Assignment 4

1. Find the following for the wave $\psi = 5\cos(2x + 3t)$
   a) wave vector
   b) wavelength
   c) frequency
   d) period
   e) phase velocity
   f) amplitude

Wave vector $k = 2$ meters^{-1}

Wavelength $\lambda = \frac{2\pi}{k} = \pi$ meters

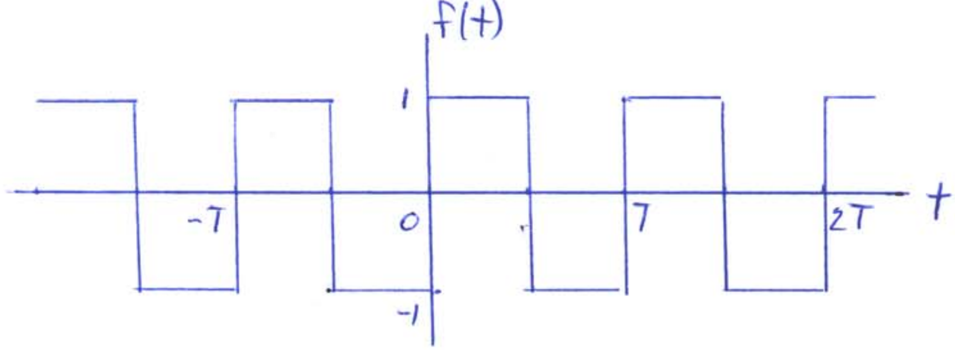
Frequency $\nu = \frac{3}{2\pi}$ sec^{-1}

Period $T = \frac{1}{\nu} = \frac{2\pi}{3}$ sec.

Phase Velocity $v_p = \frac{\omega}{k} = \frac{3}{2}$ meter/sec

Amplitude $A = 5$ meters

2. Consider a series of square wave pulses shown below.



Fourier analysis says that this pulse can expanded as:

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t) \qquad \omega \equiv \frac{2\pi}{T}$$

a) Show that $A_n = 4/n\,\pi$ where n is odd, $A_n = 0$ for n is even
$B_n = 0$ and $A_o = 0$

$f(t)$ is odd $\Rightarrow A_0$ & $A_n = 0$.

$$\therefore f(t) = \sum_{n=1}^{\infty} B_n \sin n\omega t$$

$$\int_0^T f(t) \sin m\omega t \, dt = \sum_{n=1}^{\infty} B_n \underbrace{\int_0^T \sin n\omega t \sin m\omega t \, dt}_{\equiv I} \quad (1)$$

Integral I

Case 1: $n = m$ $\quad I = \int_0^T \sin^2 n\omega t \, dt$

$$= \int_0^T \frac{1 - \cos 2n\omega t}{2} \, dt$$

$$= \left[\frac{t}{2} - \frac{\sin 2n\omega t}{4n\omega} \right]_0^T$$

$$= \frac{T}{2}$$

Case 2: $n \neq m$

$$I = \int_0^T \frac{\cos(n-m)\omega t - \cos(n+m)\omega t}{2}\, dt$$

$$= \left[\frac{\sin(n-m)\omega t}{2(n-m)\omega} - \frac{\sin(n+m)\omega t}{2(n+m)\omega} \right]_0^T$$

$$= 0$$

$$\therefore \int_0^T \sin n\omega t \sin m\omega t \, dt = \frac{T}{2} \delta_{nm}$$

$$\int_0^T f(t) \sin n\omega t \, dt = \int_0^{T/2} \sin n\omega t \, dt - \int_{T/2}^T \sin n\omega t \, dt$$

$$= \left[\frac{-\cos n\omega t}{n\omega} \right]_0^{T/2} + \left[\frac{\cos n\omega t}{n\omega} \right]_{T/2}^T$$

