar theory of diffraction was developed by Augustin Fresnel in 1818. Note that this theory predated Maxwell's work of the 1860's.

Scalar Diffraction Theory

Fresnel assumed that light is a wave. The amplitude and phase of the light wave are represented by a scalar function $\Psi(\vec{r}, t)$. $\Psi(\vec{r}, t)$ is a solution of the wave equation:

$$\nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = 0$$

where c is the speed of light. The light intensity is given by $|\Psi|^2$.

For monochromatic light $\Psi(\vec{r}, t) = \psi(\vec{r}) e^{-i\omega t}$ and the wave equation becomes:

$$\nabla^2 \psi + k^2 \psi = 0 \qquad \text{where } k \equiv \frac{\omega}{c}.$$

Comment

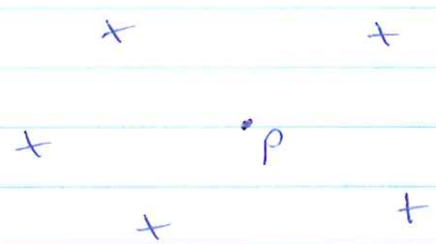
Maxwell showed light is a transverse electro-magnetic wave. The electric and magnetic fields satisfy the vector wave equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \left\{ \begin{array}{c} \vec{E} \\ \vec{B} \end{array} \right\} = 0.$$

Hence Fresnel's theory is rather crude. For example, it cannot account for the two orthogonal transverse polarizations of a light wave. It is therefore quite surprising how well the scalar theory agrees with observation.

Solution of Scalar Wave Equation

We would like to evaluate the light wavefunction ψ at some point P generated by an arbitrary distribution of point sources x.



In a region not containing any light sources, ψ satisfies the Helmholtz equation.

$$(\nabla^2 + k^2)\,\psi = 0$$

Consider a function $\chi(\vec{r})$ that also is a solution of the Helmholtz equation.

$$(\nabla^2 + k^2)\,\chi(\vec{r}) = 0.$$

Using the divergence theorem, we may write:

$$-\int_S (\psi \nabla \chi - \chi \nabla \psi)\cdot d\vec{a} = -\int_V (\psi \nabla^2 \chi - \chi \nabla^2 \psi)\, dV = 0$$

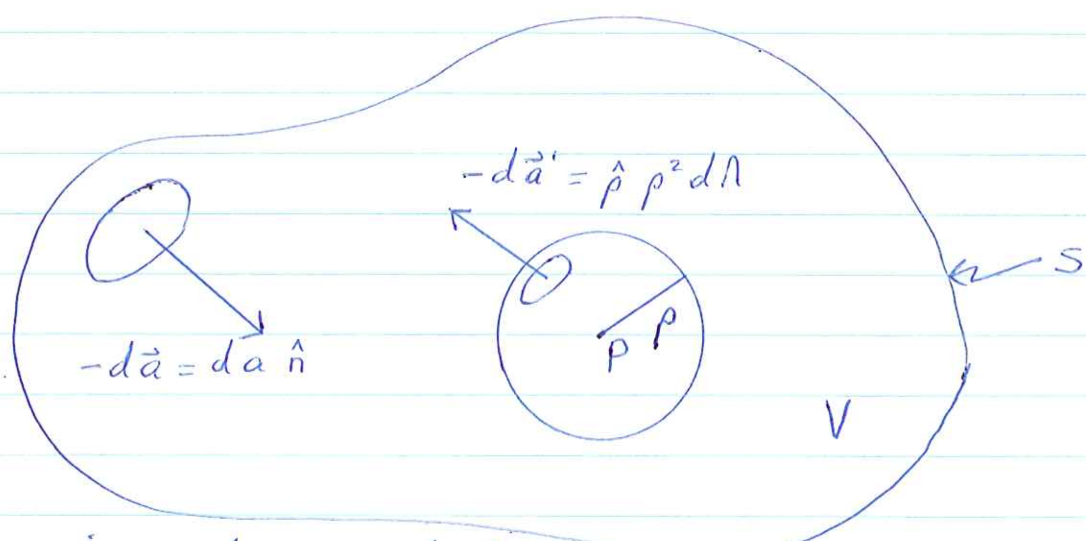
S is the surface of volume V which surrounds P but does not contain any light sources.

Next we choose $\chi(\vec{r}) = \dfrac{e^{ikr}}{r}$ which is an isotropic spherical wave. r is the distance from P. (i.e. origin of coordinates is P)

Exercise: Verify that $\dfrac{e^{ikr}}{r}$ is a solution of the Helmholtz equation.

Comment: There are solutions $\chi(\vec{r}) = \chi(r, \theta, \varphi)$ that have angular dependences. We ignore this possibility since we assume all light sources radiate isotropically.

Since χ isn't well defined at P (i.e. the origin) we shall exclude from V a small sphere of radius ρ centered at P.



$\hat{n}$ = inward normal of S

There are then two surfaces and we may write:

$$\int_S \left[\psi \, \nabla\left(\frac{e^{ikr}}{r}\right) - \frac{e^{ikr}}{r} \nabla\psi \right] \cdot (-d\vec{a})$$

$$+ \int_{\substack{\text{surface}\\ \text{of sphere}}} \left[\psi \, \nabla\left(\frac{e^{ik\rho}}{\rho}\right) - \frac{e^{ik\rho}}{\rho} \nabla\psi \right] \cdot (-d\vec{a}') = 0$$

Taking the limit as $\rho \to 0$, the second integral becomes:

$$\lim_{\rho \to 0} \int_{\substack{\text{surface}\\ \text{of sphere}}} \left[\psi \left(\frac{ik}{\rho} - \frac{1}{\rho^2}\right) e^{ik\rho} \hat{\rho} - \frac{e^{ik\rho}}{\rho} \nabla\psi \right] \cdot \hat{\rho} \, \rho^2 \, d\Lambda .$$

$$= \lim_{\rho \to 0} \int \left[-\psi \, e^{ik\rho} + O(\rho) \right] d\Lambda .$$

$$= -\psi(P) \int d\Lambda$$

$$= -4\pi \, \psi(P)$$

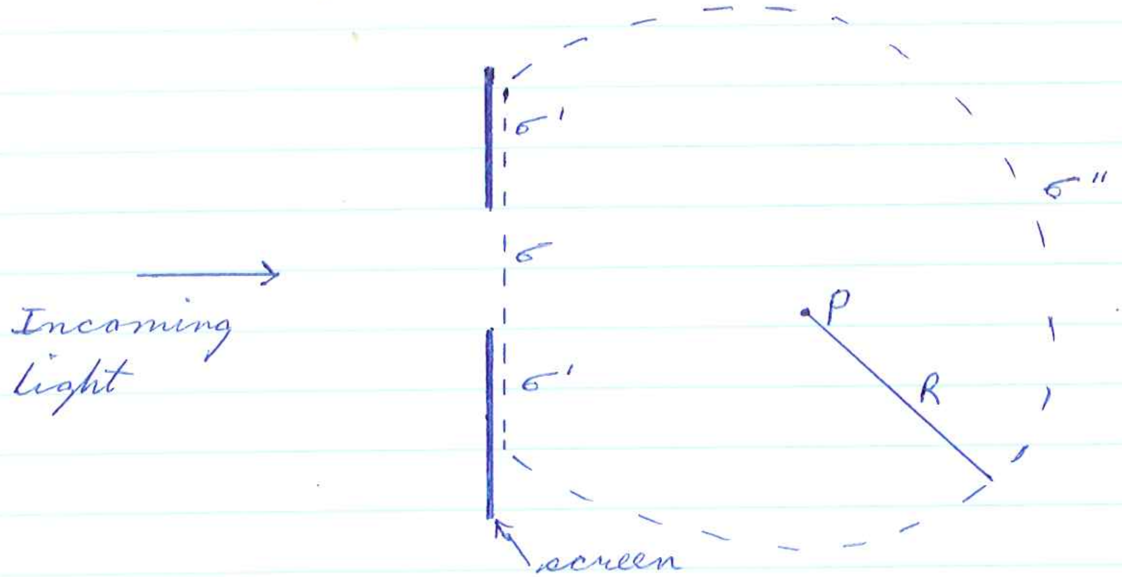
Therefore we obtain:

$$\boxed{\psi(P) = \frac{1}{4\pi} \int_S \left(\psi \, \nabla\left(\frac{e^{ikr}}{r}\right) - \frac{e^{ikr}}{r} \nabla\psi \right) \cdot \hat{n} \, da.}$$

Helmholtz - Kirchoff Integral

Kirchoff Diffraction Theory

Consider light incident on an opaque infinite plane having aperture σ as shown below.



