Phys 3010 **Assignment 5**

8.8 From lecture notes we have:

$$\theta(r) = \int \frac{\ell/r^2 \, dr}{\left[2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)\right]^{1/2}}$$

where $U = -\int kr\,dr = -\frac{kr^2}{2}$, (assuming $U(r=0)=0$).

Multiplying top & bottom of integrand by $\frac{r^3}{\ell}$ gives:

$$\theta(r) = \int \frac{r\,dr}{r^2\left[\frac{\mu k}{\ell^2} r^4 + \frac{2\mu E}{\ell^2} r^2 - 1\right]^{1/2}}$$

Next change integration variable to $x = r^2$.

$$\theta(x) = \frac{1}{2}\int \frac{dx}{x\left[ax^2 + bx + c\right]^{1/2}}$$ where $a \equiv \mu k/\ell^2$, $b \equiv 2\mu E/\ell^2$, $c \equiv -1$

$$= \frac{1}{2}\frac{1}{\sqrt{-c}} \arcsin\left\{\frac{bx + 2c}{x\sqrt{b^2 - 4ac}}\right\} + \theta_0$$ using integration table

$$\sin\left[2\sqrt{-c}\,(\theta - \theta_0)\right] = \frac{bx + 2c}{x\sqrt{b^2 - 4ac}}$$

$$= \frac{1}{\sqrt{1 - \frac{4ac}{b^2}}}\left(1 + \frac{2c}{bx}\right)$$

Substituting $x = r^2$ and for a, b & c gives:

$$\sin 2(\theta - \theta_0) = \left(1 + \frac{4\mu k}{\ell^2}\frac{\ell^4}{4\mu^2 E^2}\right)^{-1/2}\left(1 - \frac{2}{r^2}\frac{\ell^2}{2\mu E}\right)$$

$$= \left(1 + \frac{\ell^2 k}{\mu E^2}\right)^{-1/2}\left(1 - \frac{\ell^2/\mu E}{r^2}\right)$$

Define $\epsilon' \equiv \left(1 + \frac{\ell^2 k}{\mu E^2}\right)^{1/2}$ & $\alpha' \equiv \frac{\ell^2}{\mu E}$

$$\therefore \sin 2(\theta - \theta_0) = \frac{1}{\epsilon'}\left(1 - \frac{\alpha'}{r^2}\right)$$

To visualize the "orbit", we pick $\theta_0 = \frac{\pi}{4}$. (We can always define θ such that $\theta_0 = \frac{\pi}{4}$.)

$$\epsilon' \sin 2\left(\theta - \frac{\pi}{4}\right) = 1 - \frac{\alpha'}{r^2}$$

$$\frac{\alpha'}{r^2} = 1 + \epsilon' \cos 2\theta$$

$$\alpha' = r^2 + \epsilon' r^2(\cos^2\theta - \sin^2\theta)$$

Switching to Cartesian coordinates gives:

$$\alpha' = x^2 + y^2 + \epsilon'(x^2 - y^2)$$

$$1 = \frac{1+\epsilon'}{\alpha'}x^2 - \frac{(\epsilon'-1)}{\alpha'}y^2$$ where $\epsilon' > 1$ (from $\alpha' > 0$ definitions)

This is the equation of a hyperbola.

8.13 This problem investigates the effect of an additional term $\propto 1/r^3$ to the inverse square gravitational force

i.e. $F(r) = \dfrac{-k}{r^2} - \dfrac{\lambda}{r^3}$

We begin by finding the equation of motion. From lecture notes:

$$\frac{\mu}{2}\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) = E$$

Differentiating with respect to time gives:

$$\mu \dot{r}\ddot{r} - \frac{l^2}{\mu r^3}\dot{r} + \frac{dU}{dr}\dot{r} = 0$$

$$\mu\ddot{r} - \frac{l^2}{\mu r^3} - F = 0 \qquad (1)$$

Next we replace r by $u = \frac{1}{r}$.

