Assignment 5

1. Show that the real and imaginary parts of the plane wave $\psi = A e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ satisfy the three dimensional wave equation.

$$Re\,\psi = A\cos(\vec{k}\cdot\vec{r}-\omega t)$$

$$\frac{\partial\, Re\,\psi}{\partial x} = -k_x A \sin(\vec{k}\cdot\vec{r}-\omega t)$$

$$\frac{\partial^2 Re\,\psi}{\partial x^2} = -k_x^2 A\cos(\vec{k}\cdot\vec{r}-\omega t) = -k_x^2\, Re\,\psi$$

Similarly $\frac{\partial^2 Re\,\psi}{\partial y^2} = -k_y^2\, Re\,\psi \qquad \frac{\partial^2 Re\,\psi}{\partial z^2} = -k_z^2\, Re\,\psi$

$$\frac{\partial^2 Re\,\psi}{\partial t^2} = -\omega^2\, Re\,\psi$$

3D Wave Eqn. $\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$

OR $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$

$$-k_x^2\, Re\,\psi - k_y^2\, Re\,\psi - k_z^2\, Re\,\psi = \frac{-\omega^2}{v^2}\, Re\,\psi$$

$$|\vec{k}|^2 = \frac{\omega^2}{v^2}$$

Similarly one can show $Im\,\psi$ satisfies the 3D wave eqn.

2. Write down Maxwell's equations in differential form and state in words what each means.

$\nabla \cdot \vec{E} = 4\pi\rho$ Gauss Law

Electric field flux coming out of unit volume equals 4π times charge enclosed in unit volume.

$\nabla \cdot \vec{B} = 0$

Magnetic field flux out of any closed surface is zero. This is because there are no magnetic monopoles.

$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ Faraday's Law

A time dependent magnetic field flux generates a rotating electric field. This is used to produce electricity.

$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ Ampere's Law + Displacem Current

A current or time dependent electric field generates a rotating magnetic field.

3. Derive the following wave equation from Maxwell's Laws in vacuum.

$$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

In vacuum ρ + $\vec{J}$ are both zero.

$$\therefore \quad \nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{B}) = \frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \vec{E})$$

$$\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = -\frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$ using identity + Faraday's Law

$$\nabla \cdot \vec{B} = 0 \Rightarrow \quad \nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

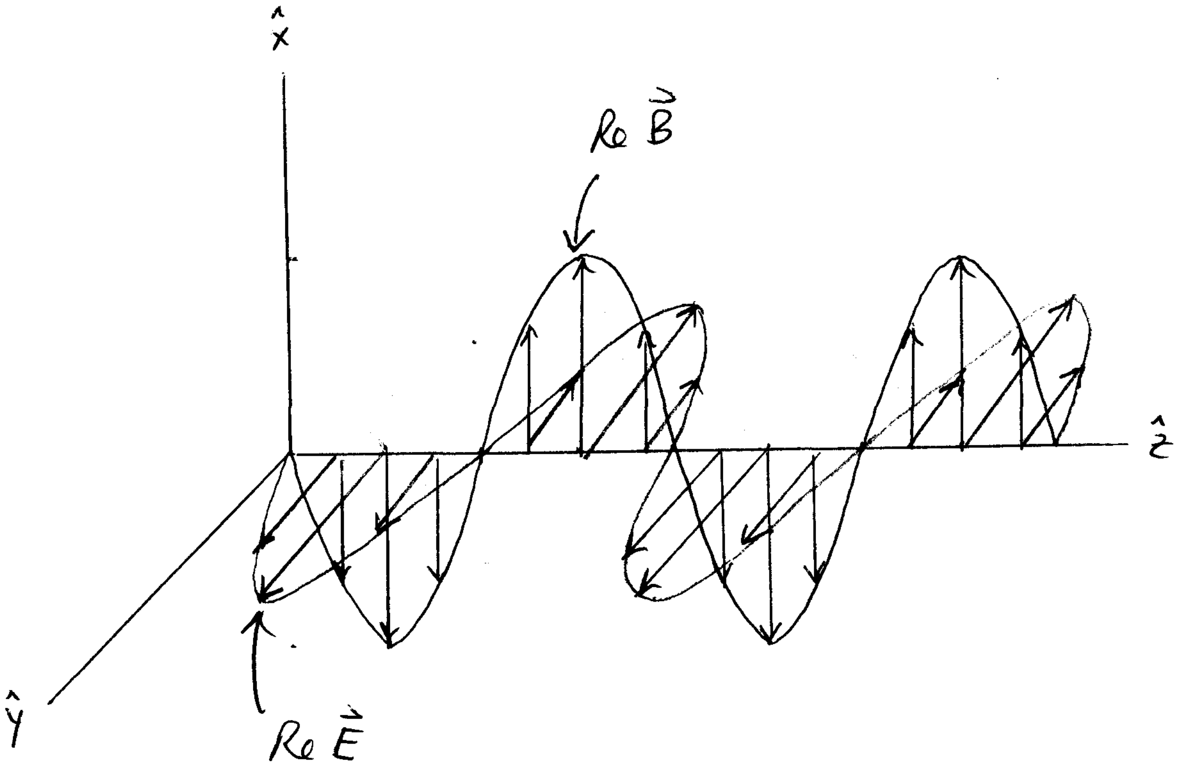
$\therefore$ $\vec{E}$ + $\vec{B}$ satisfy the same wave eqn. in vacuum

