Assignment 8 Solutions

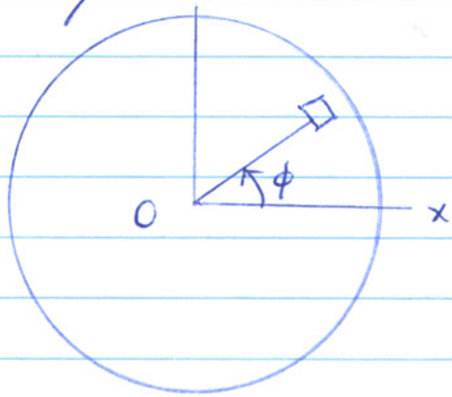
1. $I = \sum_{i=1}^{n} m_i r_i^2$

$= 2mR^2 + 2mR^2 + m(3R)^2 + m(3R)^2$

$= mR^2(2+2+9+9)$

$= 22mR^2$

2. First we need to know the moment of inertia of a disk having mass M and radius R.



Consider infinitesimal area of disk extending from ϕ to $\phi + d\phi$ and r to $r + dr$.

Mass of infinitesimal area is

$$dm = \sigma \, r \, d\phi \, dr \quad \text{where } \sigma = \frac{M}{\pi R^2}$$

$$dI = r^2 \, dm$$

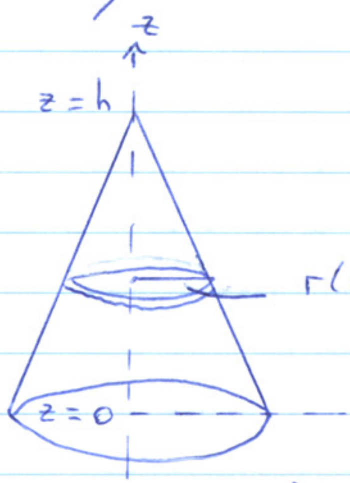
$$I = \int_0^R \int_0^{2\pi} r^2 \sigma r \, d\phi \, dr$$

$$= 2\pi\sigma \int_0^R r^3 \, dr$$

$$= 2\pi\sigma \frac{R^4}{4}$$

$$\therefore I = \frac{MR^2}{2}$$

The cone can be divided into a number of disks having thickness dz.



Mass of infinitesimal disk having radius $r(z)$ & thickness dz is

$$dm = \rho \pi r^2(z)\, dz \quad \text{where } \rho = \frac{M_{cone}}{V_{cone}} = \frac{M}{\frac{\pi}{3} R^2 h}$$

$$\therefore \; dI = \frac{1}{2} r^2\, dm$$

$$I = \frac{1}{2} \int_{z=0}^{z=h} r^2 \rho \pi r^2\, dz$$

$$= \frac{\pi}{2}\rho \int_0^h R^4 \left(1 - \frac{z}{h}\right)^4 dz.$$

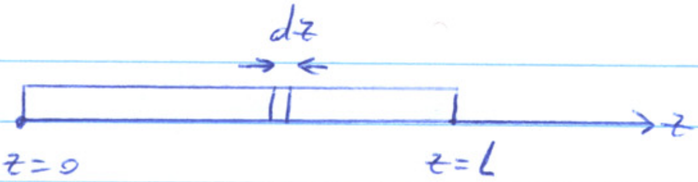
$$= \frac{\pi}{2}\rho R^4 \left[\left(-\frac{h}{5}\right)\left(1 - \frac{z}{h}\right)^5\right]_0^h$$

$$= \frac{\pi}{2}\rho R^4 \frac{h}{5}$$

$$= \frac{\pi}{2} \frac{M}{\frac{\pi}{3} R^2 h} \frac{R^4 h}{5}$$

$\therefore \; I = \dfrac{3MR^2}{10}$ is moment of inertia of cone.

