Quiz 4

Name: ______________________________

Total = 20 marks

1. (3 marks) Write down the continuity equation and state in words what it means.

See lecture notes.

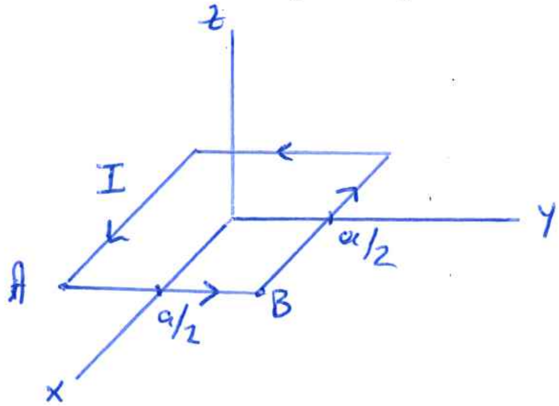
2. (2 marks) Gauge
   a) Write down the Lorentz gauge.

   See lecture notes.

   b) Write down the Coulomb gauge.

   See lecture notes.

3. (5 marks) Consider a current I in a square loop centered at the origin in the xy plane shown below



a) Find the magnetic field on the z axis just for the straight section A to B.

$$\vec{B}(\vec{r}) = \frac{I}{c} \oint \frac{d\vec{r}\,' \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$

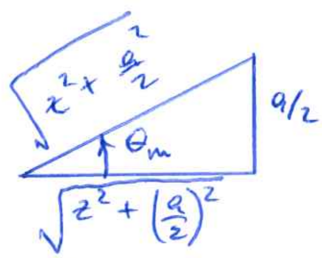
$\vec{r} = (0, 0, z)$

$\vec{r}\,'_{AB} = (\frac{a}{2}, y', 0)$

$d\vec{r}\,' = \hat{y}\, dy'$

$$= \frac{I}{c}\left(z, 0, \frac{a}{2}\right) \int_{-a/2}^{a/2} \frac{dy'}{\left[\left(\frac{a}{2}\right)^2 + y'^2 + z^2\right]^{3/2}}$$

Let $y' = \sqrt{z^2 + \left(\frac{a}{2}\right)^2}\, \tan\theta$



$$= \frac{2I}{c}\left(z, 0, \frac{a}{2}\right) \int_0^{\theta_m} \frac{\left(z^2 + \left(\frac{a}{2}\right)^2\right)^{1/2} \sec^2\theta \, d\theta}{\left(z^2 + \left(\frac{a}{2}\right)^2\right)^{3/2} \sec^3\theta}$$

$$= \frac{2I}{c} \frac{(z, 0, a/2)}{z^2 + \left(\frac{a}{2}\right)^2} \sin\theta_m$$

$$\vec{B}(0,0,z) = \frac{2I}{c} \frac{(z, 0, a/2)}{\left(z^2 + \left(\frac{a}{2}\right)^2\right)} \frac{a/2}{\sqrt{z^2 + \frac{a^2}{2}}}$$

b) Find the magnetic field on the z axis due to current in the square loop.

When considering 4 sides of loop

$\vec{B}_{TOT}(0,0,z) \parallel \hat{z}$.

$$\therefore \vec{B}_{TOT}(0,0,z) = \hat{z}\,\frac{8I}{c}\,\frac{(a/2)^2}{z^2+\left(\frac{a}{2}\right)^2}\,\frac{1}{\sqrt{z^2+\frac{a^2}{2}}}$$

$$= \hat{z}\,\frac{2Ia^2}{c}\,\frac{1}{z^2+\left(\frac{a}{2}\right)^2}\,\frac{1}{\sqrt{z^2+\frac{a^2}{2}}}$$

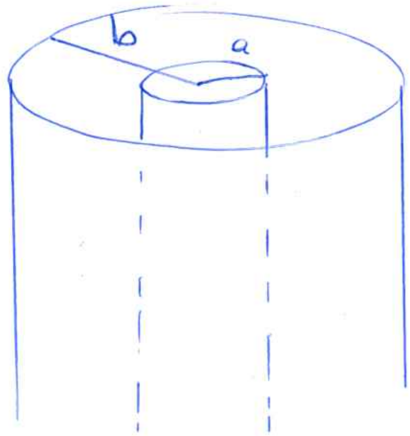
c) Find the magnetic field along the z axis for when z >>a.

$$z \gg a \Rightarrow \vec{B}_{TOT}(0,0,z) = \hat{z}\,\frac{2Ia^2}{c\,z^3}$$

$$= \hat{z}\,\frac{2m}{z^3}$$ where mag. dipole moment $m = \frac{Ia^2}{c}$

4. (5 marks) An infinitely long conducting cylinder of radius a has charge per unit length λ. It is surrounded by dielectric material ε up to radius b. At b there is a grounded conducting cylindrical shell.

a) Find the electric field everywhere



By symmetry $\vec{D} = D(r)\,\hat{r}$ ← cylindrical radial coord.

$$\int \vec{D}\cdot d\vec{a} = 4\pi \int \rho\, dV$$

$$D \cdot 2\pi r l = 4\pi \lambda l$$ for cylinder of radius r & length l.

$$\vec{D} = \frac{2\lambda}{r}\,\hat{r}$$

$r < a \qquad \vec{E} = 0$

$a < r < b \qquad \vec{E} = \dfrac{2\lambda}{\epsilon r}\,\hat{r}$

$r > b \qquad \vec{E} = 0.$

b) Find the total energy per unit length contained by the field.

$$U_{TOT} = \int \frac{\vec{E}\cdot\vec{D}}{8\pi}\, dV$$

$$= \frac{1}{8\pi}\int_a^b \left(\frac{2\lambda}{r}\right)^2 \frac{1}{\epsilon}\, 2\pi r l\, dr$$

$$= \frac{4\lambda^2\, 2\pi l}{8\pi\epsilon}\int_a^b \frac{dr}{r}$$

$$U_{TOT} = \frac{\lambda^2 l \ln(b/a)}{\epsilon}$$

∴ energy per unit length is $\dfrac{\lambda^2}{\epsilon}\ln(b/a)$

5. (5 marks) Derive an expression for the skin depth. For simplicity assume the displacement current term is negligible.

See lecture notes.