Phys 3010 **Assignment 9**

11.1 Moment of Inertia Tensor $I_{ij} = \int \rho(\vec{r}) \left[\delta_{ij} \sum_k x_k^2 - x_i x_j \right] dV$

For a homogeneous sphere of radius R

$$I_{33} = \rho \int (r^2 - z^2)\, dV$$

$$= \rho \int_0^R \int_0^\pi \int_0^{2\pi} (r^2 - r^2\cos^2\theta)\, r^2 \sin\theta \, d\phi\, d\theta\, dr$$

$$= 2\pi\rho \int_0^R r^4\, dr \int_{-1}^{1} (1-u^2)\, du \quad \text{where } u = \cos\theta$$

$$= 2\pi\rho \frac{R^5}{5} \left(u - \frac{u^3}{3} \right)_{-1}^{1}$$

$$= \frac{8\pi}{15} \rho R^5$$

$$\therefore I_{33} = \frac{2}{5} M R^2 \quad \text{using} \quad M = \rho \frac{4}{3}\pi R^3$$

By symmetry $I_{11} = I_{22} = I_{33}$. The off diagonal elements are zero. For example:

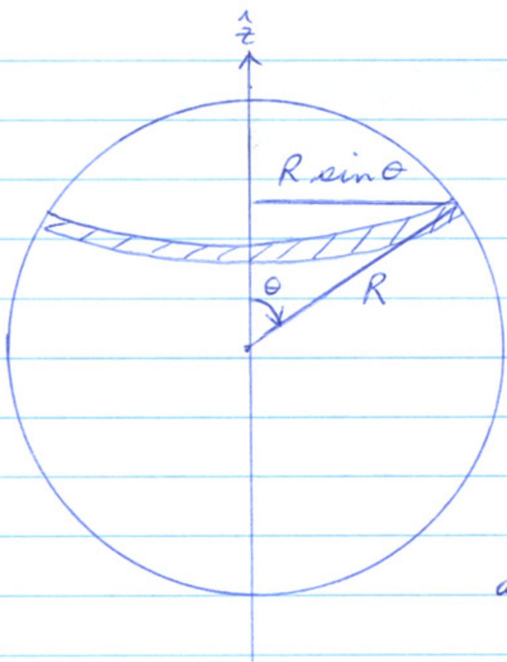
$$I_{12} = \rho \int -xy\, dV$$

$$= -\rho \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin^2\theta \sin\phi\cos\phi \; r^2 \sin\theta \, d\phi\, d\theta\, dr$$

$$= 0$$

Hence $\underset{\sim}{I}$ is a diagonal matrix and the principal moments are $I_1 = I_2 = I_3 = \frac{2}{5} M R^2$.

11.6



Consider a ring of spherical shell.

Area of ring is $R\,d\theta\ 2\pi R\sin\theta$

mass of ring is $\underset{\text{density}}{\sigma}\ 2\pi R^2\sin\theta\,d\theta.$

$\therefore$ Moment of Inertia for rotation about $\hat{z}$ axis is:

$$I = \int_0^\pi (R\sin\theta)^2\,\sigma\,2\pi R^2\sin\theta\,d\theta$$

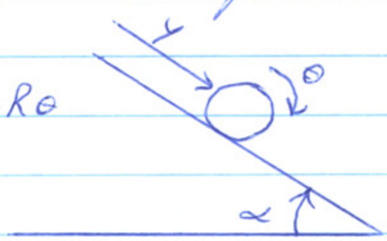
$$= \int_0^\pi \sigma R^4 \int_0^\pi \sin^3\theta\,d\theta$$

$$= \frac{8\pi}{3}\sigma R^4$$

$$\therefore I = \frac{2}{3}MR^2 \quad \text{where } M = \sigma\,4\pi R^2$$

The two spheres will now be rolled down an inclined plane.