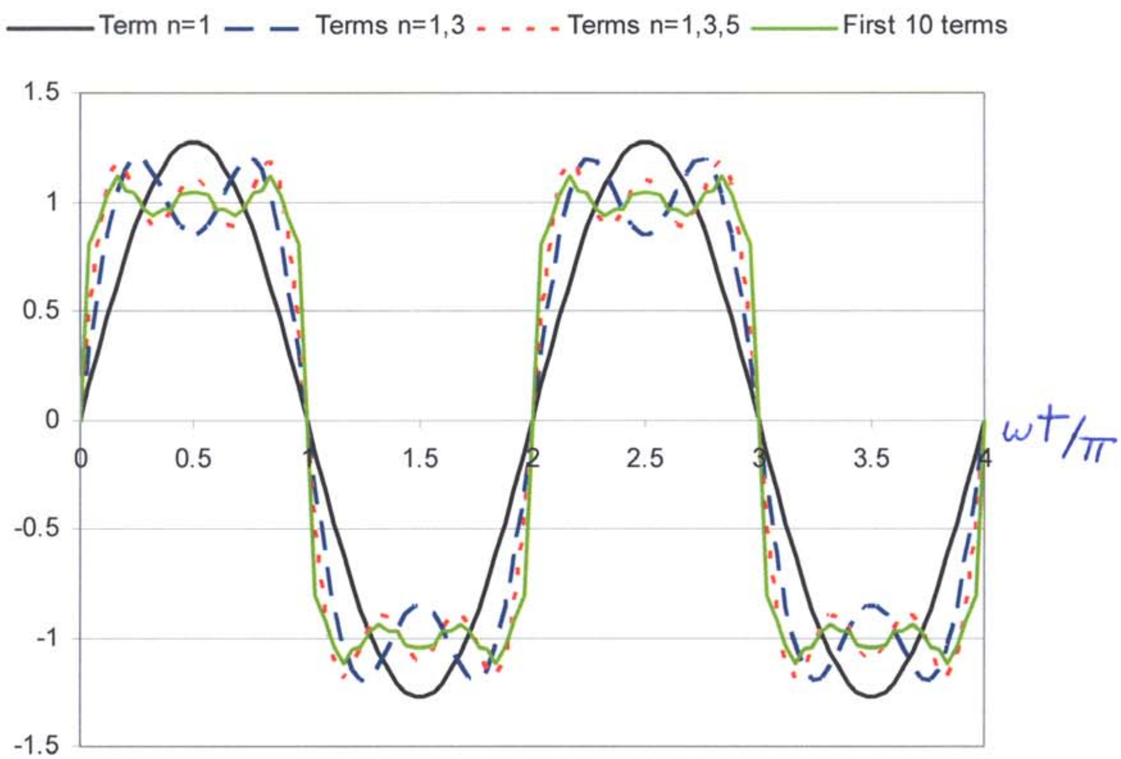
$$= \frac{1}{n\omega}\left(1 - \cos \frac{n\omega T}{2}\right) + \frac{1}{n\omega}\left(\cos n\omega T - \cos \frac{n\omega T}{2}\right)$$

$$= \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{n\omega} & \text{for } n \text{ odd.} \end{cases}$$

$$\therefore (1) \Rightarrow \quad \frac{4}{n\omega} = B_n \frac{T}{2} \quad \Rightarrow \quad B_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{n\pi} & \text{for } n \text{ odd.} \end{cases}$$

$$\therefore \quad f(t) = \frac{4}{\pi} \sum_{n=1,3,5} \frac{\sin n\omega t}{n}$$

b) Plot the first term, first 2 terms and first 3 terms in the sum.



c) Hence, a pulse of light which can convey information, is composed of many frequencies. Estimate the range of frequencies $\Delta\nu$ required to make a one femtosecond laser pulse using the Heisenberg Uncertainty Principle $\Delta\nu\,\Delta t > 2\pi$.

$\Delta t = 10^{-15}$ sec.

$$\Rightarrow \Delta\nu \simeq \frac{2\pi}{10^{-15}}$$

$$\simeq 6 \times 10^{15} \text{ Hz}.$$

Hence, a femtosecond pulse does not have a well defined frequency!

