Quiz 5

Name: ______________________________

Total = 20 marks

1. (2 marks) Explain why the electric field vanishes inside a conductor.

If $\vec{E} \neq 0$, charges move & create opposite field.

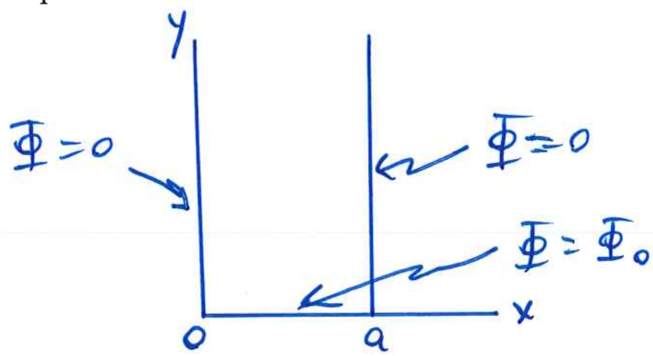
2. (4 marks) Consider a metallic, empty 3 dimensional ellipsoidal surface satisfying the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where a = 1 cm, b = 2 cm and c = 3 cm. The ellipsoid is held at a potential of 1.5 kV. Find the electric field at all points inside the ellipsoidal surface.

Obviously one solution is Φ = constant throughout volume. $\Rightarrow \vec{E} = -\nabla\Phi = 0$.
By uniqueness theorem this is only solution.

3. (6 marks) Consider an infinitely deep trench with an infinite extent the z direction where the potential on the sides is shown below.



a) Find the potential everywhere inside the tube.

$$\Phi = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} e^{-n\pi y/a}$$

For bottom $\Phi_0 = \sum_n A_n \sin\frac{n\pi x}{a}$

$$\Phi_0 \int_0^a \sin\frac{m\pi x}{a}\, dx = \sum_n A_n \int_0^a \sin\frac{m\pi x}{a} \sin\frac{n\pi x}{a}\, dx$$

$$\frac{a\Phi_0}{m\pi}\left[-\cos m\pi + 1\right] = \sum_n A_n \frac{a}{2}\delta_{nm}$$

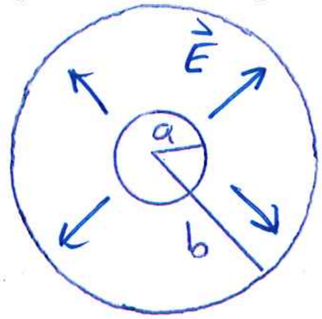
$$A_n = \frac{2}{n\pi}\Phi_0\left(1 - (-1)^n\right)$$

b) Find the electric field everywhere inside the tube.

$$\vec{E} = -\nabla\Phi$$

$$= -\sum_{n=1}^{\infty}\left(A_n \frac{n\pi}{a}\cos\frac{n\pi x}{a} e^{-n\pi y/a},\ -A_n \frac{n\pi}{a}\sin\frac{n\pi x}{a} e^{-n\pi y/a}\right)$$

4. (8 marks) Consider a conducting spherical shell of radius b that is concentric with a conducting sphere of radius a as shown below. The inner sphere is at potential Φo while the outer spherical shell is at ground potential.



Solution of Laplace's equation for azimuthally symmetric problem is:

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)} \right) P_\ell(\cos\theta)$$

Orthogonality relation of Legendre's polynomials is:

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

a) Find the potential at all points between the inner sphere and outer spherical shell.

$$0 = \Phi(b,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell b^\ell + B_\ell b^{-(\ell+1)} \right) P_\ell(\cos\theta)$$

$$\Rightarrow A_\ell = -B_b\, b^{-(2\ell+1)}$$

$$\Phi_0 = \Phi(a,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell a^\ell + B_\ell a^{-(\ell+1)} \right) P_\ell(\cos\theta)$$

$$\Phi_0 = \sum B_\ell \left(\frac{-a^\ell}{b^{2\ell+1}} + a^{-(\ell+1)} \right) P_\ell(\cos\theta)$$

$\ell = 0$ term $\Rightarrow \Phi_0 = B_0\left(\frac{-1}{b} + \frac{1}{a} \right) \Rightarrow B_0 = \frac{ab\Phi_0}{b-a}$, $A_0 = \frac{-a\Phi_0}{b-a}$

$\ell \neq 0 \Rightarrow B_\ell = 0$

$$\Rightarrow \Phi = \frac{-a\Phi_0}{b-a} + \frac{ab\Phi_0}{b-a}\frac{1}{r} = \frac{a\Phi_0}{b-a}\left(\frac{b}{r} - 1 \right)$$

b) Find an expression for the electric field between the two spheres.

$$\vec{E} = -\nabla \Phi$$

$$E_r = -\frac{\partial \Phi}{\partial r} = \frac{ab}{b-a} \frac{\Phi_0}{r^2}$$

c) Sketch the electric field.

See diagram.

surface normal is $-\hat{r}$

d) Find expressions for the surface charge densities at r = a and at r = b.

$$\sigma(r=a) = \frac{E_r(r=a)}{4\pi} = \frac{\Phi_0}{4\pi} \frac{b}{a(b-a)}$$

$$\sigma(r=b) = \frac{-E_r(r=b)}{4\pi} = -\frac{\Phi_0}{4\pi} \frac{a}{b(b-a)}$$