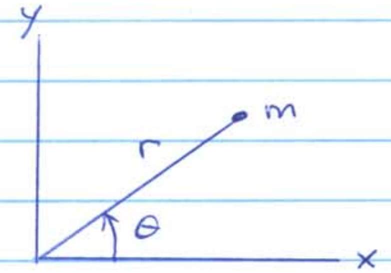
Phys 3010 **Assignment 3**

7.4 Let particle of mass m move in xy plane.



Using polar coordinates as generalized coordinates, the Kinetic Energy is:

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

The potential energy is related to the force by

$$U = -\int f(r)\,dr$$

$$= +\int A\, r^{\alpha-1}\,dr$$

$$= +\frac{A}{\alpha} r^{\alpha}$$ where integration constant is zero since $U(r=0)=0$.

$\therefore$ Lagrangian $L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{A}{\alpha} r^{\alpha}$

Lagrange equation for θ is:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$0 - \frac{d}{dt}\left(m r^2 \dot{\theta}\right) = 0.$$

$\therefore\ mr^2\dot{\theta} = l$ a constant which is the angular momentum

Lagrange equation for r is:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$$

$$mr\dot{\theta}^2 - A\,r^{\alpha-1} - m\ddot{r} = 0$$

Substituting $\dot{\theta} = \dfrac{l}{mr^2}$ gives:

$$\frac{l^2}{mr^3} - A\,r^{\alpha-1} - m\ddot{r} = 0.$$

Multiplying by $\dot{r}$ and integrating gives:

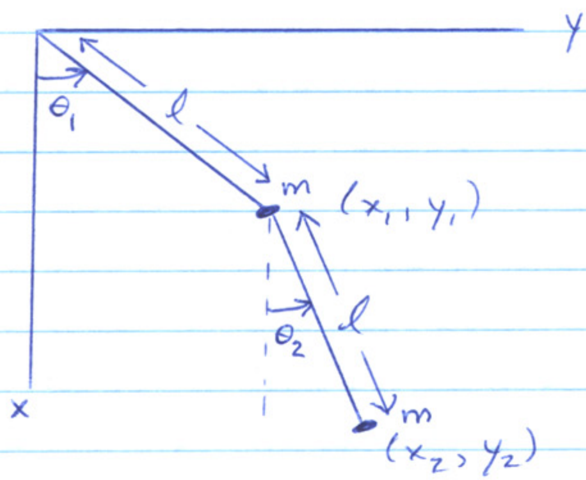
$$\int \frac{l^2}{mr^3}\dot{r}\,dt - A\int r^{\alpha-1}\dot{r}\,dt - m\int \ddot{r}\dot{r}\,dt = -E \text{ (constant}$$

$$-\frac{l^2}{2mr^2} - \frac{Ar^\alpha}{\alpha} - \frac{m}{2}\dot{r}^2 = -E.$$

$$\underbrace{\frac{m\dot{r}^2}{2} + \frac{l^2}{2mr^2}}_{\text{Kinetic Energy}} + \underbrace{\frac{Ar^\alpha}{\alpha}}_{U} = E.$$

$\therefore\ T + U = E$ a constant

7.7 Consider the double pendulum shown below where (θ_1, θ_2) are the two generalized coordinates.



One mass is at position $x_1 = \ell \cos\theta_1$
$y_1 = \ell \sin\theta_1$

Other mass is at $x_2 = \ell \cos\theta_1 + \ell \cos\theta_2$
$y_2 = \ell \sin\theta_1 + \ell \sin\theta_2$

Kinetic Energy of two masses is:

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$= \frac{m}{2}\ell^2\left[\dot{\theta}_1^2 + \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\left(\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2\right)\right]$$

$$= \frac{m}{2}\ell^2\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right]$$

Potential Energy of two masses is:

$$U = -mgx_1 - mgx_2$$
$$= -mg\ell\left[2\cos\theta_1 + \cos\theta_2\right]$$

$\therefore$ Lagrangian is:

$$L = \frac{ml^2}{2}\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] + mgl\left[2\cos\theta_1 + \cos\theta_2\right]$$

Lagrange equation for θ_1 is:

