Phys 3010 <u>Assignment 2</u>

6.1 $y(\alpha, x) = x + \alpha \sin \pi(1-x)$

$$\frac{dy}{dx} = 1 - \alpha\pi \cos \pi(1-x)$$

Path length from $(x_1, y_1) = (0,0)$ to $(x_2, y_2) = (1,1)$ is:

$$J(\alpha) = \int_{(0,0)}^{(1,1)} \sqrt{dx^2 + dy^2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \left[1 + \left(1 - \alpha\pi\cos\pi(1-x)\right)^2\right]^{1/2} dx$$

$$= \int_0^1 \left[2 - 2\pi\alpha\cos\pi(1-x) + \alpha^2\pi^2\cos^2\pi(1-x)\right]^{1/2} dx$$

Let $u = \pi(1-x)$. $\therefore du = -\pi\, dx$.

$x = 0 \Rightarrow u = \pi$

$x = 1 \Rightarrow u = 0$.

$$\therefore J(\alpha) = \frac{1}{\pi}\int^{\pi} \left[2 - 2\pi\alpha\cos u + \alpha^2\pi^2\cos^2 u\right]^{1/2} du$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi} \left[1 - \alpha\pi\cos u + \alpha^2\frac{\pi^2}{2}\cos^2 u\right]^{1/2} du$$

For small α, we can expand the integrand.

$$J(\alpha) = \frac{\sqrt{2}}{\pi} \int_0^{\pi} \left[1 + \frac{1}{2}\left(-\alpha\pi\cos u + \alpha^2 \frac{\pi^2}{2}\cos^2 u\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(-\alpha\pi\cos u + \alpha^2\frac{\pi^2}{2}\cos^2 u\right)^2 \right] du$$

Keeping terms up to $O(\alpha^2)$ gives:

$$J(\alpha) = \frac{\sqrt{2}}{\pi} \int_0^{\pi} \left[1 - \alpha\frac{\pi}{2}\cos u + \alpha^2 \frac{\pi^2}{8}\cos^2 u \right] du$$

$$= \frac{\sqrt{2}}{\pi} \left\{ \pi + 0 + \frac{\alpha^2\pi^2}{8} \frac{\pi}{2} \right\}$$

$$= \sqrt{2} + \frac{\alpha^2\pi^2}{16}\sqrt{2}$$

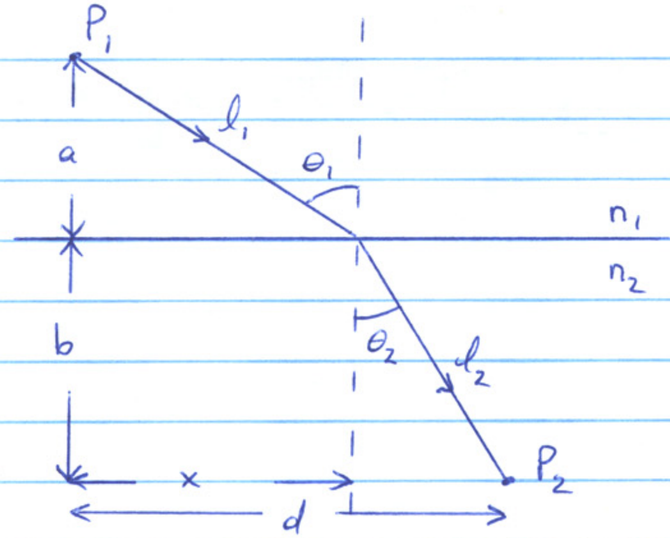
For minimum path length $0 = \frac{dJ}{d\alpha}$

$$= \frac{\alpha\pi^2}{8}\sqrt{2}$$

$\therefore \alpha = 0$.

$\therefore$ function that minimizes path length is $y(\alpha=0, x) = x$.

6.7



c = speed of light in vacuum

Define a, b, x & d as shown above.