We shall compute $\psi(P)$ using the Helmholtz-Kirchoff integral. We specify surface S to be:

$S = \sigma$ (aperture) $+ \sigma'$ (opaque screen) $+ \sigma''$ (sphere of radius R)

Next we need to know the values of ψ + $\frac{\partial \psi}{\partial n}$ on S.

Kirchoff Boundary Conditions

1) On σ, we assume ψ + $\frac{\partial \psi}{\partial n}$ have the same value as the incident wave in the absence of the screen.

2) On σ', we assume ψ + $\frac{\partial \psi}{\partial n}$ vanish since the screen is opaque.

This seems reasonable until we recall that EM waves penetrate conducting surfaces to a skindepth. Hence our solution will be invalid within a few wavelengths of the diffracting surface.

3) On σ'', we assume ψ + $\frac{\partial \psi}{\partial n}$ vanish. This occurs if R is so large that the wave has not had time to reach σ''. We hereby exclude monochromatic waves since these must exist throughout all time.

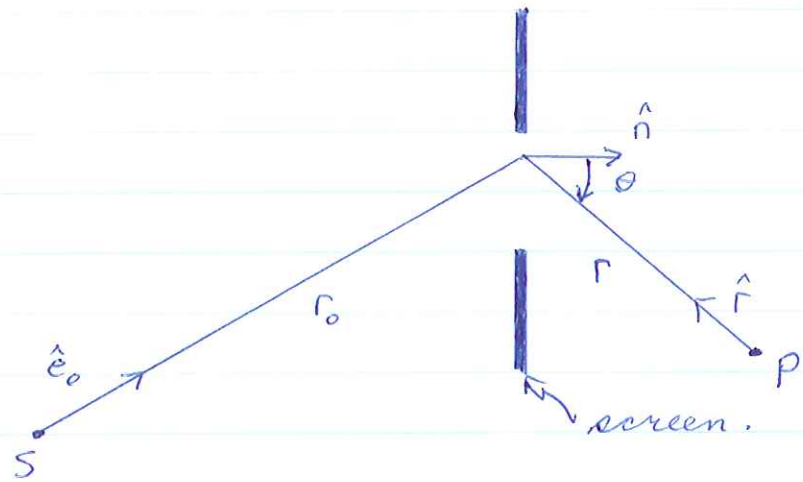
The Helmholtz - Kirchoff integral then reduces to:

$$\psi(P) = \frac{1}{4\pi} \int_{\sigma} \left(\psi_{inc} \nabla\left(\frac{e^{ikr}}{r}\right) - \frac{e^{ikr}}{r} \nabla \psi_{inc} \right) \cdot \hat{n} \, da$$

where ψ_{inc} is the incident wave at the aperture σ.

Case 1: Incident Wave = Spherical Wave

Consider a point source S, a distance r_0 from the aperture as shown below.



$\hat{e}_0 \equiv$ unit vector pointing from S to point of aperture

S emits a spherical wave of amplitude A.

$$\Rightarrow \psi_{inc} = \frac{A e^{ikr_0}}{r_0}$$

Light function at P is:

$$\psi(P) = \frac{1}{4\pi} \int_\sigma \left(\psi_{inc} \nabla\left(\frac{e^{ikr}}{r}\right) - \frac{e^{ikr}}{r} \nabla \psi_{inc} \right) \cdot \hat{n}\, da.$$

$$= \frac{1}{4\pi} \int_\sigma \left\{ \frac{A e^{ikr_0}}{r_0} \left(\frac{ik}{r} - \frac{1}{r^2} \right) e^{ikr} \hat{r} - \frac{e^{ikr}}{r} \left(\frac{ik}{r_0} - \frac{1}{r_0^2} \right) A e^{ikr_0} \hat{e}_0 \right\} \cdot \hat{n}\, da$$

Assuming $r, r_0 \gg \lambda$ we get:

$$\boxed{\psi(P) = i \frac{Ak}{4\pi} \int_\sigma \frac{e^{ik(r+r_0)}}{r\, r_0} (\hat{r} - \hat{e}_0) \cdot \hat{n}\, da}$$

Fresnel-Kirchoff Diffraction Integral

Case 2: Incident Wave = Plane Wave

Consider a plane wave ψ_0 propagating in direction $\hat{n}$.

Exercise: Replacing $A \frac{e^{ikr_0}}{r_0}$ by ψ_0 and defining

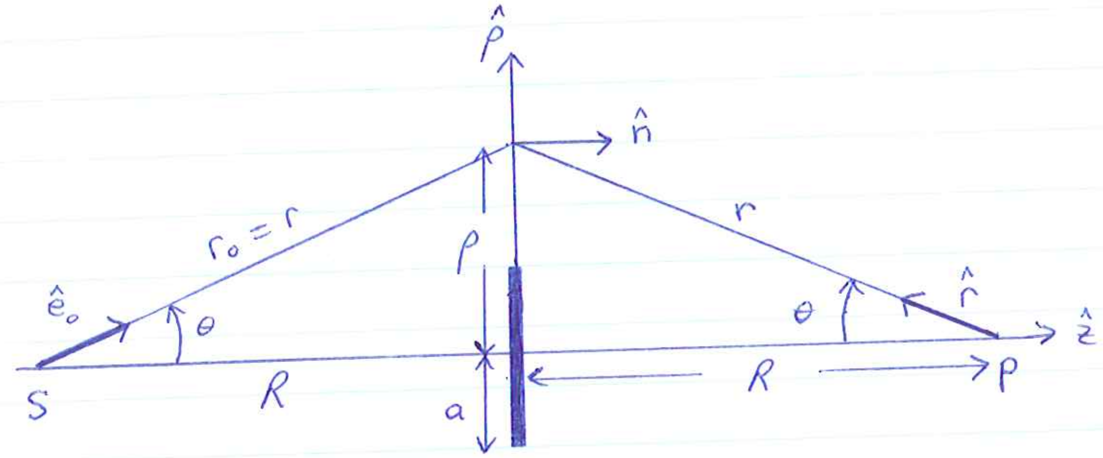
$\cos\theta = -\hat{n}\cdot\hat{r}$, show that

$$\psi(P) = -i \frac{k\psi_0}{4\pi} \int_\sigma \frac{e^{ikr}}{r} (1+\cos\theta)\, da.$$

Each point in the aperture therefore emits a spherical wave $\frac{e^{ikr}}{r}$ modified by the so-called Stokes' inclination factor $1+\cos\theta$. This angular factor is maximum in the forward direction and zero in the backwards direction.

