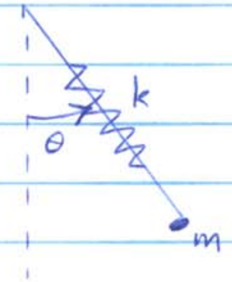
Assignment 4

7.15 Let l be length of spring which is variable and b be the unextended spring length.



Kinetic Energy of mass m is $T = \frac{m}{2}(\dot{l}^2 + l^2\dot{\theta}^2)$

Potential Energy $U = \frac{k}{2}(l-b)^2 - mgl\cos\theta$

Lagrangian $L = \frac{m}{2}(\dot{l}^2 + l^2\dot{\theta}^2) - \frac{k}{2}(l-b)^2 + mgl\cos\theta$

Lagrange Eqn. for l is:

$$\frac{\partial L}{\partial l} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{l}}\right) = 0$$

$$ml\dot{\theta}^2 - k(l-b) + mg\cos\theta - \frac{d}{dt}(m\dot{l}) = 0$$

$$\boxed{\ddot{l} - l\dot{\theta}^2 + \frac{k}{m}(l-b) - g\cos\theta = 0}$$

Lagrange Eqn. for θ is:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$-mgl\sin\theta - \frac{d}{dt}\left(ml^2\dot{\theta}\right) = 0$$

$$\frac{g}{l}\sin\theta + \frac{1}{ml^2}\left[2ml\dot{l}\dot{\theta} + ml^2\ddot{\theta}\right] = 0$$

$$\boxed{\ddot{\theta} + \frac{2}{l}\dot{l}\dot{\theta} + \frac{g}{l}\sin\theta = 0}$$

7.22 Potential Energy $U(x,t) = -\int F(x,t)\,dx$

$$= -\int \frac{k}{x^2} e^{-t/\tau} dx$$

$$= \frac{k}{x} e^{-t/\tau}$$ where we set integration constant to zero assuming $U(\infty, t) = 0$.

Lagrangian for particle of mass m is:

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{x} e^{-t/\tau}$$

Canonical Momentum $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

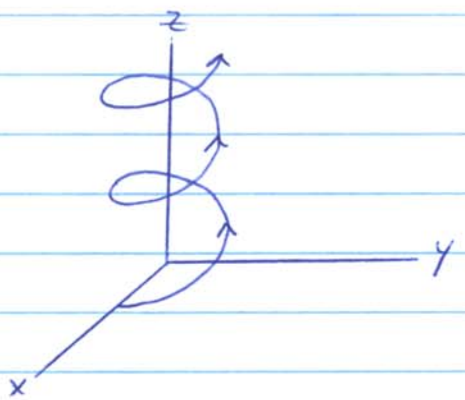
Hamiltonian $H = p_x \dot{x} - L$

$$= \frac{p_x^2}{m} - \frac{p_x^2}{2m} + \frac{k}{x} e^{-t/\tau}$$

$$= \frac{p_x^2}{2m} + \frac{k}{x} e^{-t/\tau}$$

The Hamiltonian equals the total energy (kinetic plus potential). However, the energy is not conserved since H depends on time.

7.25 Particle of mass m moves in an upward spiral.



$r = \text{constant}$
$z = k\theta$

In general a particle described by coordinates (r, θ, z) has kinetic energy:

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right)$$

But $r = \text{constant}$, $z = k\theta$ imply:

$$T = \frac{m}{2}\left(\frac{r^2}{k^2}\dot{z}^2 + \dot{z}^2\right)$$

Potential Energy $U = mgz$.