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = 0$$

$$ml^2\dot{\theta}_1\dot{\theta}_2\left[-\sin(\theta_1-\theta_2)\right] - 2mgl\sin\theta_1 - \frac{d}{dt}\left\{\frac{ml^2}{2}\left[4\dot{\theta}_1 + 2\dot{\theta}_2\cos(\theta_1-\theta_2)\right]\right\} = 0$$

$$\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + \frac{2g}{l}\sin\theta_1 + 2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1-\theta_2)$$

$$+ \dot{\theta}_2\left\{-\sin(\theta_1-\theta_2)\quad(\dot{\theta}_1-\dot{\theta}_2)\right\} = 0$$

$$\boxed{2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1-\theta_2) + \dot{\theta}_2^2\sin(\theta_1-\theta_2) + \frac{2g}{l}\sin\theta_1 = 0}$$

Lagrange equation for θ_2 is:

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = 0$$

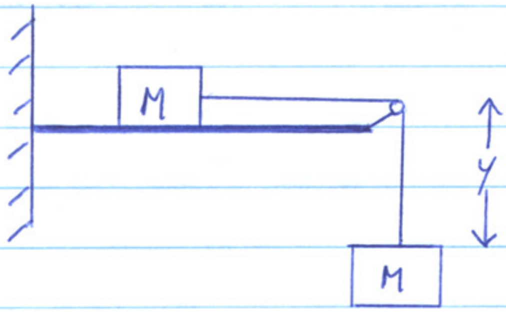
$$ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - mgl\sin\theta_2 - \frac{d}{dt}\left\{\frac{ml^2}{2}\left[2\dot{\theta}_2 + 2\dot{\theta}_1\cos(\theta_1-\theta_2)\right]\right\} = 0$$

$$-\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + \frac{g}{l}\sin\theta_2 + \ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1-\theta_2)$$

$$+ \dot{\theta}_1\left\{-\sin(\theta_1-\theta_2)\quad(\dot{\theta}_1-\dot{\theta}_2)\right\} = 0$$

$$\boxed{\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1-\theta_2) - \dot{\theta}_1^2\sin(\theta_1-\theta_2) + \frac{g}{l}\sin\theta_2 = 0}$$

7.10 a) Let system be described by generalized coordinates.



Lagrangian $L = \frac{M}{2}\dot{y}^2 + \frac{M}{2}\dot{y}^2 - (-Mgy)$

$= M\dot{y}^2 + Mgy$

Lagrangian Eqn. $\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$

$$Mg - \frac{d}{dt}(2M\dot{y}) = 0$$

$$\ddot{y} = \frac{g}{2}$$

$$\therefore y(t) = \frac{g}{4}t^2 + v_{oy}t + y_o$$

For initial conditions $v_{oy}, y_o = 0$ we get $y(t) = \frac{g}{4}t^2$.

b) Next we consider that the string of length l has mass m.

Kinetic Energy of String $T_{st} = \frac{m}{2}\dot{y}^2$

Pot. Energy of String $U_{st} = -\left(m\frac{y}{l}\right)g\frac{y}{2}$

fraction of string mass hanging down

Lagrangian of 2 masses + string is:

$$L = M\dot{y}^2 + \frac{m}{2}\dot{y}^2 + Mgy + mg\frac{y^2}{2l}$$

Lagrangian Eqn. is: $\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$

$$Mg + mg\frac{y}{l} - \frac{d}{dt}\left(2M\dot{y} + m\dot{y}\right) = 0.$$

$$(2M+m)\ddot{y} = Mg + mg\frac{y}{l}$$

$$= \frac{mg}{l}\left(y + \frac{Ml}{m}\right)$$

$$\frac{d^2y}{dt^2} = \frac{mg}{l(2M+m)}\left(y + \frac{Ml}{m}\right)$$

OR $$\frac{d^2}{dt^2}\left(y + \frac{Ml}{m}\right) = \frac{mg}{l(2M+m)}\left(y + \frac{Ml}{m}\right)$$

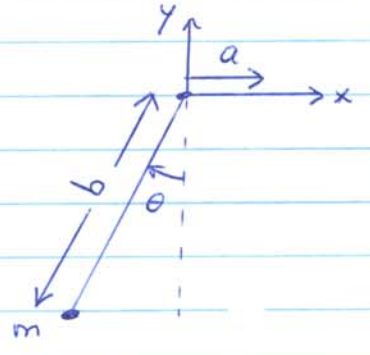
$$\therefore y + \frac{Ml}{m} = Ae^{\gamma t} + Be^{-\gamma t} \text{ where } \gamma \equiv \left(\frac{mg}{l(2M+m)}\right)^{1/2}$$

Using the initial condition $y(t=0) = 0$, $\dot{y}(t=0) = 0$ we find
$A = B = \frac{Ml}{2m}$

$$\therefore y(t) = -\frac{Ml}{m} + \frac{Ml}{2m}\left(e^{\gamma t} + e^{-\gamma t}\right)$$

$$= \frac{Ml}{m}\left(\cosh\gamma t - 1\right)$$

7.13 a) Consider pendulum where support experiences acceleration a in x direction.



