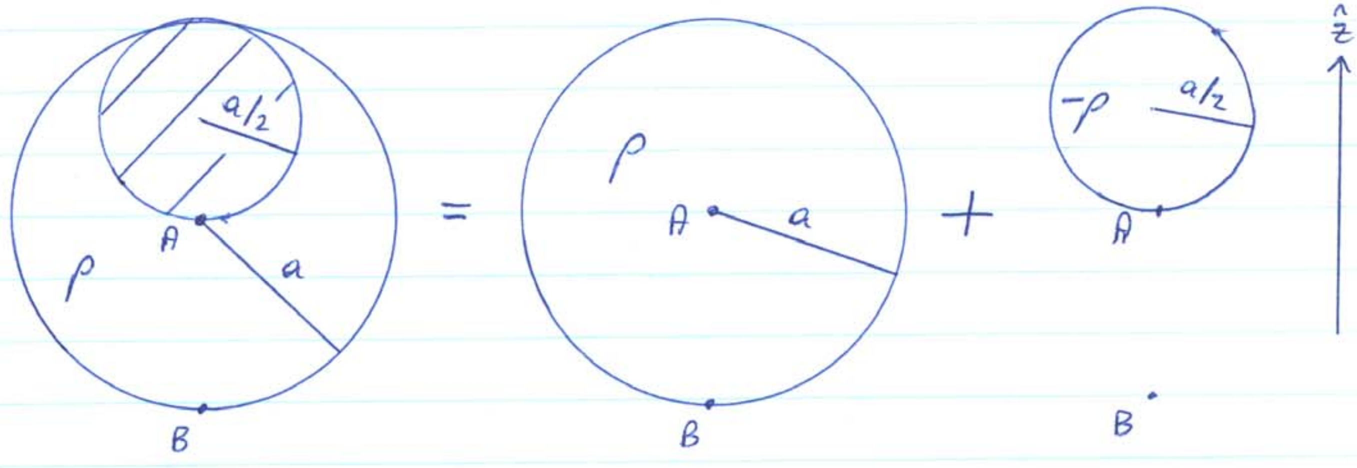
PHYS 2020 Assignment 3

1. Using the superposition principle the problem may be divided into two simpler problems.



Charge on sphere 1 $\quad q_1 = \frac{4}{3}\pi a^3 \rho$

" " 2 $\quad q_2 = -\frac{4}{3}\pi \left(\frac{a}{2}\right)^3 \rho$

Let $\vec{E}_1$ & $\vec{E}_2$ denote electric fields produced by q_1 & q_2.

$$\therefore \vec{E}(A) = \vec{E}_1(A) + \vec{E}_2(A)$$

$$= 0 + \frac{q_2(-\hat{z})}{(a/2)^2}$$ As expected $\vec{E}_2 \parallel \hat{z}$ since $q_2 < 0$.

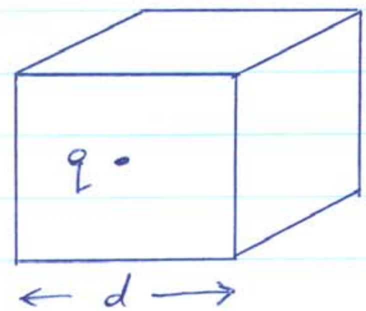
$$= \frac{2\pi}{3} a\rho\, \hat{z}$$

$$\vec{E}(B) = \vec{E}_1(B) + \vec{E}_2(B)$$

$$= \frac{q_1}{a^2}(-\hat{z}) + \frac{q_2}{(a + a/2)^2}(-\hat{z})$$

$$= \frac{-34}{27}\pi a\rho\, \hat{z}.$$

2a)