Time for light to travel from P_1 to P_2 is:

$$T = \frac{\text{distance } l_1}{\text{speed of light } v_1} + \frac{\text{distance } l_2}{\text{speed of light } v_2}$$

$$= \frac{(a^2+x^2)^{1/2}}{c/n_1} + \frac{(b^2+(d-x)^2)^{1/2}}{c/n_2}$$

$$= \frac{n_1}{c}(a^2+x^2)^{1/2} + \frac{n_2}{c}(b^2+(d-x)^2)^{1/2}$$

Fermats' Principle states light follows path that minimizes time of Travel.

$$\therefore\ 0 = \frac{dT}{dx}$$

$$= \frac{n_1}{c}\frac{1}{2}(a^2+x^2)^{-1/2}2x + \frac{n_2}{c}\frac{1}{2}(b^2+(d-x)^2)^{-1/2}[-2(d-x)]$$

$$n_1\frac{x}{(a^2+x^2)^{1/2}} = n_2\frac{(d-x)}{(b^2+(d-x)^2)^{1/2}}$$

$\therefore\ n_1 \sin\theta_1 = n_2 \sin\theta_2$ Law of Refraction

6.8 a) Rectangular parallelepiped having sides of lengths x, y + z, has volume

$$V = xyz \quad (1)$$

Volume is to be maximized subject to constraint that parallelepiped is enclosed by sphere of radius R.

$\therefore$ (parallelepiped diagonal$)^2$ = (diameter of sphere$)^2$

$$x^2 + y^2 + z^2 = 4R^2 \quad (1')$$

OR $$g(x,y,z) \equiv x^2 + y^2 + z^2 - 4R^2 = 0 \quad (2)$$

(1) is maximized subject to (2) using Lagrange multiplier λ such that:

$$\frac{\partial V}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0$$

$$\frac{\partial V}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$$

$$\frac{\partial V}{\partial z} + \lambda \frac{\partial g}{\partial z} = 0$$

which becomes:

$$yz + 2\lambda z = 0 \quad (3a)$$

$$xz + 2\lambda y = 0 \quad (3b)$$

$$xy + 2\lambda z = 0 \quad (3c)$$

$$x(3a) + y(3b) + z(3c) \Longrightarrow 3xyz + 2\lambda(x^2 + y^2 + z^2) = 0$$

Using (2) this gives: $3xyz + \lambda 8R^2 = 0.$

$$\lambda = -\frac{3xyz}{8R^2}$$

Substituting λ into (3a) gives:

$$yz - \frac{3x^2yz}{4R^2} = 0.$$

$$1 - \frac{3x^2}{4R^2} = 0$$

$$x = \frac{2}{\sqrt{3}} R$$

Similarly λ into (3b) & (3c) gives $y = z = \frac{2}{\sqrt{3}} R.$

Hence parallelepiped is a cube with side length $\frac{2R}{\sqrt{3}}$.

b) For ellipsoid (1') is replaced by

$$\frac{x^2}{4a^2} + \frac{y^2}{4b^2} + \frac{z^2}{4c^2} = 1$$

OR $g(x,y,z) = \frac{x^2}{4a^2} + \frac{y^2}{4b^2} + \frac{z^2}{4c^2} - 1$

One can then proceed as before and obtain the solution:

$$x = \frac{2}{\sqrt{3}} a, \quad y = \frac{2}{\sqrt{3}} b, \quad z = \frac{2}{\sqrt{3}} c.$$

6-10 Surface area of cylinder having radius R & height H is:

$$A = A_{TOP} + A_{Bot} + A_{side}$$

$$= 2\pi R^2 + 2\pi R H \qquad (1)$$

This is to be minimized subject to constraint

$$V = \pi R^2 H = \text{constant}$$

OR $g(R, H) = \pi R^2 H - V \qquad (2)$

(1) is minimized subject to (2) using Lagrange multiplier λ such that:

$$\frac{\partial A}{\partial R} + \lambda \frac{\partial g}{\partial R} = 0$$

$$\frac{\partial A}{\partial H} + \lambda \frac{\partial g}{\partial H} = 0$$

which becomes: $4\pi R + 2\pi H + \lambda 2\pi R H = 0 \qquad (3a)$

$2\pi R + \lambda \pi R^2 = 0 \qquad (3b)$

$(3b) \Longrightarrow \lambda = -2/R$

Substituting λ into (3a) gives: $4\pi R + 2\pi H - \frac{2}{R} 2\pi R H = 0.$

$2R + H - 2H = 0.$

$\therefore H = 2R.$