Diffraction by a Circular Disk

Consider a point source S illuminating a disk of radius a as shown below. We shall compute the light intensity behind the disk at P. In order to simplify the math, S and P are equidistant from the disk and line SP intersects the disk center.



$$\hat{r} \cdot \hat{n} = -\cos\theta$$
$$\hat{e}_0 \cdot \hat{n} = \cos\theta$$

The Fresnel-Kirchoff Integral then gives the following.

$$\psi(P) = i \frac{Ak}{4\pi} \int_\sigma \frac{e^{2ikr}}{r^2} (-2\cos\theta)\, da$$

where σ is the xy plane excluding the disk. Using polar coordinates, this becomes:

$$\psi(P) = -i \frac{Ak}{2\pi} \int_a^\infty \frac{e^{2ikr}}{r^2} \cos\theta \, 2\pi\rho \, d\rho$$

$$\psi(P) = -iAk \int_a^{\infty} \frac{e^{2ikr}}{r^2} \cos\theta \, \rho \, d\rho$$

But $r^2 = R^2 + \rho^2$

$$2r\,dr = 2\rho\,d\rho$$
$$r\,dr = \rho\,d\rho$$

When $\rho = a$, $r = \sqrt{R^2 + a^2}$
$\rho = \infty$ $\quad r = \infty$

also $\cos\theta = \dfrac{R}{r}$

$$\therefore \psi(P) = -iAk \int_{\sqrt{R^2+a^2}}^{\infty} \frac{e^{2ikr}}{r^2} \frac{R}{r} \, r\,dr$$

$$= -iAkR \int_{\sqrt{R^2+a^2}}^{\infty} \frac{e^{2ikr}}{r^2} \, dr$$

Next we integrate by parts.

$$\begin{array}{ccc} r^{-2} & & e^{2ikr} \\ & + & \\ -2r^{-3} & & \dfrac{e^{2ikr}}{2ik} \\ & - & \\ 6r^{-4} & + & \dfrac{e^{2ikr}}{(2ik)^2} \end{array}$$

$$\psi(P) = -iAkR \left\{ \left[r^{-2} \frac{e^{2ikr}}{2ik} + 2r^{-3} \frac{e^{2ikr}}{(2ik)^2} \right]_{\sqrt{R^2+a^2}}^{\infty} + \int_{\sqrt{R^2+a^2}}^{\infty} 6r^{-4} \frac{e^{2ikr}}{(2ik)^2} \, dr \right\}$$

$$\psi(P) = -iAkR\left\{ \frac{-1}{2ik}\frac{e^{2ik\sqrt{R^2+a^2}}}{R^2+a^2} - \frac{2}{(2ik)^2}\frac{e^{2ik\sqrt{R^2+a^2}}}{(R^2+a^2)^{3/2}} + \frac{6}{(2ik)^2}\int_{\sqrt{R^2+a^2}}^{\infty} \frac{e^{2ikr}}{r^4}\,dr \right\}$$

In the limit $R \gg \lambda$ or $kR \gg 1$, the first term dominates

$$\psi(P) \simeq \frac{AR}{2}\frac{e^{2ik\sqrt{R^2+a^2}}}{R^2+a^2}$$

Hence the light intensity at P is:

$$I(P) = |\psi(P)|^2 = \frac{A^2R^2}{4(R^2+a^2)^2}$$

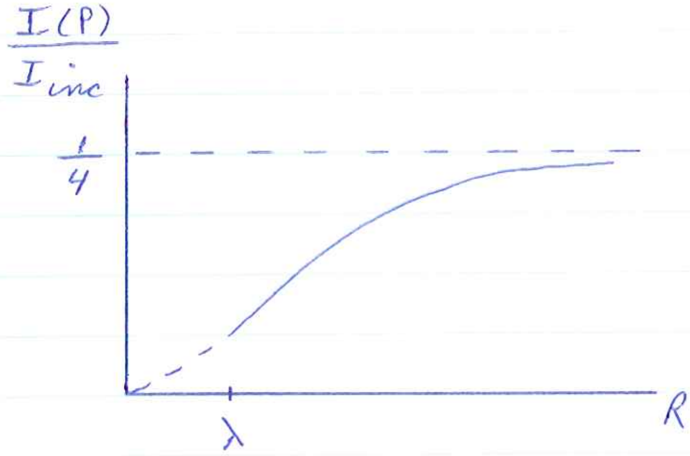
The incident light amplitude at the edge of the disk is:

$$\psi_{inc} = A\frac{e^{ik\sqrt{R^2+a^2}}}{\sqrt{R^2+a^2}}$$

which corresponds to an intensity:

$$I_{inc} = |\psi_{inc}|^2 = \frac{A^2}{R^2+a^2}.$$

$$\therefore \boxed{I(P) = \frac{I_{inc}}{4}\frac{R^2}{R^2+a^2}.}$$

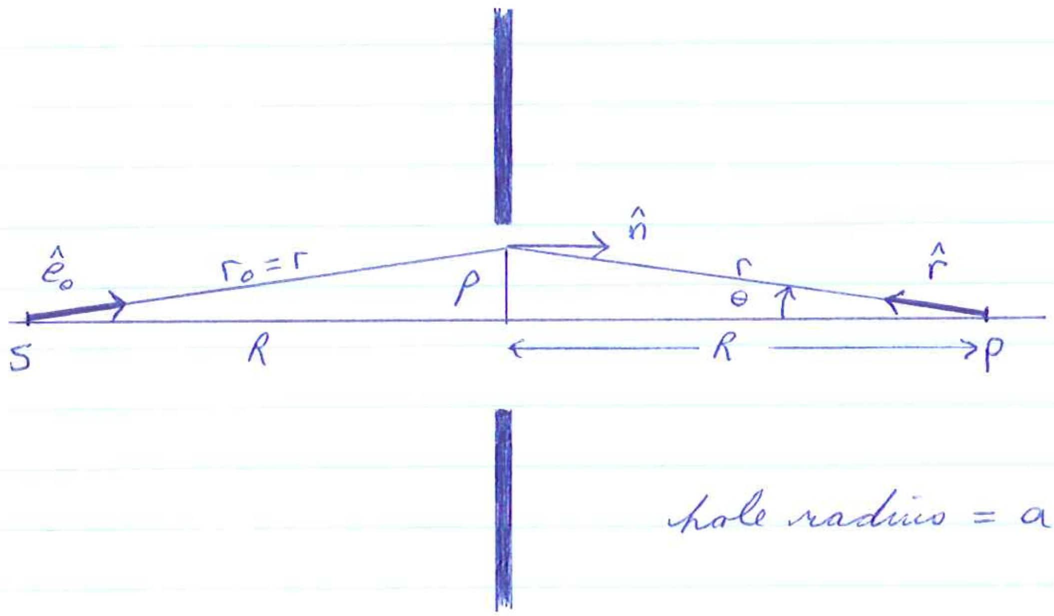


Solution does not hold when $R \lesssim \lambda$.

Note that the light intensity at P is not zero. Light passing a point near the disk edge arrives at P in phase with light coming from any other point near the disk edge. Constructive interference then occurs. Off the axis, the phases of the waves are different creating circular interference fringes.