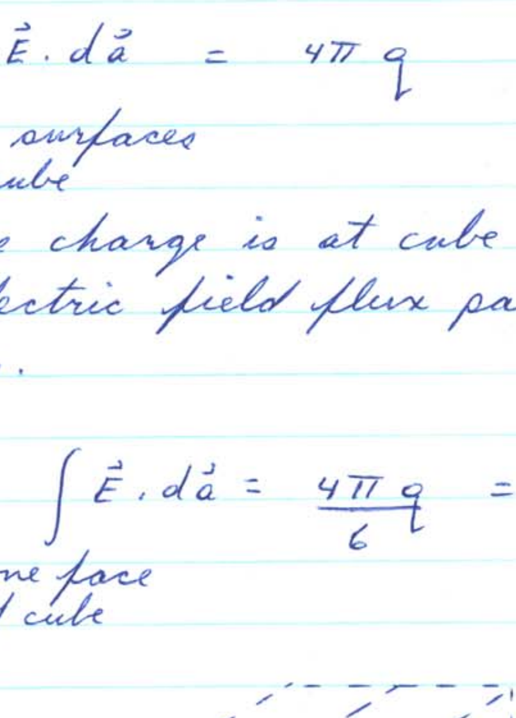


$$\int_{\substack{\text{all surfaces}\\\text{of cube}}} \vec{E}\cdot d\vec{a} = 4\pi q$$

Since charge is at cube center, an equal amount of electric field flux passes through each of 6 sides.

$$\therefore \int_{\substack{\text{one face}\\\text{of cube}}} \vec{E}\cdot d\vec{a} = \frac{4\pi q}{6} = \frac{2\pi}{3} q$$

b)



Since $\vec{E}$ extends radially outward from q, it does not cross the 3 planes that touch q at one corner. i.e. $\vec{E}\cdot d\vec{a} = 0 \Rightarrow \int_{\text{side touching } q} \vec{E}\cdot d\vec{a} = 0$

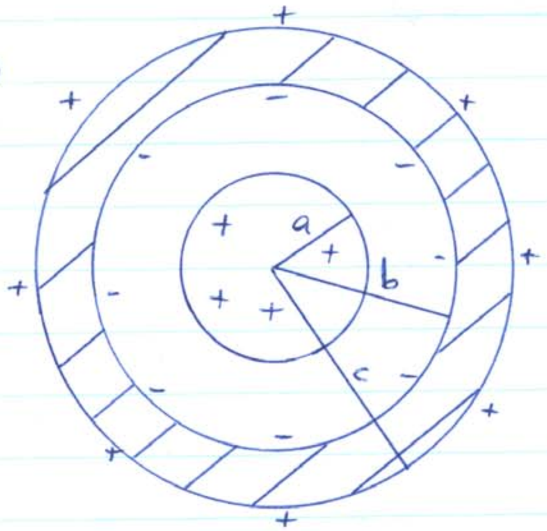
An equal amount of flux passes through each of the remaining 3 sides. (left, front & bottom)
To find this flux we construct a cube of length 2d centered about q.

$$\int_{\substack{\text{all surfaces}\\ \text{of length } 2d}} \vec{E} \cdot d\vec{a} = 4\pi q$$

$$4 \times 6 \int_{\substack{\text{a } d \times d \text{ surface}\\ \text{of cube of length } 2d}} \vec{E} \cdot d\vec{a} = 4\pi q$$

$\therefore \int \vec{E} \cdot d\vec{a} = \frac{\pi q}{6}$ of remaining 3 surfaces.

3a)



$r<a \quad \rho = \frac{\rho_0 r}{a}$ insulator

$a<r<b$ vacuum

$b<r<c$ conductor

$c<r$ vacuum

$\vec{E}=0$ inside conductor. Hence a negative surface charge must exist at $r=b$ to counteract the charged insulator. Since the conductor is neutral an equal positive charge must reside on the outer conductor surface at $r=c$.

i) $\vec{E} = E(r)\,\hat{r}$ due to spherical symmetry

$$\int_{\substack{\text{surface of}\\ \text{sphere of radius } r}} \vec{E}\cdot d\vec{a} = 4\pi \int_{\substack{\text{volume of}\\ \text{sphere of radius } r}} \rho\, dV$$