$y = R\theta$



Kinetic Energy $T = \frac{M}{2}\dot{y}^2 + \frac{I}{2}\dot{\theta}^2$

$$= \left(\frac{M}{2} + \frac{I}{2R^2}\right)\dot{y}^2$$

Potential Energy $U = -Mgy\sin\alpha$

Lagrangian $L = \left(M + \frac{I}{R^2}\right)\frac{\dot{y}^2}{2} + Mgy\sin\alpha$

Eqn. of Motion $\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$

$$Mg\sin\alpha - \left(M + \frac{I}{R^2}\right)\ddot{y} = 0$$

$$\ddot{y} = g\frac{MR^2\sin\alpha}{MR^2 + I}$$

But I (solid sphere) $<$ I (spherical shell)
$\therefore$ solid sphere has greater acceleration down inclined plane than spherical shell of same mass & radius.

11.13 Moment of Inertia tensor is:

$$I_{ij} = \sum_\alpha m_\alpha\left[\delta_{ij}\sum_k x_{\alpha k}^2 - x_{\alpha i}x_{\alpha j}\right]$$

$$\therefore I_{11} = \sum_\alpha m_\alpha\left[x_{\alpha 2}^2 + x_{\alpha 3}^2\right]$$

$$= 3mb^2 + 4m(2b^2) + 2mb^2$$
$$= 13mb^2$$

Similarly $I_{22} = 16mb^2$ & $I_{33} = 15mb^2$.

$$I_{12} = I_{21} = -\sum_\alpha m_\alpha x_{\alpha 1}x_{\alpha 2}$$

$$= -3m\,0 - 4mb^2 - 2m(-b^2)$$
$$= -2mb^2$$

Similarly $I_{23} = I_{32} = 4mb^2$ & $I_{13} = I_{31} = mb^2$.

$$\therefore \underset{\sim}{I} = mb^2\begin{pmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix}$$

The principal moments of inertia are found solving:

$$\begin{vmatrix} 13-\lambda & -2 & 1 \\ -2 & 16-\lambda & 4 \\ 1 & 4 & 15-\lambda \end{vmatrix} = 0$$

OR $\lambda^3 - 44\lambda^2 + 622\lambda - 2820 = 0$

This has solutions (found numerically) $\lambda_1 = 10$, $\lambda_2 = 14.35$ & $\lambda_3 = 19.65$

$\therefore$ principal moments of inertia are $I_1 = 10\,mb^2$, $I_2 = 14.35\,mb^2$ & $I_3 = 19.65\,mb^2$.

Principal axes are found solving $(\underset{\sim}{I} - \underset{\sim}{1} I_i)\vec{\omega} = 0$.

$$I_i = I_1 \Rightarrow \begin{pmatrix} 13-10 & -2 & 1 \\ -2 & 16-10 & 4 \\ 1 & 4 & 15-10 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

OR
$$3\omega_1 - 2\omega_2 + \omega_3 = 0 \qquad (1)$$
$$-2\omega_1 + 6\omega_2 + 4\omega_3 = 0 \qquad (2)$$
$$\omega_1 + 4\omega_2 + 5\omega_3 = 0. \qquad (3)$$

$(1) \Rightarrow \omega_3 = -3\omega_1 + 2\omega_2$ Substituting this into (2) gives:

$$-2\omega_1 + 6\omega_2 + 4(-3\omega_1 + 2\omega_2) = 0.$$
$$\therefore \omega_1 = \omega_2$$

Then $(3) \Rightarrow \omega_3 = -\omega_2$.

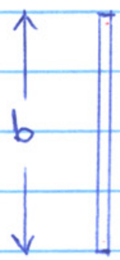
$\therefore$ principal axis associated with I_1 is $\frac{1}{\sqrt{3}}(\hat{x} + \hat{y} - \hat{z})$

Similarly principal axes associated with I_2 is $-.81\hat{x}+.29\hat{y}-.52\hat{z}$

" " I_3 " $-.14\hat{x}+.77\hat{y}+.63\hat{z}$

Note that the 3 principal axes are perpendicular.