Diffraction by a Circular Aperture

Consider a point source illuminating an infinite plane having a circular hole as shown below. As in the previous example, the source S and observation point P are equidistant from the plane and lie on the symmetry axis.



hole radius = a

Exercise: Show that the Fresnel-Kirchoff integral gives:

$$\psi(P) = -ikAR \int_R^{\sqrt{R^2+a^2}} \frac{e^{2ikr}}{r^2}\, dr.$$

Exercise: Integrating by parts and using the approximation $R \gg \lambda$ show that we get:

$$\psi(P) = -\frac{AR}{2}\left\{ \frac{e^{2ik\sqrt{R^2+a^2}}}{R^2+a^2} - \frac{e^{2ikR}}{R^2} \right\}$$

$$\text{or } \psi(P) = \frac{-A}{2R}\left\{\frac{\exp 2ikR\left[1+\frac{a^2}{R^2}\right]^{1/2}}{1+a^2/R^2} - e^{2ikR}\right\}$$

We shall now assume that $R \gg a$.

$$\psi(P) \simeq \frac{-A}{2R}\left\{\exp 2ikR\left[1+\frac{a^2}{2R^2}+\dots\right] - e^{2ikR}\right\}$$

$$= \frac{-A}{2R}\, e^{2ikR}\left[e^{ika^2/R} - 1\right]$$

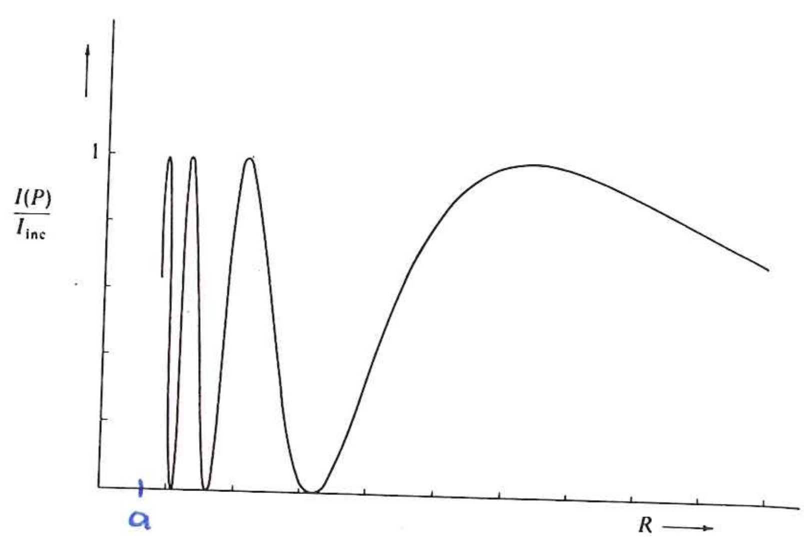
$$= \frac{-A}{2R}\, e^{2ikR}\, e^{ika^2/2R}\left[e^{ika^2/2R} - e^{-ika^2/2R}\right]$$

$$= \frac{-iA}{R}\, e^{2ikR}\, e^{ika^2/2R}\,\sin\frac{ka^2}{2R}$$

Hence the intensity $I(P) = |\psi(P)|^2 = \frac{A^2}{R^2}\sin^2\frac{ka^2}{2R}$

Incident intensity at hole center $I_{inc} = \frac{A^2}{R^2}$

and $\boxed{I(P) = I_{inc}\sin^2\frac{ka^2}{2R}}$



Babinet's Principle

The two diffracting objects we have just considered, the disk and the infinite plane with a hole are complementary. There is however, no simple relation between the intensities.

i.e. $I(P)$ due to disk of radius a + $I(P)$ due to hole of radius a $\neq$ $I(P)$ no diff. object

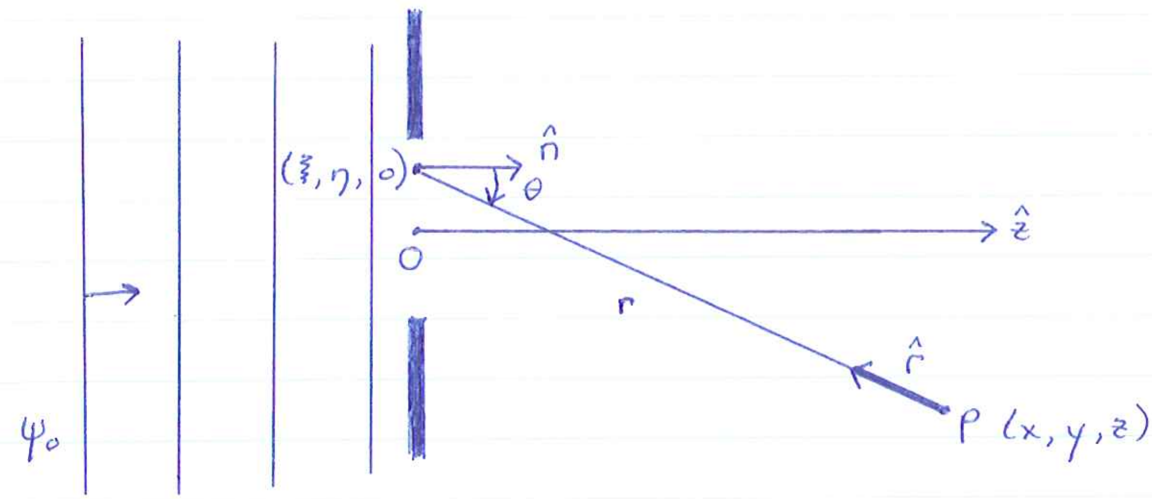
One can show (Exercise) that:

$\psi(P)$ due to disk of radius a + $\psi(P)$ due to hole of radius a = $\psi(P)$ no diff. object

This is called Babinet's Principle.

Fraunhofer Diffraction

Consider a plane wave normally incident on a planar diffracting surface as shown below.



$(\xi, \eta, 0)$ is an arbitrary point in the aperture

$\cos\theta \equiv -\hat{n} \cdot \hat{r}$

r = distance from P to $(\xi, \eta, 0)$

The light function at P was previously shown to be:

$$\psi(P) = -\frac{i k \psi_0}{4\pi} \int_\sigma \frac{e^{ikr}}{r} (1 + \cos\theta)\, da.$$

We shall only consider near off axis points P such that $\cos\theta \simeq 1$. i.e. $x, y \ll z$

$$\therefore \psi(P) = -\frac{i k \psi_0}{2\pi} \int_\sigma \frac{e^{ikr}}{r}\, da.$$

We next find an approximate expression for r.

$$r \equiv \left[(x-\xi)^2 + (y-\eta)^2 + z^2 \right]^{1/2}$$

$$= \left[x^2 + y^2 + z^2 - 2(x\xi + y\eta) + \xi^2 + \eta^2 \right]^{1/2}$$

$$= \left[R^2 - 2(x\xi + y\eta) + \xi^2 + \eta^2 \right]^{1/2}$$

where R is distance of P from origin

$$r = R \left[1 - \frac{2(x\xi + y\eta)}{R^2} + \frac{\xi^2 + \eta^2}{R^2} \right]^{1/2}$$

$$= R \left[1 - \frac{2(\alpha\xi + \beta\eta)}{R} + \frac{\xi^2 + \eta^2}{R^2} \right]^{1/2}$$

where $\alpha \equiv \frac{x}{R}$ & $\beta \equiv \frac{y}{R}$

We shall assume that $R \gg d$ the aperture size.

