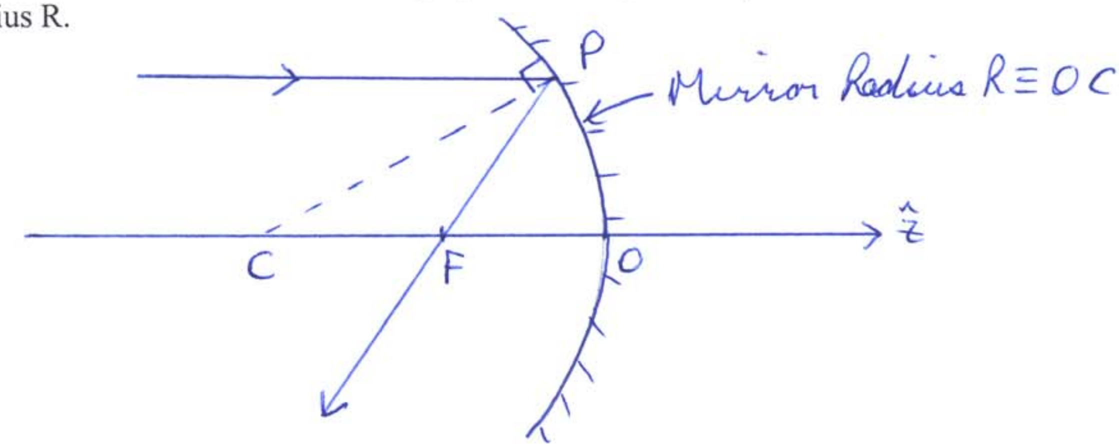
Assignment 3

1. Derive the 2 x 2 matrix describing light bouncing off a spherical mirror of radius R.



at mirror $\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$

$r_2 = r_1 \Rightarrow A = 1, \ B = 0$

Consider incident ray parallel to $\hat{z}$ axis.

$$r_2' = C r_1 + D r_1'$$

$$\frac{-r_1}{f} = C r_1 + D\,0$$

$$C = \frac{-1}{f} \quad \text{where } f = \frac{R}{2}$$

Similarly consider incident ray travelling reverse of ray shown above.

$$0 = C\, r_1 + D \frac{r_1}{f}$$

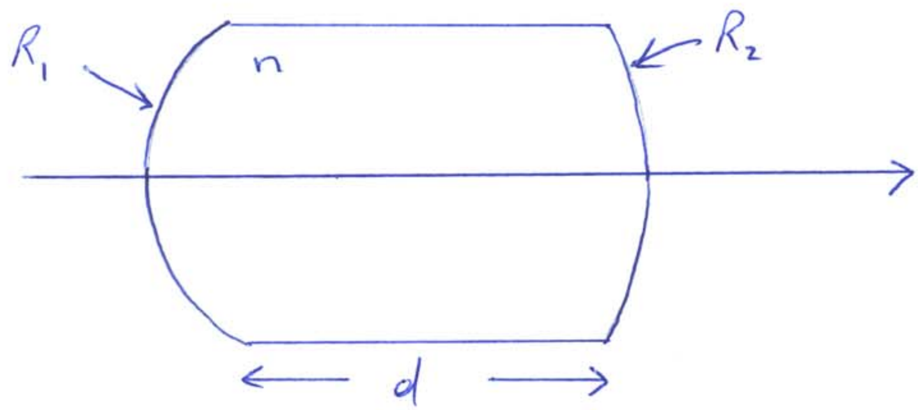
$$= \frac{-r_1}{f} + D \frac{r_1}{f}$$

$\therefore D = 1$ $\longrightarrow$ $\therefore \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix}$

2. Consider a convex lens having a thickness d and whose sides have radii of curvature R_1 and R_2.

a) Find the 2 x 2 matrix describing this lens.

b) What is the change in focal length for the case where the two radii equal R and d = 0.1 R, compared to the case where the lens thickness is neglected?



$$M = M \text{ for } R_2 \quad M \text{ for } d \quad M \text{ for } R_1$$

$$= \begin{pmatrix} 1 & 0 \\ (1-n)\frac{1}{R_2} & n \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{n-1}{n}\left(\frac{-1}{R_1}\right) & \frac{1}{n} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{(n-1)}{n}\frac{d}{R_1} & \frac{d}{n} \\ -(n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{(n-1)^2}{n}\frac{d}{R_1 R_2} & 1 - \frac{(n-1)}{n}\frac{d}{R_2} \end{pmatrix}$$