$$r<a \qquad E(r)\,4\pi r^2 = 4\pi \int_0^r \frac{\rho_0 r}{a}\, 4\pi r^2\, dr$$

$$= 4\pi \quad 4\pi \frac{\rho_0}{a} \int_0^r r^3\, dr$$

$$= 4\pi \quad \frac{4\pi \rho_0}{a} \quad \frac{r^4}{4}$$

$$\therefore \vec{E}(r) = \frac{\pi \rho_0 r^2}{a}\ \hat{r}$$

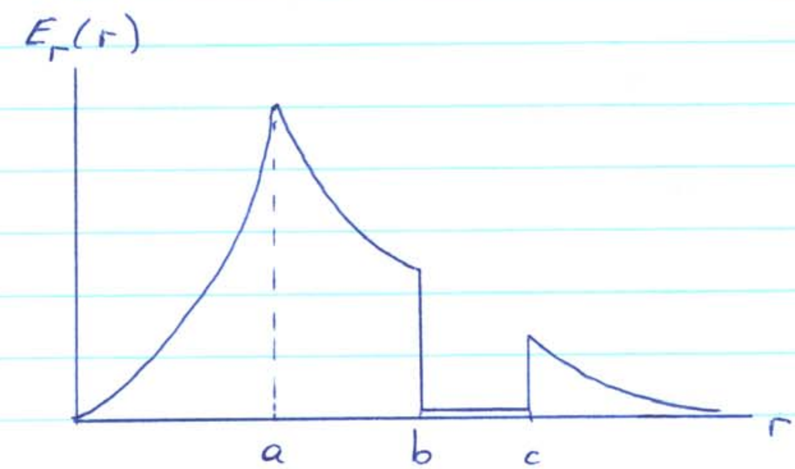
Note that charge on insulator $q = \pi \rho_0 a^3$.

$a < r < b \qquad E(r)\, 4\pi r^2 = 4\pi q$

$$\vec{E}(r) = \frac{q}{r^2}\,\hat{r}$$

$b < r < c \qquad \vec{E} = 0$ inside conductor

$r > c \qquad \vec{E} = \frac{q}{r^2}\,\hat{r}.$



ii) Electric Field Outside Conductor $\vec{E} = 4\pi\sigma\,\hat{n}$

where $\hat{n}$ = direction perpendicularly outward from surface.

$\sigma(r=b) = \frac{\vec{E}(b)}{4\pi} \cdot (-\hat{r})$ (Outward direction from conductor at $r=b$ is $-\hat{r}$)

$$= \frac{1}{4\pi}\,\frac{q}{b^2}\,\hat{r}\cdot(-\hat{r})$$

$$= -\frac{q}{4\pi b^2}$$

$$= -\frac{\rho_0 a^3}{4 b^2}$$

$$\sigma(r=c) = \frac{\vec{E}(c)\cdot\hat{r}}{4\pi}$$

$$= \frac{1}{4\pi}\frac{q}{c^2}\,\hat{r}\cdot\hat{r}$$

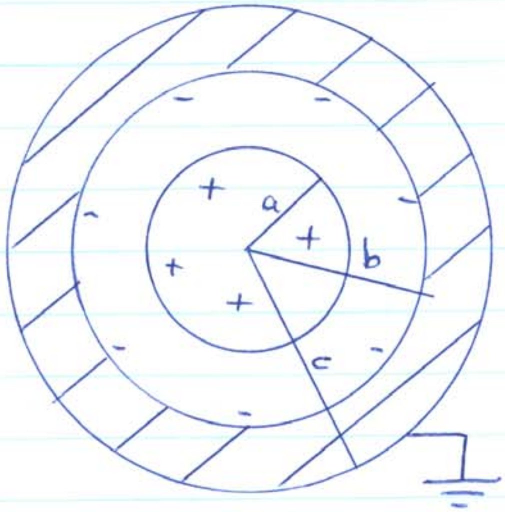
$$= \frac{\rho_0 a^3}{4c^2}$$

Note that total amount of charge on conductor

$= \sigma(r=b)\cdot 4\pi b^2 + \sigma(r=c)\cdot 4\pi c^2$

$= 0$.

b)



Since $\vec{E}=0$ inside a conductor there still is the negative charge at the inner conductor surface. However since the conductor is connected to ground the outer positive charge flows away into ground.

$$\therefore \sigma(r=b) = -\frac{\rho_0 a^3}{4b^2} \quad \text{(same as before)}$$

$$\sigma(r=c) = 0.$$

Note that $\sigma(r=b)\cdot 4\pi b^2 = -\pi\rho_0 a^3$

$= -q$ (q = charge of insulator)

