Assignment 1 Solutions

1. $(\vec{a} \times \vec{b})^2 = a^2 b^2 - (\vec{a} \cdot \vec{b})^2$

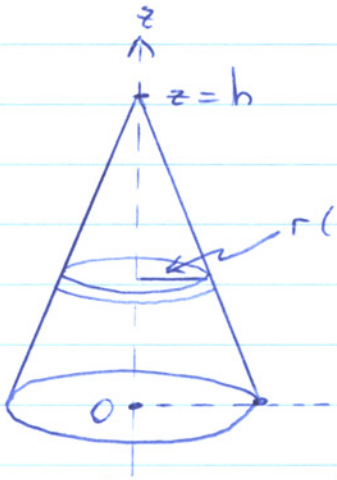
$$|\vec{a} \times \vec{b}|^2 = a^2 b^2 - a^2 b^2 \cos^2\theta$$

$$= a^2 b^2 (1 - \cos^2\theta)$$

$$= a^2 b^2 \sin^2\theta$$

$$\therefore \vec{a} \times \vec{b} = ab \sin\theta$$

2.



Consider a disk of radius $r(z)$ at height z having thickness dz.

Volume of disk $= \pi r^2(z)\, dz$.

Volume of cone $V = \int_{z=0}^{z=h} \pi r^2(z)\, dz$.

$$= \int_0^h \pi R^2 \left(1 - \frac{z}{h}\right)^2 dz.$$

$$= \pi R^2 \int_0^h \left[1 - \frac{2z}{h} + \frac{z^2}{h^2}\right] dz.$$

$$= \pi R^2 \left[z - \frac{z^2}{h} + \frac{z^3}{3h^2}\right]_0^h$$

$$= \pi R^2 h \left[1 - 1 + \tfrac{1}{3}\right] = \pi R^2 h/3$$

3. $y = \operatorname{Arcsec} x$

$x = \sec y.$

$$\frac{dx}{dx} = \frac{d \sec y}{dx}$$

$$1 = \frac{d}{dx} (\cos y)^{-1}$$

$$= (-1)(\cos y)^{-2} (-\sin y) \frac{dy}{dx}.$$

$$\frac{dy}{dx} = \left(\frac{\sin y}{(\cos y)^2} \right)^{-1}$$

$$= \left[\tan y \sec y \right]^{-1}$$

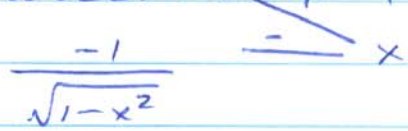
$$\therefore \frac{d \operatorname{Arcsec} x}{dx} = \left[x \sqrt{x^2 - 1} \right]^{-1}$$

since $\sin^2 y + \cos^2 y = 1$

$\tan^2 y + 1 = \sec^2 y$

$\tan y = \sqrt{\sec^2 y - 1}$

4. $\int \operatorname{Arccos} x \, dx.$



$$= x \operatorname{Arccos} x + \int \frac{x \, dx}{\sqrt{1 - x^2}}$$

Let $x = \sin \theta$

$dx = \cos \theta \, d\theta.$

$$= x \operatorname{Arccos} x + \int \frac{\sin \theta \cos \theta \, d\theta}{\sqrt{1 - \sin^2 \theta}}$$

$$= x \operatorname{Arccos} x + \int \sin \theta \, d\theta.$$

$$= x \operatorname{Arccos} x - \cos \theta + C.$$

$$= x \operatorname{Arccos} x - \sqrt{1 - \sin^2 \theta} + C$$

$$= x \operatorname{Arccos} x - \sqrt{1 - x^2} + C$$

5. $\int_0^\infty x^4 e^{-2x}\,dx$

x^4	+	e^{-2x}
$4x^3$	−	$\frac{e^{-2x}}{-2}$
$12x^2$	+	$\frac{e^{-2x}}{4}$
$24x$	−	$\frac{e^{-2x}}{-8}$
24	+	$\frac{e^{-2x}}{16}$
0	−	$\frac{e^{-2x}}{-32}$

$$= \left[x^4 \frac{e^{-2x}}{-2} - 4x^3 \frac{e^{-2x}}{4} + 12x^2 \frac{e^{-2x}}{-8} - 24x \frac{e^{-2x}}{16} + 24 \frac{e^{-2x}}{-32} \right]_0^\infty$$

$$= \frac{-24}{-32}$$

$$= \frac{3}{4}$$

6. $\int e^{-x} \sin x\,dx = -e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\sin x\,dx$

$\sin x$	+	e^{-x}
$\cos x$	−	$-e^{-x}$
$-\sin x$	+	e^{-x}

$$2\int e^{-x}\sin x\,dx = -e^{-x}(\sin x + \cos x)$$

$$\int e^{-x}\sin x\,dx = -\frac{e^{-x}}{2}(\sin x + \cos x) + C$$

7. $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$ $\qquad u = \sin x$, $du = \cos x\,dx$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\sin x| + C$$

8. $\int x^2 \ln 3x \, dx$

$\ln 3x$		x^2
$\frac{1}{x}$	$-$	$\frac{x^3}{3}$

$= \frac{x^3}{3} \ln 3x - \int \frac{1}{x} \frac{x^3}{3} \, dx$

$= \frac{x^3}{3} \ln 3x - \frac{1}{3} \int x^2 dx$

$= \frac{x^3}{3} \ln 3x - \frac{1}{3} \frac{x^3}{3} + C$

$= \frac{x^3}{3} \ln 3x - \frac{x^3}{9} + C$

9. $\int \frac{\sec^2 t \, dt}{(2+3\tan t)(1-2\tan t)}$ $\quad u = \tan t$, $du = \sec^2 t \, dt$

$= \int \frac{du}{(2+3u)(1-2u)}$

let $\frac{1}{(2+3u)(1-2u)} = \frac{A}{2+3u} + \frac{B}{1-2u}$

$1 = A(1-2u) + B(2+3u)$

$u = \frac{1}{2} \Rightarrow 1 = 0 + B\left(2 + \frac{3}{2}\right)$

$= \frac{7}{2} B$

$B = \frac{2}{7}$

$u = -\frac{2}{3} \Rightarrow 1 = A\left(1 + \frac{4}{3}\right)$

$= \frac{7}{3} A$

$A = \frac{3}{7}$

$$\therefore \int \frac{\sec^2 t \, dt}{(2+3\tan t)(1-2\tan t)} = \int \left[\frac{3/7}{2+3u} + \frac{2/7}{1-2u} \right] du$$

$$= \frac{1}{7} \ln(2+3u) - \frac{1}{7} \ln(1-2u) + C$$

$$= \ln \left(\frac{2+3\tan t}{1-2\tan t} \right)^{1/7} + C$$

10. $\int_0^\infty \frac{r \, dr}{(4+r^2)^{3/2}}$ $\quad u = 4 + r^2$, $\quad du = 2r \, dr$.

$$= \int_{u=4}^{u=\infty} \frac{du/2}{u^{3/2}}$$

$$= \frac{1}{2} \int_4^\infty u^{-3/2} \, du$$

$$= \frac{1}{2} \left[\frac{u^{-1/2}}{-1/2} \right]_4^\infty$$

$$= 4^{-1/2}$$

$$= \frac{1}{2}$$