# Assignment 6

1 a) $$Q_{ij} \equiv \int_V \left[ 3x_i x_j - \delta_{ij} r^2 \right] \rho(\vec{r})\, d^3r$$

$$= \int_V \left[ 3x_j x_i - \delta_{ji} r^2 \right] \rho(\vec{r})\, d^3r$$

$\therefore Q_{ij} = Q_{ji} \Rightarrow \underset{\sim}{Q}$ is symmetric.

b) $$\mathrm{Tr}\, \underset{\sim}{Q} \equiv Q_{11} + Q_{22} + Q_{33}$$

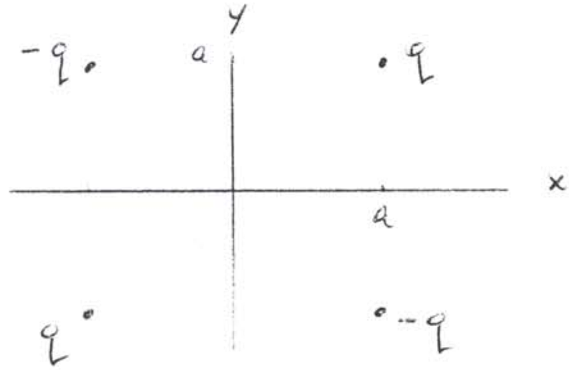
$$= \int \rho(\vec{r})\, d^3r \left[ 3x_1^2 - r^2 + 3x_2^2 - r^2 + 3x_3^2 - r^2 \right]$$

$$= \int \rho(\vec{r})\, d^3r \left[ 3(x_1^2 + x_2^2 + x_3^2) - 3r^2 \right]$$

$$= \int \rho(\vec{r})\, d^3r \left[ 3(x^2 + y^2 + z^2) - 3r^2 \right]$$

$\therefore \mathrm{Tr}\, \underset{\sim}{Q} = 0.$

2)




$$Q_{ij} = \sum_k q_k \left[ 3 x_{ki} x_{kj} - \delta_{ij} r_k^2 \right]$$

$$Q_{11} = \sum_k q_k \left[ 3 x_{k1}^2 - r_k^2 \right]$$

$$= 2q \left[ 3a^2 - (\sqrt{2}\, a)^2 \right] - 2q \left[ 3a^2 - (\sqrt{2}\, a)^2 \right]$$

$$= 0.$$

*Similarly* $Q_{22} = Q_{33} = 0$

$$Q_{12} = \sum_k q_k \left[ 3 x_{k1} x_{k2} \right]$$

$$= 3 \left[ q a^2 - q(-a^2) + q a^2 - q(-a^2) \right]$$

$$= 12 q a^2.$$

*Also* $Q_{13} = Q_{23} = 0$.

$$\therefore \ \underset{\sim}{Q} = \begin{pmatrix} 0 & 12 q a^2 & 0 \\ 12 q a^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3) $$U_{Quad} = -\frac{Q_{33}}{4} \frac{\partial E_z}{\partial z}.$$

Electron electric field $E = \frac{q}{r^2}$

$$\frac{\partial E}{\partial r} = \frac{-2q}{r^3}$$

$$\therefore U_{Quad} \sim \frac{Q_{33}}{4} \frac{2q}{r^3}$$

$$= \frac{Q_{33}\, q}{2r^3}$$

$$= \frac{(.00282 \times 10^{-24} cm^2)(4.8 \times 10^{-10} esu)^2}{2\ (1.5 \times 10^{-8}\ cm)^3}$$

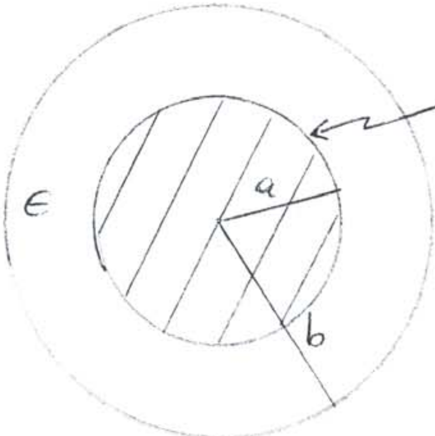
where we set $r \simeq$ Bohr radius

$$= 2.6 \times 10^{-21}\ erg.$$

$$U_{Ryd} = 13.6\ eV$$
$$= 2.18 \times 10^{-11}\ erg.$$

$$\therefore \frac{U_{Quad}}{U_{Ryd}} \simeq 1 \times 10^{-10}.$$

4 a)




Q sits on surface of spherical conductor.