3. Superposition Principle

a) Show that if Ψ_1 and Ψ_2 are solutions of the 3 dimensional wave equation that their sum also is a solution.

$$\nabla^2(\psi_1 + \psi_2) = \nabla^2\psi_1 + \nabla^2\psi_2$$

$$= \frac{1}{v^2}\frac{\partial^2\psi_1}{\partial t^2} + \frac{1}{v^2}\frac{\partial^2\psi_2}{\partial t^2}$$

$$= \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(\psi_1 + \psi_2)$$

$\therefore$ $\psi_1 + \psi_2$ is also a solution.

b) This may seem trivial but show that the superposition principle does not hold for the following differential equation.

$$\frac{d^2\psi}{dx^2} = \psi^2$$

Let ψ_1 & ψ_2 be solns. i.e. $\dfrac{d^2\psi_1}{dx^2} = \psi_1^2$

$$\frac{d^2\psi_2}{dx^2} = \psi_2^2$$

But $$\frac{d^2}{dx^2}(\psi_1 + \psi_2) = \frac{d^2\psi_1}{dx^2} + \frac{d^2\psi_2}{dx^2}$$

$$= \psi_1^2 + \psi_2^2$$

$$\neq (\psi_1 + \psi_2)^2$$

$\therefore$ $\psi_1 + \psi_2$ is not a solution.

4. Damped harmonic oscillator

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt}$$

a) Consider a solution $x = A\,e^{\lambda x}$. Solve for λ. (Result will be complex) This approach is much simpler than using $x = A\cos\omega t + B\sin\omega t$.

b) Write down the general solution for the case of weak damping $km \gg \gamma^2$.

c) What is the solution for the case the mass is initially at rest at distance x_o? Plot this solution.

$$x = Ae^{\lambda x} \Rightarrow \quad m\lambda^2 = -k - \gamma\lambda$$

$$m\lambda^2 + \gamma\lambda + k = 0.$$

$$\lambda^2 + \frac{\gamma}{m}\lambda + \frac{k}{m} = 0$$

$$\lambda = \frac{-\frac{\gamma}{m} \pm \sqrt{\frac{\gamma^2}{m^2} - \frac{4k}{m}}}{2}$$

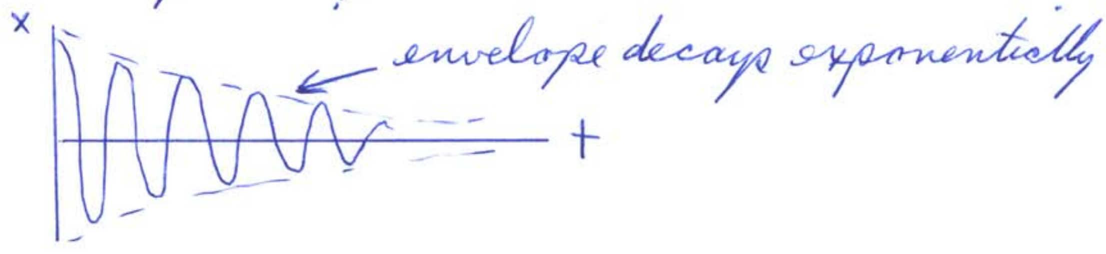
$$\lambda = \frac{-\gamma}{2m} \pm i\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$$

$$\simeq \frac{-\gamma}{2m} \pm i\omega_o \qquad \omega_o \equiv \sqrt{k/m}$$

$$\therefore\ x = e^{-\gamma t/2m}\left[A\,e^{i\omega_o t} + B\,e^{-i\omega_o t}\right]$$

$$= e^{-\gamma t/2m}\left[(A+B)\cos\omega_o t + i(A-B)\sin\omega_o t\right]$$

Taking real part yields: $x = x_o\,e^{-\gamma t/2m}\cos\omega_o t$



5. Show that the group velocity v_g is related to the phase velocity v by the following equation. Note that for the case of normal dispersion $v_g < c$.

$$v_g = \frac{c}{n + \omega \frac{dn}{d\omega}}$$

Group Velocity $v_g \equiv \frac{d\omega}{dk}$

Phase Velocity $v_p = \frac{c}{n} = \frac{\omega}{k}$.

or $\omega = \frac{ck}{n}$

$$\therefore \frac{d\omega}{dk} = \frac{c}{n} - \frac{ck}{n^2}\frac{dn}{dk}$$

$$= \frac{c}{n} - \frac{ck}{n^2}\frac{dn}{d\omega}\cdot\frac{d\omega}{dk}$$

$$\frac{d\omega}{dk}\left(1 + \frac{ck}{n^2}\frac{dn}{d\omega}\right) = \frac{c}{n}$$

$$v_g = \frac{c/n}{1 + \frac{ck}{n^2}\frac{dn}{d\omega}}$$

$$\therefore v_g = \frac{c}{n + \omega\frac{dn}{d\omega}}$$