Assignment 3

1a) electric field inside a conductor $\vec{E} = 0$.

Gauss Law $\Rightarrow$ $\rho = \dfrac{\nabla \cdot \vec{E}}{4\pi} = 0$.

b) $$\Phi(b) - \Phi(a) = -\int_a^b \vec{E} \cdot d\vec{l}$$

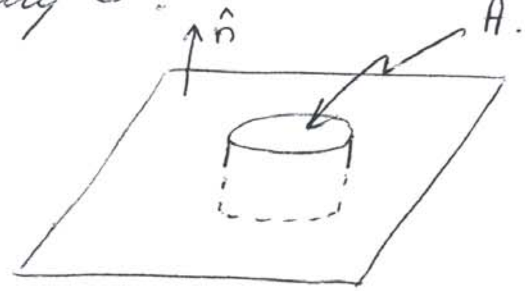
$$= 0$$

$\Rightarrow \Phi(b) = \Phi(a)$ for any 2 points a & b inside conductor.

c). Just outside a conductor, there can be no tangential fields since charge would move until an equal opposing field was established.

$\therefore \vec{E}$ is $\perp$ to conductor surface.

Next consider a conductor with surface charge density σ.



Consider a pillbox of negligible height and area A

Gauss Law $$\int_{\substack{\text{surface}\\ \text{of pillbox}}} \vec{E} \cdot d\vec{S} = 4\pi \int_{\text{pillbox}} \rho \, dV.$$

$$\int_{Top} \vec{E}\cdot d\vec{s} + \underbrace{\int_{sides} \vec{E}\cdot d\vec{s}}_{\substack{=0 \text{ since} \\ \vec{E} \perp \text{conductor} \\ \text{surface.}}} + \underbrace{\int_{Bottom} \vec{E}\cdot d\vec{s}}_{\vec{E}=0 \text{ in cond.}} = 4\pi\sigma A.$$

$$\int_{Top} E\, da = 4\pi\sigma A.$$

$$E A = 4\pi\sigma A$$

$$E = 4\pi\sigma.$$

One can argue that $\int_{Top} E\, da \neq EA$. However we can take $\lim_{A\to 0}$ on both sides of the equation.

$\therefore\ \vec{E} = 4\pi\sigma\,\hat{n}$ holds in general.

2a) Because of spherical symmetry $\vec{E} = E(r)\,\hat{r}$.

Gauss Law $\displaystyle\int_S \vec{E}\cdot d\vec{s} = 4\pi \int_V \rho\, dV.$

Let S be sphere of radius r.

$$\therefore\ E(r)\, 4\pi r^2 = 4\pi \int \rho\, dV.$$

$r < a$. $$E(r)\, 4\pi r^2 = 4\pi q.$$

$$\vec{E}(r) = \frac{q}{r^2}\,\hat{r}$$

$a < r < b$ $\vec{E} = 0$ inside conductor.

$r > b$ $$E(r)\, 4\pi r^2 = 4\pi \int \rho\, dV.$$

$$= 4\pi q.$$

$$\vec{E}(r) = \frac{q}{r^2}\,\hat{r}.$$

b) Component of electric field $\perp$ to cond. surface of radius $r = a$ is

$$E_\perp(r = a) = \frac{q}{a^2}\,\hat{r}\cdot(-\hat{r})$$

normal unit vector to surface

$$= -\frac{q}{a^2}$$

$\therefore$ charge density on $r=a$ cond. surface

$$\sigma(r=a) = \frac{E_\perp(r=a)}{4\pi}$$

$$= \frac{-q}{4\pi a^2}.$$

Component of electric field $\perp$ to cond. surface of radius $r=b$ is:

$$E_\perp(r=b) = \frac{q}{b^2}\,\hat{r}\cdot\hat{r}$$

$$= \frac{q}{b^2}.$$

normal unit vector to surface

$\therefore$ charge density $\sigma(r=b) = \dfrac{q}{4\pi b^2}$.

c) Let $\Phi(r=\infty) = 0$.

$r \geq b$ $\quad \Phi(r) - \Phi(\infty) = -\int_\infty^r \vec{E}\cdot d\vec{l}$ $\quad$ (Choose $d\vec{l} = d\vec{r}$)

$$\Phi(r) = -\int_\infty^r \frac{q}{r^2}\,dr$$

$$= -q\left[\frac{-1}{r}\right]_\infty^r$$

$$= \frac{q}{r}.$$

$a \leq r \leq b$ $\quad \Phi = \dfrac{q}{b}$ = const. inside conductor.