4. Sketch the electric and magnetic fields corresponding to a plane wave propagating in the $\hat{z}$ direction having electric field in the $\hat{y}$ direction.

$$\vec{E} = \hat{y} E_o e^{i(kz-\omega t)}$$

Light travels in direction $\vec{E} \times \vec{B}$ which we are told is $\hat{z}$.

$$\therefore \vec{B} = -\hat{x} B_o e^{i(kz-\omega t)}$$



5. Circular Polarization

a) $\hat{\epsilon}_+ \cdot \hat{\epsilon}_+ = (\hat{x}\cos\omega t + \hat{y}\sin\omega t)\cdot(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$

$= \cos^2\omega t + \sin^2\omega t$

$\therefore \langle \hat{\epsilon}_+ \cdot \hat{\epsilon}_+ \rangle = 1$

$\hat{\epsilon}_+ \cdot \hat{\epsilon}_- = \cos^2\omega t - \sin^2\omega t$

$\langle \hat{\epsilon}_+ \cdot \hat{\epsilon}_- \rangle = \langle \cos^2\omega t \rangle - \langle \sin^2\omega t \rangle$

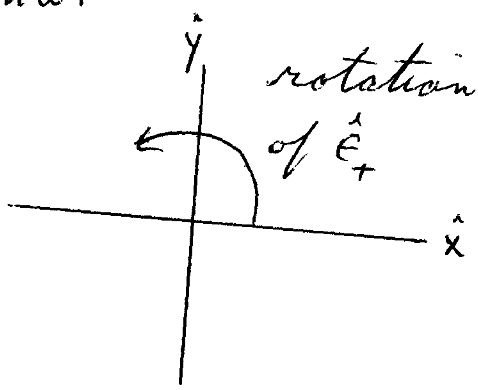
$= \frac{1}{2} - \frac{1}{2}$

$= 0$

Similarly $\langle \hat{\epsilon}_- \cdot \hat{\epsilon}_- \rangle = 1$ & $\langle \hat{\epsilon}_- \cdot \hat{\epsilon}_+ \rangle = 0$

b) $\hat{\epsilon}_+ = \hat{x}\cos\omega t + \hat{y}\sin\omega t$

ωt	$\hat{\epsilon}_+$
0	$\hat{x}$
$\pi/2$	$\hat{y}$
π	$-\hat{x}$
$\vdots$	$\vdots$



rotation direction of $\hat{\epsilon}_+$

c) Similarly $\hat{\epsilon}_-$ rotates clockwise.

d) $\vec{E} = \hat{x} E_0 \cos\omega t$

$= \frac{E_0}{2}(\hat{\epsilon}_+ + \hat{\epsilon}_-)$

$\therefore$ half of linearly polarized wave rotates in clockwise & half in counterclockwise direction.