$$\therefore r \simeq R \left[1 - \frac{(\alpha\xi + \beta\eta)}{R} + \frac{\xi^2 + \eta^2}{2R^2} + \cdots \right]$$

$$r = R - (\alpha\xi + \beta\eta) + \frac{\xi^2 + \eta^2}{2R}$$

In the expression for $\psi(P)$, we approximate r in the denominator by R. In the exponent we include the second term. Hence we assume that:

$$k \frac{(\xi^2 + \eta^2)}{2R} \simeq \frac{k d^2}{R} \ll 2\pi$$

or $$R \gg \frac{k d^2}{2\pi}$$

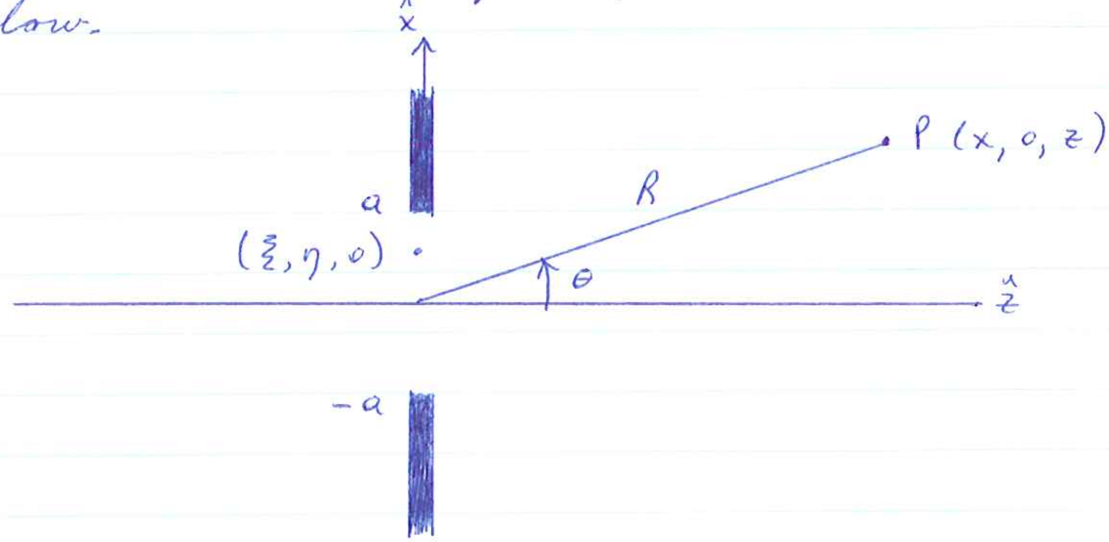
$$R \gg \frac{d^2}{\lambda}$$

This is called the Fraunhofer limit.

$$\Rightarrow \boxed{\psi(P) = -i\,\psi_0 \frac{e^{ikR}}{\lambda R} \int_{\sigma} e^{-ik(\alpha\xi + \beta\eta)}\, d\xi\, d\eta}$$

Fraunhofer Diffraction by an Incident Slit

Consider an infinitely long slit of width 2a as shown below.



$$\psi(P) = \underbrace{-i\psi_0 \frac{e^{ikR}}{\lambda R} \int_{-\infty}^{\infty} e^{-ik\beta\eta}\, d\eta}_{\equiv C} \int_{-a}^{a} e^{-ik\alpha\xi}\, d\xi$$

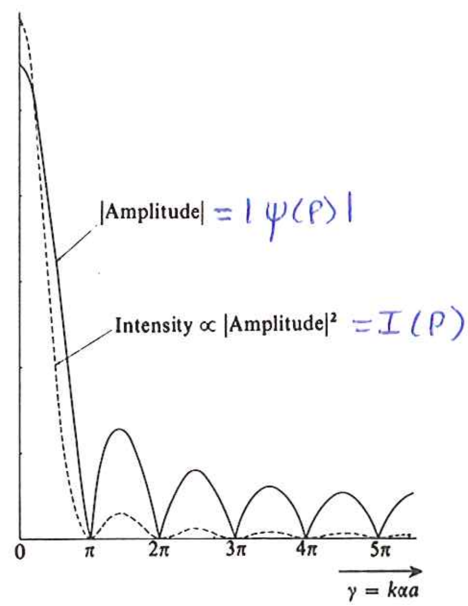
$$= C \left. \frac{e^{-ik\alpha\xi}}{-ik\alpha} \right|_{-a}^{+a}$$

$$\psi(P) = \frac{C}{-ik\alpha}\left(e^{-ik\alpha a} - e^{ik\alpha a}\right)$$

$$= \frac{2C}{k\alpha} \sin k\alpha a$$

$$= 2Ca \frac{\sin\gamma}{\gamma} \quad \text{where } \gamma \equiv k\alpha a.$$

$\therefore$ light intensity $I(P) = 4C^2a^2\left(\frac{\sin\gamma}{\gamma}\right)^2$ which is plotted below.



Most of the diffracted light is contained in the central on axis maximum. This has an angular width $\Delta\theta$ corresponding to $\Delta\gamma = 2\pi$.

$$\gamma = ka\alpha$$

$$= \frac{2\pi}{\lambda} a \frac{x}{R}$$

$$= \frac{2\pi}{\lambda} a \sin\theta.$$

$$\therefore \Delta \gamma = \frac{2\pi a}{\lambda} \cos\theta \, \Delta\theta$$

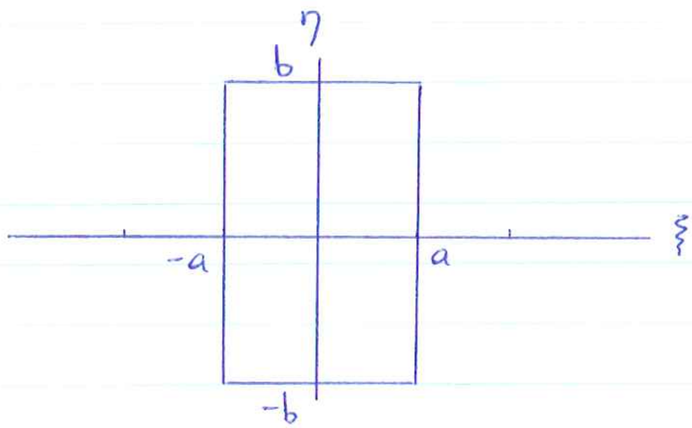
$$\simeq \frac{2\pi a}{\lambda} \Delta\theta \quad \text{since } \theta \simeq 0.$$

$$\Rightarrow \boxed{\Delta\theta \simeq \frac{\lambda}{a}.}$$

Hence we see that diffraction is only significant for small slits.

Fraunhofer Diffraction by a Rectangular Aperture

Consider a plane wave incident on the rectangular slit shown below.



Light wavefunction at P is:

$$\psi(P) = \underbrace{-i\psi_0 \frac{e^{ikR}}{\lambda R}}_{\equiv C} \int_{-a}^{a} e^{-ik\alpha\xi}\, d\xi \int_{-b}^{b} e^{-ik\beta\eta}\, d\eta$$

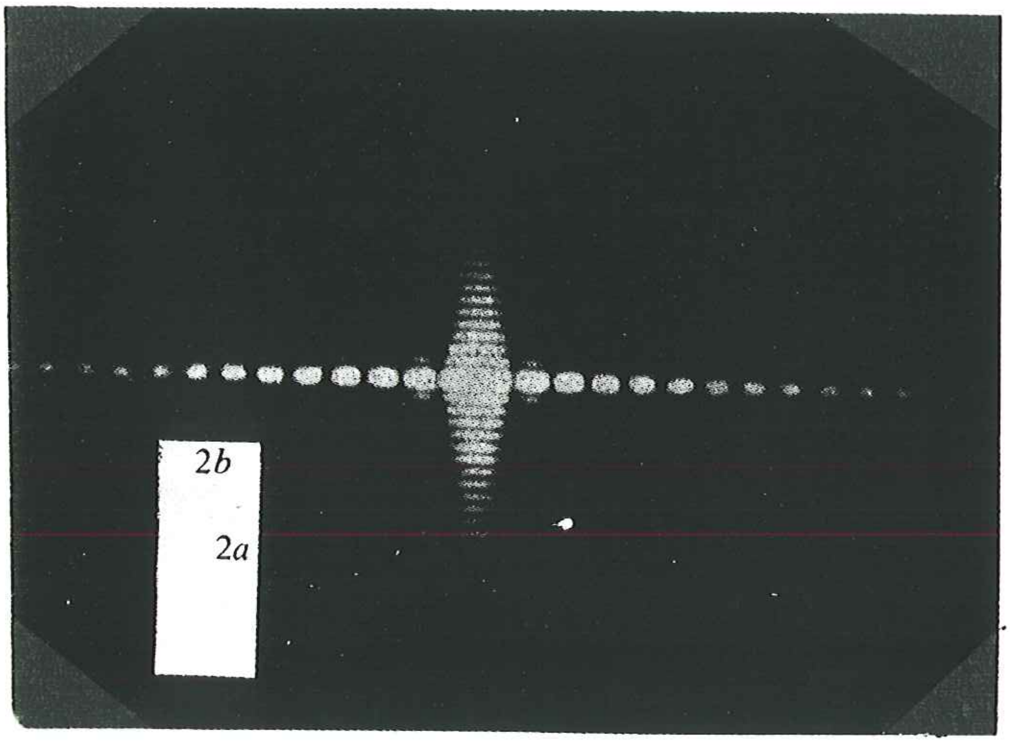
$$= C \left. \frac{e^{-ik\alpha\xi}}{-ik\alpha} \right|_{-a}^{a} \left. \frac{e^{-ik\beta\eta}}{ik\beta} \right|_{-b}^{b}$$