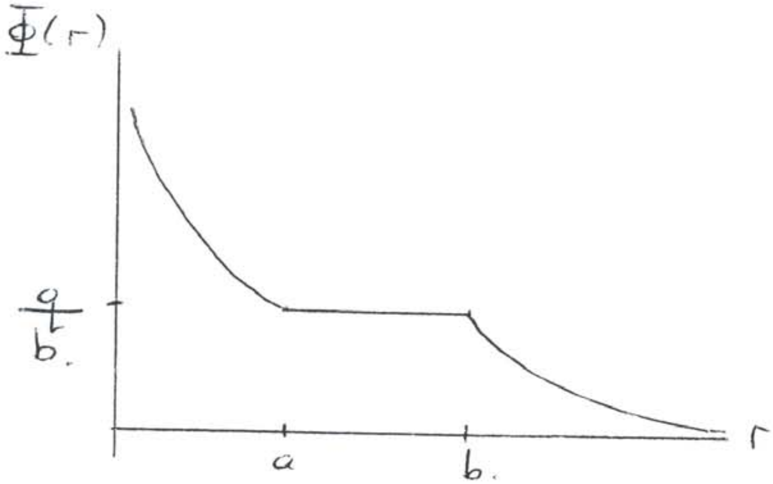
$r < a \qquad \Phi(r) - \Phi(a) = -\int_a^r \vec{E} \cdot d\vec{l} \qquad (\text{Choose } d\vec{l} = d\vec{r})$

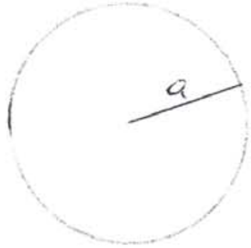
$$\Phi(r) - \frac{q}{b} = -\int_a^r \frac{q}{r^2}\, dr$$

$$= -q \left[\frac{-1}{r} \right]_a^r$$

$$= \frac{q}{r} - \frac{q}{a}$$

$$\Phi(r) = q \left[\frac{1}{r} + \frac{1}{b} - \frac{1}{a} \right]$$



3a) 

$$\rho = \begin{cases} \rho_0 & r < a \\ 0 & r > a \end{cases}$$

Gauss Law $\displaystyle\int_S \vec{E}\cdot d\vec{s} = 4\pi \int_V \rho\, dV$

Due to spherical symmetry $\vec{E} = E(r)\,\hat{r}$.

Let S be sphere of radius r.

$r < a$. $\qquad E(r)\, 4\pi r^2 = 4\pi \rho_0 \frac{4}{3}\pi r^3.$

$$\vec{E}(r) = \frac{4\pi}{3}\rho_0 \vec{r}$$

$r > a$ $\qquad \vec{E}(r) = \dfrac{4\pi}{3}\dfrac{a^3 \rho_0}{r^2}\,\hat{r}.$

b) $\displaystyle \Phi(r) - \Phi(0) = -\int_0^r \vec{E}\cdot d\vec{l}$ $\qquad$ (Choose $d\vec{l} = d\vec{r}$).

$r \le a$ $\qquad \displaystyle \Phi(r) = -\int_0^r \frac{4\pi}{3}\rho_0\, r\, dr.$

$$= -\frac{4\pi}{3}\rho_0 \left.\frac{r^2}{2}\right|_0^r$$

$$= -\frac{2\pi}{3}\rho_0 r^2.$$

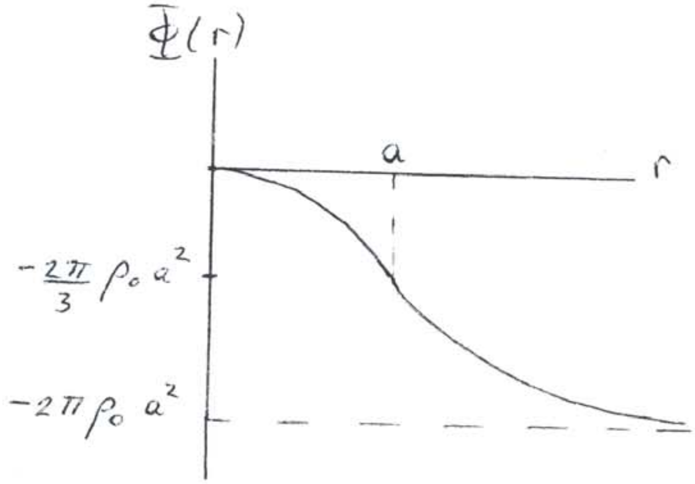
$r \ge a$ $\qquad \displaystyle \Phi(r) - \Phi(a) = -\int_a^r \vec{E}\cdot d\vec{l}.$

$$= -\frac{4\pi}{3} a^3 \rho_0 \int_a^r \frac{dr}{r^2}.$$

$$\Phi(r) - \left(-\frac{2\pi}{3}\rho_0 a^2\right) = -\frac{4\pi}{3} a^3 \rho_0 \left[\frac{-1}{r}\right]_a^r$$

$$= \frac{4\pi}{3} a^3 \rho_0 \frac{1}{r} - \frac{4\pi}{3} a^2 \rho_0 .$$

$$\Phi(r) = \frac{4\pi}{3} a^3 \rho_0 \left(\frac{1}{r} - \frac{3}{2a}\right).$$



c) $a = 2\text{ cm}$, $\rho_0 = \frac{3}{2\pi}$ esu/cm^3.