3a)



Consider infinitesimal part of bar from z to $z+dz$.

$$dm = \lambda\, dz \quad \text{where } \lambda = \frac{M}{L} \text{ is linear density}$$

$$dI = z^2\, dm$$

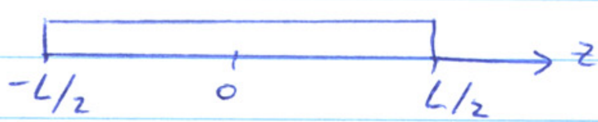
$$I = \int_{z=0}^{z=L} z^2 \lambda\, dz.$$

$$= \lambda \left.\frac{z^3}{3}\right|_0^L$$

$$= \frac{\lambda L^3}{3}$$

$$\therefore I_{end} = \frac{ML^2}{3}$$

b)



$$dI = z^2\, dm$$

$$I = \int_{-L/2}^{L/2} z^2 \lambda\, dz$$

$$= \lambda \left[\frac{z^3}{3}\right]_{-L/2}^{L/2}$$

$$= \frac{\lambda}{3}\, \frac{2L^3}{8}$$

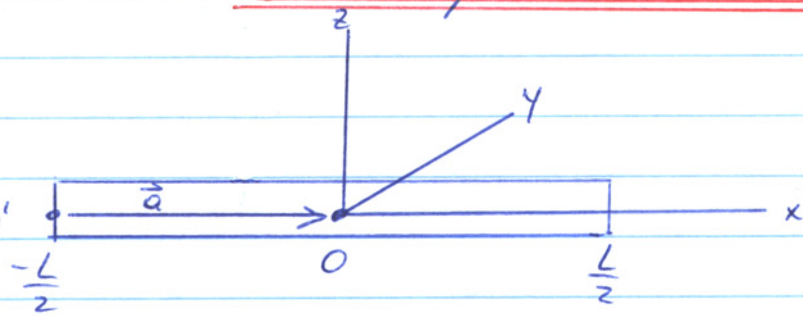
$$\therefore I_{Mid.} = \frac{ML^2}{12}$$

c) Note $I_{end} = I_{mid} + M\left(\frac{L}{2}\right)^2$

in agreement with parallel axis theorem for moment of inertia.

Moment of Inertia Tensor (Assign. 8 Cont'd.)
For Infinitesimal Bar.

4.



$\vec{a} = \left(\frac{L}{2}, 0, 0\right)$

$\lambda \equiv \frac{M}{L}$

$$I_{ij} = \int \rho \, dV \left\{ \delta_{ij} \sum x_k^2 - x_i x_j \right\}$$

$$I_{xx} \equiv I_{11} = \int_{-L/2}^{L/2} \lambda \, dx \left\{ x^2 + y^2 + z^2 - x^2 \right\}_{y=z=0 \text{ for bar}}$$

$$I_{xx} = 0$$

$$I_{yy} \equiv I_{22} = \int_{-L/2}^{L/2} \lambda \, dx \left\{ x^2 + y^2 + z^2 - y^2 \right\}_{y=z=0 \text{ for bar}}$$

$$= \int_{-L/2}^{L/2} \lambda \, dx \; x^2$$

$$I_{yy} = \frac{ML^2}{12}$$

Similarly $I_{zz} \equiv I_{33} = \frac{ML^2}{12}$

Also $I_{xy} = I_{xz} = I_{yz} = 0.$

$$\therefore \underset{\sim}{I} = \frac{ML^2}{12} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i.e. Moment of inertia is same if we rotate bar about y or z axes.

Steiner Parallel Axis Theorem

$$J_{ij} = I_{ij} + M\left\{ \delta_{ij}\, a^2 - a_i a_j \right\}$$

$$J_{xx} = I_{xx} + M\left\{ a^2 - a_x^2 \right\} = I_{xx}$$

$$J_{yy} = I_{yy} + M\left\{ a^2 - a_y^2 \right\} = I_{yy} + \frac{ML^2}{4}$$

$$J_{zz} = I_{zz} + M\left\{ a^2 - a_z^2 \right\} = I_{zz} + \frac{ML^2}{4}$$

Also $J_{xy} = I_{xy}$, $J_{xz} = I_{xz}$, $J_{yz} = I_{yz}$.

$$\therefore \underset{\sim}{J} = \frac{ML^2}{12}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{ML^2}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{ML^2}{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$