First we find electric displacement using

$$\int_S \vec{D} \cdot d\vec{a} = 4\pi \int_V \rho_f \, dV.$$

Spherical symmetry $\Rightarrow \vec{D}(\vec{r}) = D(r)\,\hat{r}$.
Consider a spherical surface of radius r.

$$r < a \qquad \vec{D} = 0$$
$$r > a \qquad \vec{D} = \frac{Q}{r^2}\,\hat{r}.$$

Electric Field $\vec{E} = \dfrac{\vec{D}}{\epsilon}$

$$r < a \qquad \vec{E} = 0$$
$$b > r > a \qquad \vec{E} = \frac{Q}{\epsilon r^2}\,\hat{r}$$
$$r > b \qquad \vec{E} = \frac{Q}{r^2}\,\hat{r}$$

Polarization $\vec{P} = \dfrac{\vec{D} - \vec{E}}{4\pi}$

$$r < a \text{ or } r > b \qquad \vec{P} = 0$$

$$a < r < b \qquad \vec{P} = \frac{1}{4\pi}\left[1 - \frac{1}{\epsilon}\right]\frac{Q}{r^2}\,\hat{r} = \frac{\epsilon - 1}{4\pi\epsilon}\frac{Q}{r^2}\,\hat{r}.$$

Bound Charge Densities

$\sigma_b(r=a) = \vec{P}\cdot(-\hat{r})$
$= -\frac{\epsilon-1}{4\pi\epsilon}\frac{Q}{a^2}$

$\sigma_b(r=b) = \vec{P}\cdot\hat{r}$
$= \frac{\epsilon-1}{4\pi\epsilon}\frac{Q}{b^2}$.

b) $\varphi(r) - \varphi(\infty) = -\int_\infty^r \vec{E}\cdot d\vec{l}$

Since integral is path independent we pick the radial path.

$r > b \quad \varphi(r) = -\int_\infty^r \frac{Q}{r^2}\,dr.$

$= \frac{Q}{r}$

$b > r > a \quad \varphi(r) - \varphi(b) = -\int_b^r \frac{Q}{\epsilon r^2}\,dr$

$= \frac{Q}{\epsilon}\left[\frac{1}{r} - \frac{1}{b}\right]$

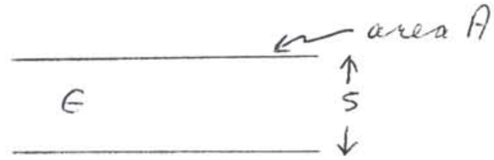
$\varphi(r) = \frac{Q}{b} + \frac{Q}{\epsilon}\left[\frac{1}{r} - \frac{1}{b}\right]$

$= Q\left[\frac{1}{\epsilon r} + \frac{1}{b}\left(1 - \frac{1}{\epsilon}\right)\right]$

$r < a \quad \varphi = \varphi(a) = Q\left[\frac{1}{\epsilon a} + \frac{1}{b}\left(1 - \frac{1}{\epsilon}\right)\right]$

= constant inside conductor.

5)




Let $\sigma$ be charge per unit area on plates.
Neglecting edge effects, electric displacement $D = 4\pi\sigma$.
between plates.

$\therefore$ electric field $E = \frac{D}{\epsilon} = \frac{4\pi\sigma}{\epsilon}$

$\therefore$ voltage between plates $V = Es$.
Capacitance $C = \frac{Q}{V} = \frac{\epsilon A}{4\pi s}$.