$$= 4Cab \frac{\sin k\alpha a}{k\alpha a} \frac{\sin k\beta b}{k\beta b}$$

$$= 4Cab \frac{\sin \gamma_a}{\gamma_a} \frac{\sin \gamma_b}{\gamma_b} \quad \text{where } \gamma_a \equiv k\alpha a, \quad \gamma_b \equiv k\beta b$$

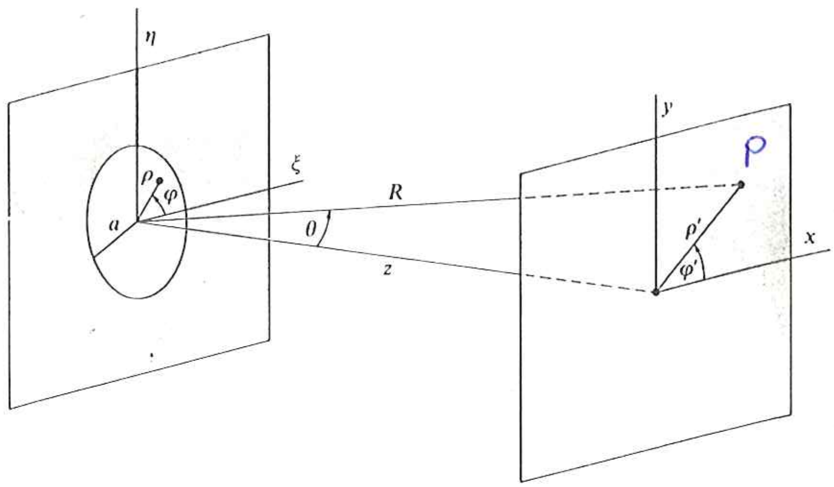
Diffracted light intensity $I(P) = 16\,C^2 a^2 b^2 \left(\frac{\sin \gamma_a}{\gamma_a}\right)^2 \left(\frac{\sin \gamma_b}{\gamma_b}\right)^2$

The diffracted intensity is a rectangular array of bright spots. Most of the light is contained in the on axis maxima as is shown below.



Fraunhofer Diffraction by a Circular Aperture

Consider a plane wave incident on a circular aperture of radius a as shown below.



The light amplitude at P is:

$$\psi(P) = \underbrace{-i\psi_0 \frac{e^{ikR}}{\lambda R}}_{\equiv C} \int_{\sigma} e^{-ik(\alpha\xi + \beta\eta)}\, d\xi\, d\eta$$

We now convert to polar coordinates.

$$\xi = \rho\cos\varphi \qquad\qquad x = \rho'\cos\varphi'$$
$$\eta = \rho\sin\varphi \qquad\qquad y = \rho'\sin\varphi'$$

Then $\alpha \equiv \dfrac{x}{R} = \dfrac{\rho'}{R}\cos\varphi'$

$\beta \equiv \dfrac{y}{R} = \dfrac{\rho'}{R}\sin\varphi'$

Now $\dfrac{\rho'}{R} = \sin\theta$. For points P nearly on axis,

θ is small. $\therefore \theta \simeq \sin\theta$
$= \frac{\rho'}{R}$

$$\Rightarrow \quad \alpha = \theta \cos\varphi'$$
$$\beta = \theta \sin\varphi'$$

$$\therefore \alpha\xi + \beta\eta = \theta\cos\varphi' \rho\cos\varphi + \theta\sin\varphi' \rho\sin\varphi$$

$$= \rho\theta\left[\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'\right]$$

$$= \rho\theta \cos(\varphi - \varphi')$$

Since there is cylindrical symmetry about the z axis, the intensity $I(P)$ is independent of φ'. We therefore set $\varphi' = 0$.

$$\therefore \psi(P) = C \int_0^a \int_0^{2\pi} e^{-ik\rho\theta\cos\varphi} \, d\varphi \, \rho \, d\rho$$

This can be simplified using the Bessel function $J_0(u)$.

$$J_0(u) = \frac{1}{2\pi} \int_0^{2\pi} e^{iu\cos\varphi} \, d\varphi$$

Hence $\psi(P) = C \int_0^a 2\pi J_0(-k\rho\theta) \rho \, d\rho$.

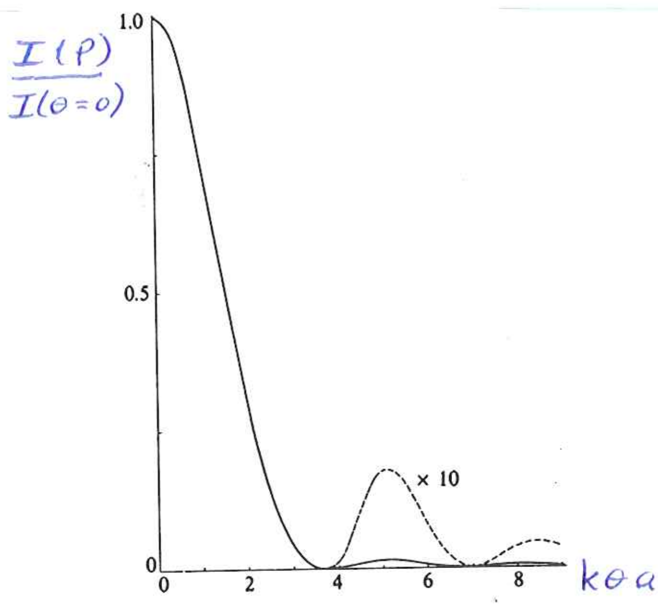
$$= 2\pi C \int_0^a J_0(k\rho\theta) \rho \, d\rho \quad \text{since } J_0(u) = J_0(-u)$$

Next we use $\int u\, J_0(u)\, du = u\, J_1(u)$ to obtain:

$$\psi(P) = 2\pi C a^2 \frac{J_1(k\theta a)}{k\theta a}.$$

$\therefore$ light intensity $\boxed{I(P) = \pi^2 C^2 a^4 \left(\frac{2 J_1(k\theta a)}{k\theta a}\right)^2}$

This intensity normalized to intensity when $\theta = 0$ is plotted below.