i) total charge on sphere $\frac{4\pi}{3} a^3 \rho_0$

$$= \frac{4\pi}{3} (2\text{ cm})^3 \frac{3}{2\pi} \text{ esu/cm}^3$$

$$= 16 \text{ esu}.$$

ii) $$\vec{E}(10\text{ cm}) = \frac{16\text{ esu}}{(10\text{ cm})^2} \hat{r}$$

$$= .16\ \hat{r}\ \text{esu/cm}^2.$$

$$= .16\ \hat{r}\ \text{statvolt/cm}.$$

$$\Phi(10\,cm) = 16\,esu\left(\frac{1}{10\,cm} - \frac{3}{2\times 2\,cm}\right)$$

$$= -10.4 \;\; esu/cm$$

$$= -10.4 \;\; statvolts$$

iii) Work done in moving 5 esu from infinity to 10 cm from center of sphere is

$$W_{10,\infty} \equiv 5\,esu\left[\Phi(10\,cm) - \Phi(\infty)\right]$$

$$= 5\,esu\left[-10.4 \text{ statvolts} - (-2\pi)\rho_0 a^2\right]$$

$$= 5\,esu\left[-10.4 \text{ statvolts} + 2\pi \frac{3}{2\pi}\frac{esu}{cm^3}\cdot(2\,cm)^2\right]$$

$$= 5\,esu\left[-10.4 + 12\right] \text{ statvolts}$$

$$\therefore W_{10,\infty} = 8 \text{ ergs}.$$

Force between charge & sphere is

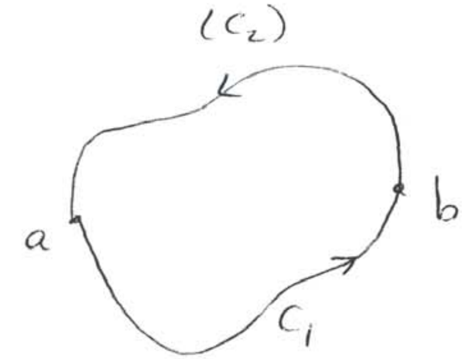
$$5\,esu \times E(r = 10\,cm)$$

$$= 5\,esu \times .16\ \frac{\text{statvolt}}{\text{cm}}.$$

$$= .80 \text{ dynes}.$$

The force is repulsive since both charges are positive.

4 a)



Let a and b be two different points on a closed path.

$$\oint \vec{E}\cdot d\vec{l} = \int_{a\,(C_1)}^{b} \vec{E}\cdot d\vec{l} + \int_{b\,(C_2)}^{a} \vec{E}\cdot d\vec{l}$$

$$= -\Phi(b) + \Phi(a) - \Phi(a) + \Phi(b)$$

$$\therefore \oint \vec{E}\cdot d\vec{l} = 0.$$

b) $\oint \vec{E}\cdot d\vec{l} = 0.$ for any closed path

$\int_S (\nabla \times \vec{E})\cdot d\vec{a} = 0$ for any arbitrary surface S

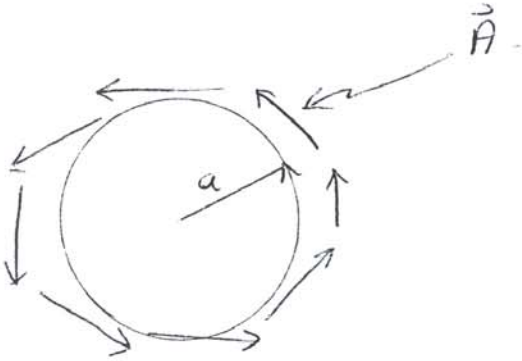
$$\Longrightarrow \nabla \times \vec{E} = 0.$$

c) If $\vec{E} = -\nabla \Phi$

$$= -\left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right)$$

then $\nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial \Phi}{\partial x} & -\frac{\partial \Phi}{\partial y} & -\frac{\partial \Phi}{\partial z} \end{vmatrix} = \vec{0}$

d)



$$\oint_{\substack{\text{circle perimeter}\\\text{of radius } a}} \vec{A}\cdot d\vec{l} \neq 0.$$