11.20



Upright rod has total energy

$$E = \text{Potential Energy}$$
$$= mg\frac{b}{2} \qquad (1)$$

When rod hits floor, total energy

$$E = \text{Kinetic Energy}$$
$$= \frac{I}{2}\omega^2 \qquad (2)$$

Moment of Inertia of uniform rod about an end is

$$I = \int_0^b x^2 \sigma\, dx = \sigma \frac{b^3}{3} = \frac{mb^2}{3} \quad (3) \quad \text{where } m = \sigma b$$

Combining (1), (2) & (3) gives:

$$mg\frac{b}{2} = \frac{mb^2}{6}\omega^2$$

$\therefore\ \omega = \sqrt{\dfrac{3g}{b}}$ is angular velocity of rod when hitting the floor.

11.29 For a vertical top ($\theta = 0$, $\dot{\theta} = 0$) we have the following

$(11.153 \text{ \& } 11.154) \Longrightarrow \quad p_\phi = p_\psi = I_3(\dot{\phi} + \dot{\psi})$

$(11.159) \Longrightarrow \quad p_\phi = p_\psi = I_3 \omega_3$

$(11.160) \Longrightarrow \quad E' = Mgh$

This problem seeks to examine the stability of the top near $\theta = 0$. Substituting Mgh for E' in 11.160 gives:

$$Mgh = \frac{I_1}{2}\dot{\theta}^2 + \frac{I_3^2\omega_3^2(1-\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta$$

We now set $z \equiv \cos\theta$. $\therefore \dot{z} = -\sin\theta\,\dot{\theta}$

$$\dot{\theta}^2 = \frac{\dot{z}^2}{\sin^2\theta} = \frac{\dot{z}^2}{1-\cos^2\theta} = \frac{\dot{z}^2}{1-z^2}$$

$$\therefore Mgh = \frac{I_1}{2}\frac{\dot{z}^2}{1-z^2} + \frac{I_3^2\omega_3^2(1-z)^2}{2I_1(1-z^2)} + Mghz$$

$$Mgh(1-z) = \frac{I_1}{2}\frac{\dot{z}^2}{1-z^2} + \frac{I_3^2\omega_3^2(1-z)^2}{2I_1(1-z^2)}$$

Multiplying by $2I_1(1-z^2)$ gives:

$$2I_1 Mgh(1-z)(1-z^2) = I_1^2\dot{z}^2 + I_3^2\omega_3^2(1-z)^2$$

$$I_1^2\dot{z}^2 = (1-z)^2\left[2I_1 Mgh(1+z) - I_3^2\omega_3^2\right]$$

$$\dot{z}^2 = \frac{(1-z)^2}{I_1^2}\left[2I_1 Mgh(1+z) - I_3^2\omega_3^2\right] \quad (1)$$

Obviously $\dot{z}^2 \geq 0$.

Case 1: ω_3 is large such that $2I_1 Mgh(1+z) - I_3^2\omega_3^2 < 0$

$$\Rightarrow z = 1 \Rightarrow \theta = 0$$

$\therefore$ motion is stable if $4I_1 Mgh - I_3^2\omega_3^2 < 0$

Case 2: ω_3 is small such that $2I_1 Mgh(1+z) - I_3^2\omega_3^2 > 0$

$\Rightarrow \dot{z}^2 > 0 \Rightarrow \theta$ increases from 0. This continues until $z = z_0$ such that $2I_1 Mgh(1+z_0) - I_3^2\omega_3^2 = 0$.

$\therefore$ there is nutation between $z=1$ & $z=z_0$.

$\therefore$ there is a critical angular frequency ω_c such that
$\omega_3 > \omega_c \Rightarrow$ stable motion
$\omega_3 < \omega_c \Rightarrow$ nutation

where $\quad 2I_1 Mgh\, 2 - I_3^2\omega_c^2 = 0$

$$\omega_c = \frac{2}{I_3}\sqrt{MghI_1}.$$

Hence, if top starts with $\omega_3 > \omega_c$ & $\theta = 0°$, motion is stable. As friction slows the top ω_3 reduces and at $\omega_3 = \omega_c$, nutation starts.