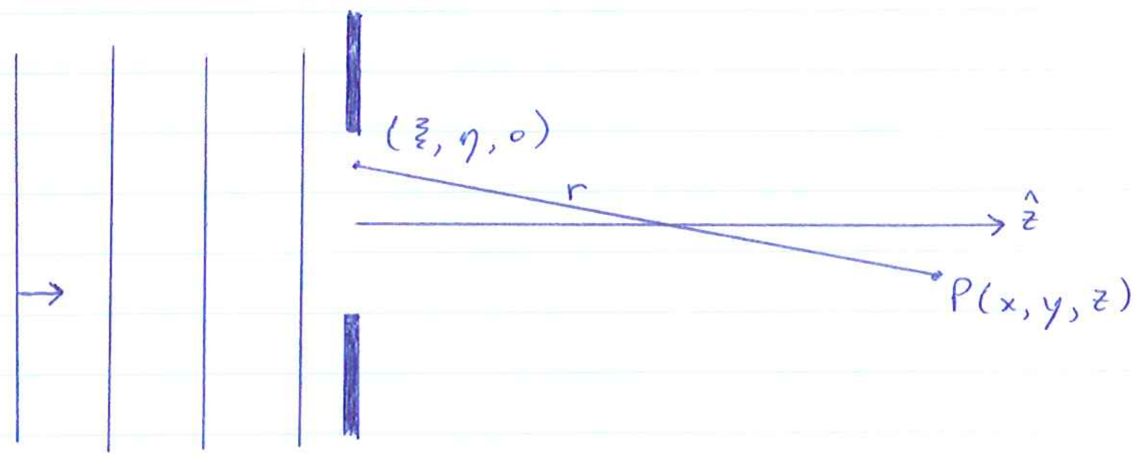
Most of the diffracted light is contained in the central maximum, as is the case with the infinite rectangular slit. A comparison of the maxima and minima positions for the infinite slit and circular aperture is given in the following table.

Salient Points of the Diffraction Patterns for the Infinite Slit and for the Circular Aperture

	Slit		Circular aperture	
	$k\alpha a$	$\left(\frac{\sin k\alpha a}{k\alpha a}\right)^2$	$k\theta a$	$\left(\frac{2J_1(k\theta a)}{k\theta a}\right)^2$
First maximum	0	1	0	1
First minimum	$\pi = 3.14$	0	$1.22\pi = 3.83$	0
Second maximum	$1.43\pi = 4.49$	0.0472	$1.64\pi = 5.14$	0.0175
Second minimum	$2\pi = 6.28$	0	$2.23\pi = 7.02$	0
Third maximum	$2.46\pi = 7.72$	0.0169	$2.68\pi = 8.42$	0.0042
Third minimum	$3\pi = 9.42$	0	$3.24\pi = 10.17$	0
Fourth maximum	$3.47\pi = 10.90$	0.0083	$3.67\pi = 11.62$	0.0016

Fresnel Diffraction

The evaluation of the light intensity near the aperture where the Fraunhofer limit $R \gg \frac{d^2}{\lambda}$ does not apply is called Fresnel diffraction.
Consider a plane wave ψ_0 incident on a planar diffracting surface.



For points P nearly on the z axis, we have previously shown that the light amplitude is given by:

$$\psi(P) = -i \frac{k\psi_0}{2\pi} \int_\sigma \frac{e^{ikr}}{r}\, da$$

where $r = \left[(x-\xi)^2 + (y-\eta)^2 + z^2 \right]^{1/2}$

$$= z \left[1 + \frac{(x-\xi)^2}{z^2} + \frac{(y-\eta)^2}{z^2} \right]^{1/2}$$

$$\simeq z \left[1 + \frac{(x-\xi)^2}{2z^2} + \frac{(y-\eta)^2}{2z^2} \right]$$

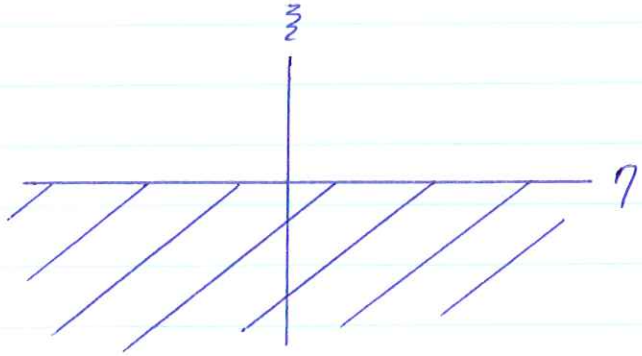
assuming $z \gg$ size of diffracting aperture

The light amplitude then becomes:

$$\psi(P) = -i \frac{k\psi_0}{2\pi} \frac{e^{ikz}}{z} \int_\sigma \exp \frac{ik}{2z}\left[(x-\xi)^2+(y-\eta)^2\right] d\xi d\eta$$

Example

We shall now illustrate Fresnel diffraction for a semi-infinite opaque plane shown below.



The light amplitude then becomes:

$$\psi(P) = C \int_0^\infty \exp \frac{ik}{2z} (x-\xi)^2 \, d\xi$$

where C contains the terms in front of the integral and the η integral.

Next we make a change of integration variable.

$$u \equiv \sqrt{\frac{k}{\pi z}} \, (x-\xi)$$

or $\quad x-\xi = \sqrt{\frac{\pi z}{k}} \, u.$

$$d\xi = -\sqrt{\frac{\pi z}{k}} \, du.$$

When $\xi = 0 \quad u = \sqrt{\frac{k}{\pi z}}\, x \equiv u_0$

$\xi = \infty \quad u = -\infty.$

$$\therefore \psi(P) = C \sqrt{\frac{\pi z}{k}} \int_{-\infty}^{u_0} \exp\left(i \frac{\pi}{2} u^2\right) du.$$

Relation of C To Incident Intensity

If P is far from the diffracting edge such that $x \to \infty$ or $u_0 \to \infty$ then we expect $\psi(P) = \psi_0$, the incident plane wave.

$$\lim_{u_0 \to \infty} \psi(P) = C \sqrt{\frac{\pi z}{k}} \int_{-\infty}^{\infty} \exp\left(\frac{i\pi}{2} u^2\right) du.$$

$$= C \sqrt{\frac{\pi z}{k}} \,(1+i)$$

Exercise: Show $\int_{-\infty}^{\infty} \exp\left(i \frac{\pi}{2} u^2\right) du = 1+i.$

Incident intensity $I_0 = |\psi_0|^2 = \frac{2 C^2 \pi z}{k}$

Hence we find the light intensity at P to be given by:

$$\boxed{I(P) = \frac{I_0}{2} \left| \int_{-\infty}^{u_0} \exp\left[i \frac{\pi}{2} u^2\right] du \right|^2}$$

Comment

The assumption made initially that P is at position $z \gg$ size of diffracting aperture does not hold for a semi-infinite aperture! Hence it is surprising that the results agree so well with observation!

Evaluation of Integral

$$I(P) = \frac{I_0}{2} \left| \int_{-\infty}^{u_0} \cos \frac{\pi}{2} u^2 \, du + i \int_{-\infty}^{u_0} \sin \frac{\pi}{2} u^2 \, du \right|^2$$

$$= \frac{I_0}{2} \left\{ \left[C(u_0) - C(-\infty) \right]^2 + \left[S(u_0) - S(-\infty) \right]^2 \right\}$$

where the so called Fresnel integrals

$$C(u) \equiv \int_0^u \cos \frac{\pi u^2}{2} \, du$$

$$S(u) \equiv \int_0^u \sin \frac{\pi u^2}{2} \, du.$$

In general $C(u)$ + $S(u)$ must be evaluated numerically.

Exercise: Derive the following properties for the Fresnel integrals.