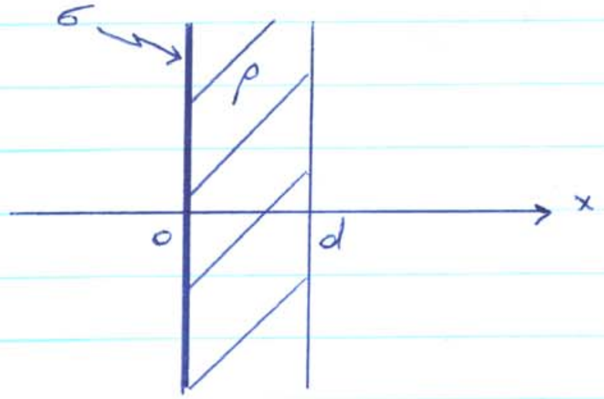
$\therefore$ when $r>c$ using Gauss law we get:

$$E(r)\,4\pi r^2 = 4\pi\left[q + (-q)\right]$$

$$\vec{E}(r) = 0.$$

For $r<c$, electric field is the same as in part a.

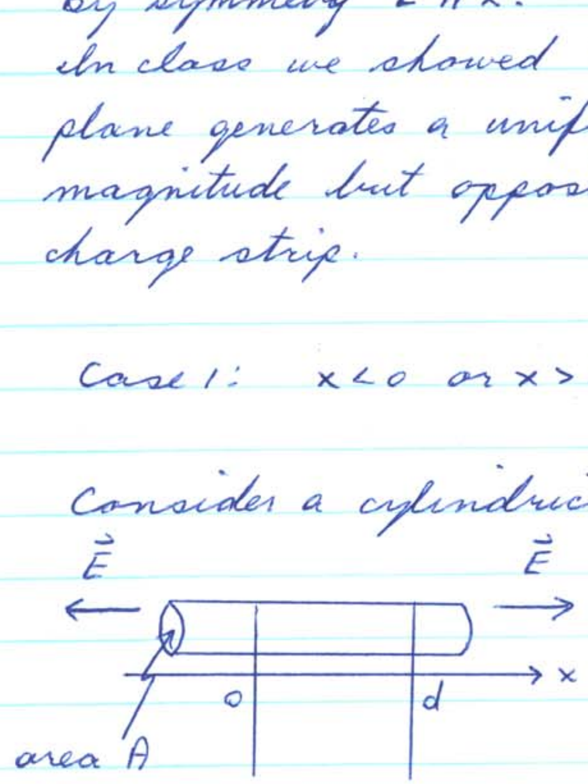
4.



By symmetry $\vec{E} \parallel \hat{x}$.
In class we showed that an infinite uniformly charged plane generates a uniform field. Hence $\vec{E}$ has the same magnitude but opposite direction on either side of the charge strip.

Case 1: $x<0$ or $x>d$.

Consider a cylindrical pillbox straddling the strip



$$\underset{\substack{\text{cylinder}\\ \text{surface}}}{\int} \vec{E}\cdot d\vec{a} = 4\pi \underset{\substack{\text{cylinder}\\ \text{volume}}}{\int} \rho\, dV.$$

$$EA + EA = 4\pi\,(\sigma + \rho d)\,A$$

$$E = 2\pi\,(\sigma + \rho d)$$

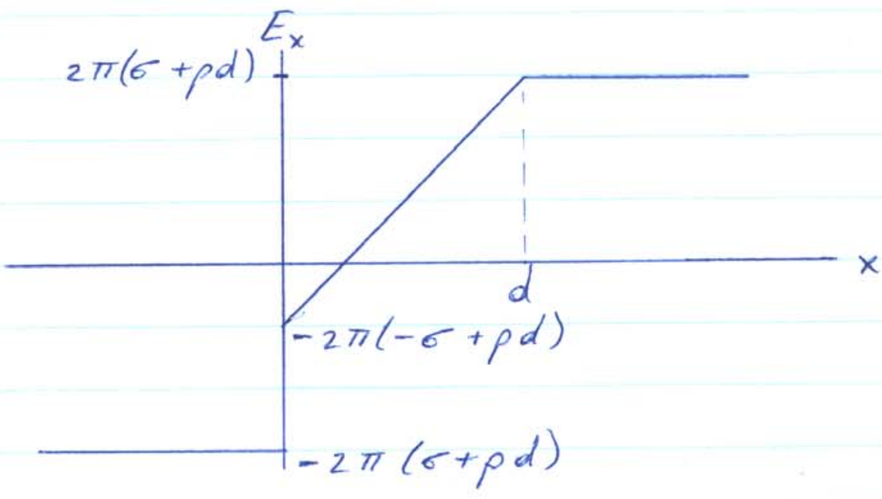
This result expresses that uniformly charged infinite planes attract or repel charges depending on which side of the plane the charge is.