Comparing to thin lens result, we find

$$\frac{-1}{f} = -(n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{(n-1)^2}{n}\frac{d}{R_1 R_2}$$

For $R_1 = R_2 = R \implies \frac{-1}{f} = -(n-1)\frac{2}{R} + \frac{(n-1)^2}{n}\frac{d}{R^2}$ (1)

Let $f = f_0 + \Delta f$ where $f_0 \equiv \frac{1}{n-1}\frac{R}{2}$ thin lens result

$$\therefore \frac{-1}{f} = \frac{-1}{f_0 + \Delta f}$$

$$= \frac{-1}{f_0}\left(1 + \frac{\Delta f}{f_0}\right)^{-1}$$

$$\simeq \frac{-1}{f_0} + \frac{\Delta f}{f_0^2} \qquad (2)$$

Comparing (1) & (2) gives:

$$\frac{\Delta f}{f_0^2} = \frac{(n-1)^2}{n}\frac{d}{R^2}$$

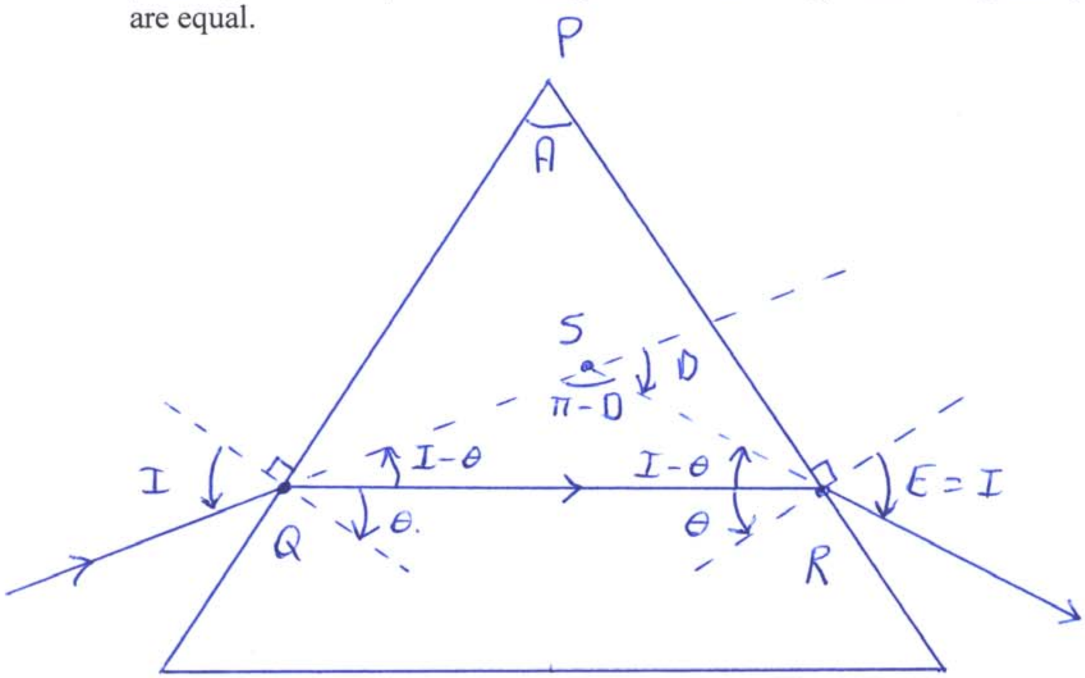
$$\frac{\Delta f}{f_0} = \frac{n-1}{n}\frac{d}{2R}$$

$$= \frac{1.5-1}{1.5}\frac{0.1}{2}$$

$$\therefore \frac{\Delta f}{f_0} = 0.017$$

3. Consider a refracting prism shown below. The face opposite the apex (top of prism) is called the base. The total angle by which light changes direction is called the angle of deviation D.

a) Show that n_{prism} = sin (A+D)/2 / sin A/2 when light passes through the prism symmetrically such that angles of incidence (I) and emergence (E) are equal.



$$\Delta PQR \Rightarrow A + \left(\frac{\pi}{2} - \theta\right)2 = \pi$$

$$\theta = \frac{A}{2} \quad (1)$$

$$\Delta SQR \Rightarrow (\pi - D) + (I - \theta)2 = \pi$$

$$-D + 2I - 2\theta = 0$$

$$I = \frac{D + 2\theta}{2}$$

Using (1) we get: $I = \frac{D + A}{2}$ (2)