Position of mass m is $x = \frac{1}{2} a t^2 - b \sin\theta$

$$y = -b \cos\theta$$

Velocity components of mass m are: $\dot{x} = at - b\dot{\theta}\cos\theta$

$$\dot{y} = b\dot{\theta}\sin\theta$$

Lagrangian $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy$

$$L = \frac{m}{2}\left[a^2t^2 - 2at\, b\dot{\theta}\cos\theta + b^2\dot{\theta}^2\right] + mgb\cos\theta$$

Lagrangian eqn. for θ is:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$-m\, at\, b\, \dot{\theta}(-\sin\theta) - mgb\sin\theta - \frac{d}{dt}\left(-matb\cos\theta + b^2 m\dot{\theta}\right) = 0$$

$$at\,\dot{\theta}\sin\theta - g\sin\theta + \frac{d}{dt}\left(at\cos\theta - b\dot{\theta}\right) = 0$$

$$at\,\dot{\theta}\sin\theta - g\sin\theta + a\cos\theta - at\,\dot{\theta}\sin\theta - b\ddot{\theta} = 0$$

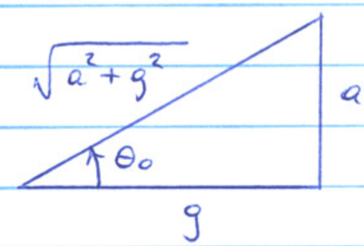
$$\boxed{\ddot{\theta} + \frac{g}{b}\sin\theta - \frac{a}{b}\cos\theta = 0}$$

b) We are next interested in oscillations of the pendulum about the equilibrium point. The latter ($\theta = \theta_0$) is found by setting $\ddot{\theta} = 0$.

$$\therefore \quad \frac{g}{b}\sin\theta_0 - \frac{a}{b}\cos\theta_0 = 0.$$

$$g\sin\theta_0 = a\cos\theta_0$$

$$\tan\theta_0 = \frac{a}{g}$$



We now set $\theta = \theta_0 + \delta(t)$ where $\delta(t)$ represents small oscillations.

$$\begin{aligned}\therefore \sin\theta &= \sin(\theta_0 + \delta) \\ &= \sin\theta_0\cos\delta + \sin\delta\cos\theta_0 \\ &\simeq \sin\theta_0 + \delta\cos\theta_0\end{aligned}$$

$$\begin{aligned}\cos\theta &= \cos(\theta_0 + \delta) \\ &= \cos\theta_0\cos\delta - \sin\theta_0\sin\delta \\ &\simeq \cos\theta_0 - \delta\sin\theta_0\end{aligned}$$

Substituting $\sin\theta$ & $\cos\theta$ into the eqn. of motion gives:

$$\ddot{\delta} + \frac{g}{b}\left(\sin\theta_0 + \delta\cos\theta_0\right) - \frac{a}{b}\left(\cos\theta_0 - \delta\sin\theta_0\right) = 0$$

$$\ddot{\delta} + \frac{g}{b}\left(\frac{a}{\sqrt{a^2+g^2}} + \delta\frac{g}{\sqrt{a^2+g^2}}\right) - \frac{a}{b}\left(\frac{g}{\sqrt{a^2+g^2}} - \delta\frac{a}{\sqrt{a^2+g^2}}\right) = 0$$

$$\therefore \ \ddot{\delta} = -\frac{\delta}{b} \frac{a^2+g^2}{\sqrt{a^2+g^2}}$$

$$\boxed{\ddot{\delta} = -\frac{\sqrt{a^2+g^2}}{b} \delta}$$

This has solution $\delta(t) = A\cos\omega t + B\sin\omega t$ where

$\omega = \frac{(a^2+g^2)^{1/4}}{b^{1/2}}$ is angular frequency

and $T = \frac{2\pi}{\omega} = \frac{2\pi\, b^{1/2}}{(a^2+g^2)^{1/4}}$ is the oscillation period.