$\therefore$ $E(x) = -2\pi(\text{charge per cm}^2 \text{ to right of } x) + 2\pi(\text{charge per cm}^2 \text{ to left of } x)$

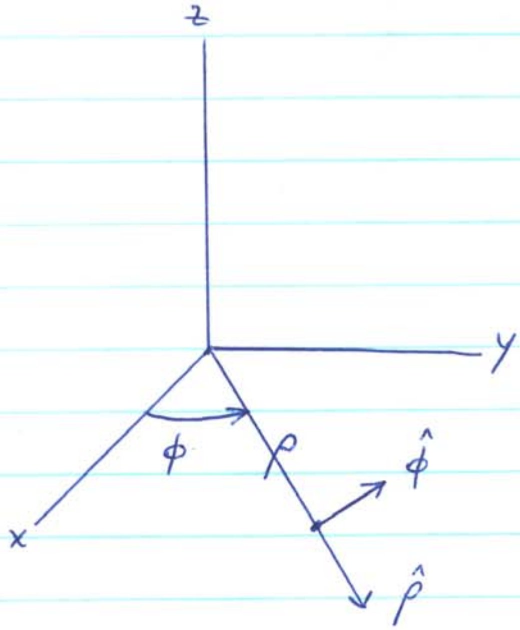
Case 2: $0 < x < d$.

$$E(x) = -2\pi \rho(d-x) + 2\pi(\sigma + \rho x)$$

$$= 2\pi(\sigma - \rho d + 2x).$$



5 i)



$z = z$
$y = \rho \sin\phi$
$x = \rho \cos\phi$

ii) $\hat{z} = \hat{z}$ (1)

$\hat{\rho} = (\cos\phi, \sin\phi, 0) = \hat{x}\cos\phi + \hat{y}\sin\phi$ (2)

$\hat{\phi} = (-\sin\phi, \cos\phi, 0) = -\hat{x}\sin\phi + \hat{y}\cos\phi.$ (3)

Solving (2) + (3) for $\hat{x}$ + $\hat{y}$ we get:

$\hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$ (4)
$\hat{y} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi$ (5).

iii) $E_x = \vec{E}\cdot\hat{x}$

$= \vec{E}\cdot(\hat{\rho}\cos\phi - \hat{\phi}\sin\phi)$

$$\boxed{E_x = E_\rho\cos\phi - E_\phi\sin\phi}$$

$E_y = \vec{E}\cdot\hat{y}$
$= \vec{E}\cdot(\hat{\rho}\sin\phi + \hat{\phi}\cos\phi)$

$$\boxed{E_y = E_\rho\sin\phi + E_\phi\cos\phi}$$

iv) $$\frac{\partial E_y}{\partial y} = \frac{\partial E_y}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial E_y}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial E_y}{\partial z}\frac{\partial z}{\partial y}$$

$$\frac{\partial E_z}{\partial z} = \frac{\partial E_z}{\partial \rho}\frac{\partial \rho}{\partial z} + \frac{\partial E_z}{\partial \phi}\frac{\partial \phi}{\partial z} + \frac{\partial E_z}{\partial z}\frac{\partial z}{\partial z}.$$

v) $$\rho = \sqrt{x^2+y^2}$$

$$\frac{\partial \rho}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{x}{\rho} = \cos\phi.$$

$$\frac{\partial \rho}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y = \frac{y}{\rho} = \sin\phi$$