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta}$$

$$= \frac{l}{\mu r^2}\frac{d}{d\theta} \qquad \text{using } l = \mu r^2\dot{\theta} = \text{constant (from lecture notes)}$$

$$= \frac{l u^2}{\mu}\frac{d}{d\theta}$$

$$\therefore \ddot{r} = \frac{d}{dt}\left(\frac{dr}{dt}\right)$$

$$= \frac{l u^2}{\mu}\frac{d}{d\theta}\left(\frac{l u^2}{\mu}\frac{dr}{d\theta}\right)$$

$$= \frac{l u^2}{\mu}\frac{d}{d\theta}\left(-\frac{l}{\mu}\frac{du}{d\theta}\right)$$

$$= -\frac{l^2}{\mu^2}u^2\frac{d^2u}{d\theta^2}$$

$$\therefore (1) \Longrightarrow \quad \mu \left(\frac{-\ell^2}{\mu^2}\right) u^2 \frac{d^2u}{d\theta^2} - \frac{\ell^2 u^3}{\mu} - F = 0$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{F}{u^2}$$

Substituting for F gives:

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}\left(-k u^2 - \lambda u^3\right)$$

$$= \frac{\mu}{\ell^2}\left(k + \lambda u\right)$$

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{\mu\lambda}{\ell^2}\right) u = \frac{\mu k}{\ell^2}$$

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{\mu\lambda}{\ell^2}\right)\left(u - \frac{\mu k}{\ell^2}\frac{1}{1 - \frac{\mu\lambda}{\ell^2}}\right) = 0$$

Define $v \equiv u - \frac{\mu k}{\ell^2}\frac{1}{1 - \frac{\mu\lambda}{\ell^2}}$ and $\beta^2 \equiv 1 - \frac{\mu\lambda}{\ell^2}$.

$$= u - \frac{\mu k}{\ell^2 - \mu\lambda}$$

$$\therefore \frac{d^2v}{d\theta^2} + \beta^2 v = 0 \qquad (2)$$

Case 1) $\beta^2 > 0$ or $\lambda < \frac{\ell^2}{\mu}$

$(2) \Longrightarrow v(\theta) = A\cos(\beta\theta + \delta)$ where θ can be defined such that integ. constant $\delta = 0$

$$\therefore \frac{1}{r} = A\cos\beta\theta + \frac{\mu k}{\ell^2 - \mu\lambda}$$

If $\beta = 1$ $(\lambda = 0)$, we get the usual elliptical orbit.
If $\beta \neq 1$ then $r(\theta) \neq r(\theta + 2\pi)$ and the ellipse precesses.

Case 2) $\beta^2 = 0$ or $\lambda = \dfrac{\ell^2}{\mu}$

$$\frac{d^2u}{d\theta^2} = \frac{\mu k}{\ell^2}$$

$$u = \frac{\mu k}{\ell^2}\frac{\theta^2}{2} + A\theta + B$$

$$\frac{1}{r} = \frac{\mu k}{2\ell^2}\theta^2 + A\theta + B$$

As θ increases, $r \to 0$. i.e. particle spirals toward origin.

Case 3) $\beta^2 < 0$ or $\lambda > \dfrac{\ell^2}{\mu}$

$$\frac{d^2v}{d\theta^2} + \beta^2 v = 0$$

$$v = A\cosh(|\beta|\theta + \delta)$$ Again, define θ so that $\delta = 0$.

$$\frac{1}{r} = A\cosh|\beta|\theta + \frac{\mu k}{\ell^2 - \mu\lambda}$$

As θ increases, $r \to 0$. i.e. particle spirals toward origin.