1) $C(-u) = -C(u)$ and $S(-u) = -S(u)$

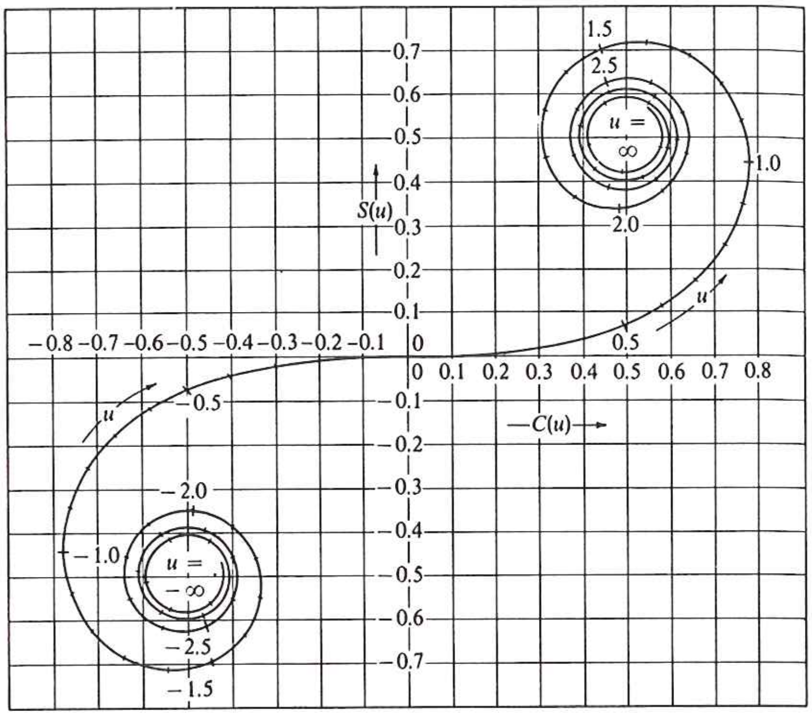
2) $C(0) = S(0) = 0$

3) $C(\infty) = S(\infty) = 1/2$

$C(-\infty) = S(-\infty) = -1/2$

Cornu's Spiral

We now define a curve $\vec{Q}(u) \equiv (C(u), S(u))$ called Cornu's Spiral and shown below.

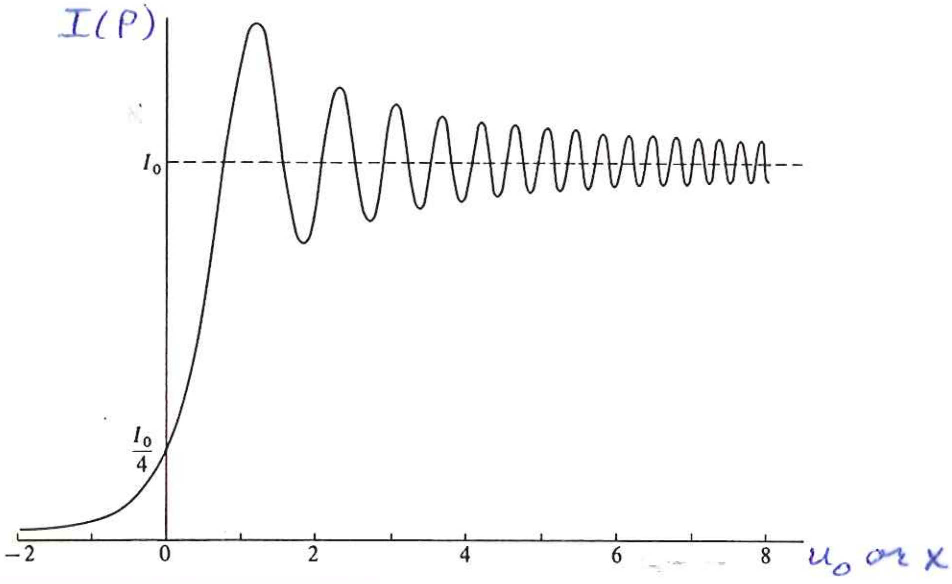


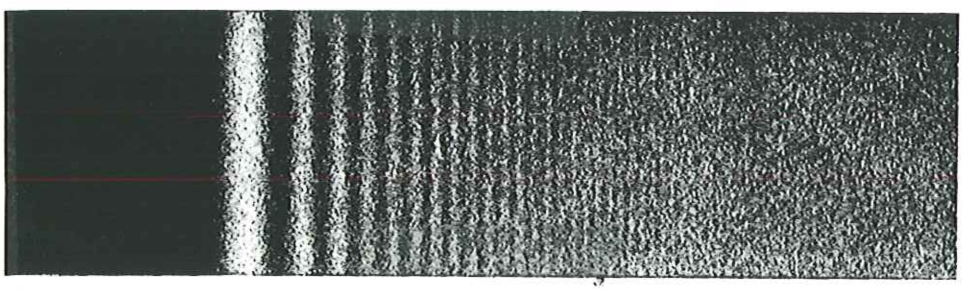
The intensity at P is then simply:

$$I(P) = \frac{I_0}{2} \left| \vec{Q}(u_0) - \vec{Q}(-\infty) \right|^2$$

i.e. $I(P) \propto$ length of line from $\vec{Q}(u_0)$ to point $\vec{Q}(-\infty) = \left(-\frac{1}{2}, -\frac{1}{2}\right)$.

The intensity is plotted in the next figure and compared to actual observation below that. Note the excellent agreement!





Exercise: Show the following

1) $I(u_0 = -\infty) = 0$

2) $I(u_0 = 0) = \frac{I_0}{4}$

3) $I(u_0 = +\infty) = I_0$

Assignment

1) Verify that Babinet's Principle holds for the diffracting disk and for the aperture in an infinite plane.

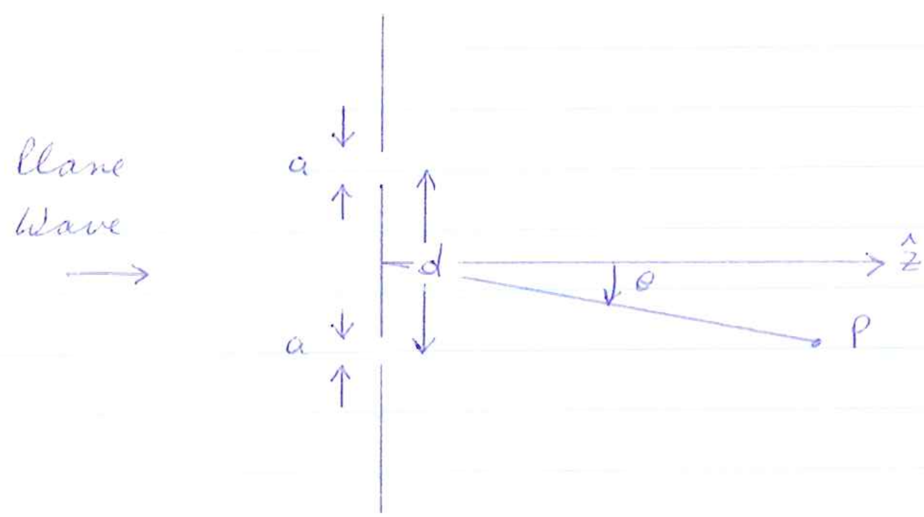
2) Diffraction from a single slit of infinite length

a) Why is the central maximum twice as wide as the other maxima?

b) Evaluate the Fraunhofer limit for a 1 mm wide slit illuminated by a HeNe laser $\lambda = 6328\,\text{Å}$.

c) A screen is located 4 m. from the slit described in b. What is the distance between the two minima next to the central maximum?

3) A plane wave is incident on a double slit.



Assuming Fraunhofer diffraction, find the intensity at P and plot it versus θ. Discuss how d and a affect I (P).