$$\frac{\partial \rho}{\partial z} = 0.$$

$$\tan\phi = \frac{y}{x}.$$

$$\sec^2\phi \frac{\partial \phi}{\partial x} = \frac{-y}{x^2}$$

$$= -\frac{\sin\phi}{\rho\cos^2\phi}.$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{\rho}$$

$$\sec^2\phi \frac{\partial \phi}{\partial y} = \frac{1}{x}$$

$$= \frac{1}{\rho\cos\phi}.$$

$$\frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho}.$$

Obviously $\frac{\partial \phi}{\partial z} = 0.$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0 \qquad \frac{\partial z}{\partial z} = 1.$$

vi) $\nabla . \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial \rho} \cos\phi + \frac{\partial E_x}{\partial \phi} \frac{(-\sin\phi)}{\rho} + 0 \quad \text{from (iv) + (v)}$$

$$= \frac{\partial}{\partial \rho}\left(E_\rho \cos\phi - E_\phi \sin\phi \right) \cos\phi$$

$$+ \frac{\partial}{\partial \phi}\left(E_\rho \cos\phi - E_\phi \sin\phi \right) \frac{(-\sin\phi)}{\rho} \quad \text{using (iii)}$$

$$\boxed{\begin{aligned}\frac{\partial E_x}{\partial x.} &= \left(\frac{\partial E_\rho}{\partial \rho} \cos\phi - \frac{\partial E_\phi}{\partial \rho} \sin\phi \right) \cos\phi \\ &+ \left(\frac{\partial E_\rho}{\partial \phi} \cos\phi - E_\rho \sin\phi - \frac{\partial E_\phi}{\partial \phi} \sin\phi - E_\phi \cos\phi \right) \frac{(-\sin\phi)}{\rho}\end{aligned}}$$

$$\frac{\partial E_y}{\partial y} = \frac{\partial E_y}{\partial \rho} \sin\phi + \frac{\partial E_y}{\partial \phi} \frac{\cos\phi}{\rho} + 0 \quad \text{from (iv) + (v).}$$

$$= \frac{\partial}{\partial \rho}\left(E_\rho \sin\phi + E_\phi \cos\phi \right) \sin\phi$$

$$+ \frac{\partial}{\partial \phi}\left(E_\rho \sin\phi + E_\phi \cos\phi \right) \frac{\cos\phi}{\rho}$$

$$\boxed{\begin{aligned}\frac{\partial E_y}{\partial y} &= \left(\frac{\partial E_\rho}{\partial \rho}\sin\phi + \frac{\partial E_\phi}{\partial \rho}\cos\phi\right)\sin\phi \\ &+ \left(\frac{\partial E_\rho}{\partial \phi}\sin\phi + E_\rho\cos\phi + \frac{\partial E_\phi}{\partial \phi}\cos\phi - E_\phi\sin\phi\right)\frac{\cos\phi}{\rho}\end{aligned}}$$

$$\therefore \nabla\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\begin{aligned} &= \frac{\partial E_\rho}{\partial \rho}\left(\cos^2\phi + \sin^2\phi\right) \\ &\quad + \frac{\partial E_\phi}{\partial \rho}\left(-\sin\phi\cos\phi + \cos\phi\sin\phi\right) \\ &\quad + \frac{\partial E_\rho}{\partial \phi}\left(-\cos\phi\frac{\sin\phi}{\rho} + \sin\phi\frac{\cos\phi}{\rho}\right) \\ &\quad + \frac{\partial E_\phi}{\partial \phi}\left(\frac{\sin^2\phi}{\rho} + \frac{\cos^2\phi}{\rho}\right) \\ &\quad + E_\rho\left(\frac{\sin^2\phi}{\rho} + \frac{\cos^2\phi}{\rho}\right) \\ &\quad + E_\phi\left(\cos\phi\frac{\sin\phi}{\rho} - \sin\phi\frac{\cos\phi}{\rho}\right) \\ &\quad + \frac{\partial E_z}{\partial z}\end{aligned}$$

$$\boxed{\begin{aligned}&\nabla\cdot\vec{E} = \frac{\partial E_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} + \frac{E_\rho}{\rho} + \frac{\partial E_z}{\partial z}. \\ \text{or } &\nabla\cdot\vec{E} = \frac{1}{\rho}\frac{\partial(\rho E_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}.\end{aligned}}$$