Snell's Law: $\sin I = n \sin \theta$

$$n = \frac{\sin\left(\frac{D+A}{2}\right)}{\sin\frac{A}{2}}$$ using (1) + (2)

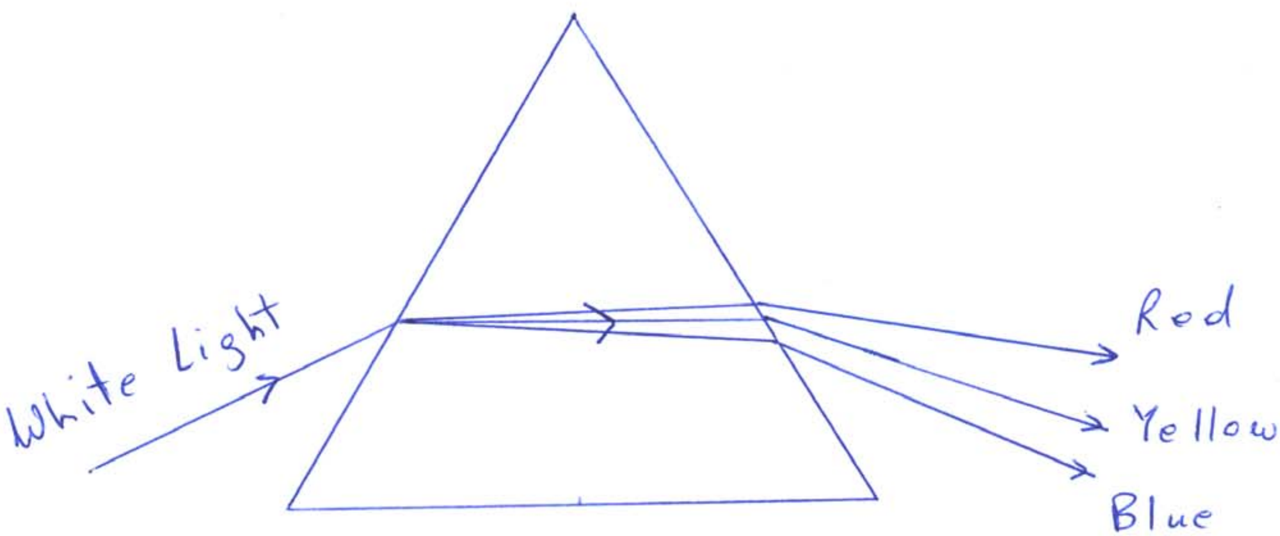
b) Find the deviation angles for blue and red light having indices of refraction 1.652 and 1.618 respectively.

$$n = \frac{\sin\left(\frac{60^\circ + D}{2}\right)}{\sin 30^\circ}$$

Blue light $n = 1.652 \Rightarrow D = 51.4^\circ$

Red light $n = 1.618 \Rightarrow D = 48.0^\circ$

c) Sketch what happens when white light is incident on the prism.



4. It can be shown that the radial distance of a light ray travelling along the z axis of an optical fiber is described by the following equation.

$$\frac{d^2 r}{dz^2} = \frac{1}{n(r)} \frac{dn}{dr}$$

For the case where the index of refraction $n(r) = n_o(1 - Ar^2)$ and $Ar^2 << 1$ find a solution of the differential equation for $r(z)$.

For $Ar^2 << 1 \Longrightarrow \quad \dfrac{d^2 r}{dz^2} = \dfrac{1}{n_o}(-2An_o r)$

$$\frac{d^2 r}{dz^2} = -2Ar$$

$$\therefore r(z) = r_o \cos kz + r_o' \sin kz \qquad (1)$$

where $k \equiv \sqrt{2A}$ and $r(0) \equiv r_o, \dfrac{dr}{dz}(0) \equiv r_o'$

Note $r'(z) = -r_o k \sin kz + r_o' k \cos kz \qquad (2)$

$$\therefore (1) \,\&\, (2) \Longrightarrow \begin{pmatrix} r(z) \\ r'(z) \end{pmatrix} = \begin{pmatrix} \cos kz & \sin kz \\ -k \sin kz & k \cos kz \end{pmatrix} \begin{pmatrix} r_o \\ r_o' \end{pmatrix}$$

5. Explain how a rainbow is created. Hint: Look in some textbooks.

For example see pg. 908 of Fundamentals of Physics (Edition 8) By Halliday, Resnick & Walker.