Lagrangian $L = \frac{m}{2}\left(\frac{r^2}{k^2} + 1\right)\dot{z}^2 - mgz$

Canonical Momentum $p_z = \frac{\partial L}{\partial \dot{z}}$

$$= m\left(\frac{r^2}{k^2} + 1\right)\dot{z}$$

Hamiltonian $H = p_z\dot{z} - L$

$$= p_z \frac{p_z}{m\left(\frac{r^2}{k^2}+1\right)} - \frac{p_z^2}{2m\left(\frac{r^2}{k^2}+1\right)} + mgz$$

$$= \frac{p_z^2}{2m\left(\frac{r^2}{k^2}+1\right)} + mgz$$

One of Hamilton's equations is:

$$-\frac{\partial H}{\partial z} = \dot{p}_z$$

$$-mg = \dot{p}_z \qquad (1)$$

Other Hamilton equation is:

$$\frac{\partial H}{\partial p_z} = \dot{z}$$

$$\frac{p_z}{m\left(\frac{r^2}{k^2}+1\right)} = \dot{z} \qquad (2)$$

Taking the derivative of (2) and substituting in (1) gives the equation of motion:

$$\boxed{\ddot{z} = -\frac{g}{\frac{r^2}{k^2}+1}}$$

7.30 a) The total time derivative of $g(q_k, p_k, t)$ is:

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left[\frac{\partial g}{\partial q_k} \frac{\partial q_k}{\partial t} + \frac{\partial g}{\partial p_k} \frac{\partial p_k}{\partial t} \right]$$

Using $\dot{q}_k = \frac{\partial H}{\partial p_k}$ and $\dot{p}_k = -\frac{\partial H}{\partial q_k}$ we get:

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left[\frac{\partial g}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial H}{\partial q_k} \right]$$

$$\therefore \frac{dg}{dt} = \frac{\partial g}{\partial t} + [g, H]$$

b) $$[q_j, H] = \sum_k \left[\frac{\partial q_j}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial q_j}{\partial p_k} \frac{\partial H}{\partial q_k} \right]$$

$$= \sum_k \left[\delta_{jk} \frac{\partial H}{\partial p_k} - 0 \right]$$ using $\frac{\partial q_j}{\partial q_k} = \delta_{jk}$ + $\frac{\partial q_j}{\partial p_k} = 0$

$$= \frac{\partial H}{\partial p_j}$$

$\therefore [q_j, H] = \dot{q}_j$ using Hamilton's eqn.

$$[p_j, H] = \sum_k \left[\frac{\partial p_j}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial p_j}{\partial p_k} \frac{\partial H}{\partial q_k} \right]$$

$$= \sum_k \left[0 - \delta_{jk} \frac{\partial H}{\partial q_k} \right]$$ using $\frac{\partial p_j}{\partial q_k} = 0$ + $\frac{\partial p_j}{\partial p_k} = \delta_{jk}$

$$= -\frac{\partial H}{\partial q_j}$$

$\therefore [p_j, H] = \dot{p}_j$ using Hamilton's eqn.

c) $$[p_k, p_j] = \sum_\ell \left\{ \underbrace{\frac{\partial p_k}{\partial q_\ell}}_{=0} \frac{\partial p_j}{\partial p_\ell} - \frac{\partial p_k}{\partial p_\ell} \underbrace{\frac{\partial p_j}{\partial q_\ell}}_{=0} \right\}$$

$$\therefore [p_k, p_j] = 0$$

$$[q_k, q_j] = \sum_\ell \left\{ \frac{\partial q_k}{\partial q_\ell} \underbrace{\frac{\partial q_j}{\partial p_\ell}}_{=0} - \underbrace{\frac{\partial q_k}{\partial p_\ell}}_{=0} \frac{\partial q_j}{\partial q_\ell} \right\}$$

$$\therefore [q_k, q_j] = 0$$

d) $$[q_k, p_j] = \sum_\ell \left\{ \frac{\partial q_k}{\partial q_\ell} \frac{\partial p_j}{\partial p_\ell} - \frac{\partial q_k}{\partial p_\ell} \frac{\partial p_j}{\partial q_\ell} \right\}$$

$$= \sum_\ell \left(\delta_{k\ell} \delta_{j\ell} - 0 \right)$$

$$\therefore [q_k, p_j] = \delta_{kj}$$

e) $$\frac{dg}{dt} = \frac{\partial g}{\partial t} + [g, H]$$

If $g = g(p_k, q_k)$ *i.e.* $\frac{\partial g}{\partial t} = 0$ *and* g *commutes with* H *i.e.* $[g, H] = 0$ *then:*

$$\frac{dg}{dt} = 0.$$

$\therefore$ g *is a constant of the motion.*