8.14 From lecture notes, equation of motion is:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{2\mu r^4}{l^2}\left[E - U - \frac{l^2}{2\mu r^2}\right] \quad (1)$$

We are given $r = k\theta^2$.

$$\therefore \frac{dr}{d\theta} = 2k\theta \qquad \left(\frac{dr}{d\theta}\right)^2 = 4k^2\theta^2 = 4kr$$

Substituting into (1) gives:

$$4kr = \frac{2\mu r^4}{l^2}\left[E - U - \frac{l^2}{2\mu r^2}\right]$$

$$U = E - \frac{2kl^2}{\mu}\frac{1}{r^3} - \frac{l^2}{2\mu r^2}$$

$\therefore$ Force $F = -\dfrac{dU}{dr}$

$$F(r) = -\frac{l^2}{\mu}\left[\frac{6k}{r^4} + \frac{1}{r^3}\right]$$

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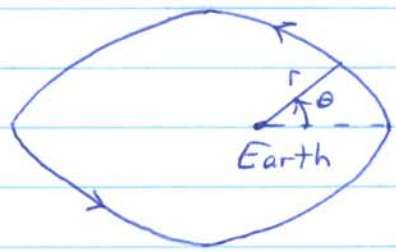

Orbit Equation is $\dfrac{\alpha}{r} = 1 + \epsilon \cos\theta.$ (1)

at closest approach $\dfrac{\alpha}{r_{min}} = 1 + \epsilon$ (2)

at furthest approach $\dfrac{\alpha}{r_{max}} = 1 - \epsilon$ (3)

$(2)+(3) \Rightarrow \quad \alpha\left(\dfrac{1}{r_{min}} + \dfrac{1}{r_{max}}\right) = 2$

$\alpha = 2\left(\dfrac{1}{r_{min}} + \dfrac{1}{r_{max}}\right)^{-1}$

$r_{min} = R_{Earth} + 300\,km$
$= 6{,}700\ km$

$r_{max} = R_{Earth} + 3500$
$= 9{,}900\ km.$

$\therefore \alpha = 7992\ km.$

Substituting α & r_{min} into (2) gives $\epsilon = .193$

Hence at $\theta = 90°$, (1) gives distance r to be:

$\dfrac{7992}{r} = 1 + 0$

$r = 7992\ km.$

$\therefore$ at $\theta = 90°$, object is at height 1592 km.

8.25 Let subscript a (p) denote apogee (perigee).

Conservation of Angular Momentum $\quad m r_a v_a = m r_p v_p \quad (1)$

Conservation of Energy $\quad \frac{m}{2} v_a^2 - \frac{G M_E m}{r_a} = \frac{m}{2} v_p^2 - \frac{G M_E m}{r_p} \quad (2)$

Solving (1) for v_a and substituting into (2) gives:

$$\frac{m}{2} \frac{r_p^2 v_p^2}{r_a^2} - \frac{G M_E m}{r_a} = \frac{m}{2} v_p^2 - \frac{G M_E m}{r_p}$$

$$\frac{r_p^2 v_p^2}{2 r_a^2} - \frac{G M_E}{r_a} - \frac{v_p^2}{2} + \frac{G M_E}{r_p} = 0$$

$$r_a^2 \left(\frac{G M_E}{r_p} - \frac{v_p^2}{2} \right) - r_a G M_E + \frac{r_p^2 v_p^2}{2} = 0.$$

$$\therefore r_a = \frac{G M_E \pm \left[G^2 M_E^2 - 2 r_p^2 v_p^2 \left(\frac{G M_E}{r_p} - \frac{v_p^2}{2} \right) \right]^{1/2}}{2 \left(\frac{G M_E}{r_p} - \frac{v_p^2}{2} \right)}$$

Using $v_p = 28{,}070$ km/hr.
$G = 6.67 \times 10^{-11}$ Nt m^2/kg^2
$M_E = 6 \times 10^{24}$ kg.
$r_p = 220$ km $+ R_E = 6620$ km.

one obtains $r_a = 6658$ km or 288 km above Earth's surface.

$$(1) \Longrightarrow v_a = \frac{r_p v_p}{r_a}$$

$$= 27{,}780 \text{ km/hr.}$$

Kepler's 3rd Law (see lecture notes) gives:

$$\tau^2 = \frac{4\pi^2 \mu}{k} a^3$$

$$= \frac{4\pi^2}{G M_E} \left(\frac{r_a + r_p}{2} \right)^2$$

$$\tau = 1.5 \text{ hrs.}$$