Assignment 3

1) $L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$

Converting to spherical coordinates we get;

$$\frac{\partial}{\partial y} = \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial r}{\partial y}\frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial x} = \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial r}{\partial x}\frac{\partial}{\partial r}$$

Using $\left.\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta \end{aligned}\right\} \Rightarrow$ $r = \sqrt{x^2+y^2+z^2}$

$\cos\theta = \frac{z}{r}$

$\tan\phi = \frac{y}{x}$

one can show; $\frac{\partial r}{\partial x} = \sin\theta\cos\phi \qquad \frac{\partial r}{\partial y} = \sin\theta\sin\phi$

$\frac{\partial\theta}{\partial x} = \frac{\cos\theta\cos\phi}{r} \qquad \frac{\partial\theta}{\partial y} = \frac{\cos\theta . \sin\phi}{r}$

$\frac{\partial\phi}{\partial x} = \frac{-\sin\phi}{r\sin\theta} \qquad \frac{\partial\phi}{\partial y} = \frac{\cos\phi}{r\sin\theta}$

$$\therefore L_z = -i\hbar r\sin\theta\cos\phi\left(\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \sin\theta\sin\phi\frac{\partial}{\partial r}\right)$$

$$+ i\hbar r\sin\theta\sin\phi\left(\frac{-\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} + \sin\theta\cos\phi\frac{\partial}{\partial r}\right.$$

$$= -i\hbar\left(\cos^2\phi + \sin^2\phi\right)\frac{\partial}{\partial\phi} = -i\hbar\frac{\partial}{\partial\phi}$$

2 a) $L_- Y_{11} = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)(-1)\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$

$$= -\hbar\sqrt{\frac{3}{8\pi}}\, e^{-i\phi}\left(-\cos\theta\, e^{i\phi} + i\cos\theta\,(i)e^{i\phi}\right)$$

$$= \hbar\sqrt{\frac{3}{2\pi}}\cos\theta.$$

Also $L_- Y_{11} = \hbar\sqrt{1(2)-1.0}\,Y_{10} = \sqrt{2}\,\hbar\, Y_{10}$

$$\Rightarrow Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta.$$

Applying L_- to $Y_{10} \Rightarrow Y_{1-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$.

b) $\int Y^*_{2-2}\, Y_{22}\, d\Omega$.

$$= \int_0^{2\pi}\int_0^{\pi}\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\phi}\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\phi}\sin\theta\, d\theta\, d\phi.$$

$$= \frac{15}{32\pi}\underbrace{\int_0^{2\pi} e^{4i\phi}\, d\phi}_{=0.}\int_0^{\pi}\sin^5\theta\, d\theta.$$

$= 0.$

Similarly:

$$\int Y_{21}^{*} \, Y_{21} \, d\Omega$$

$$= \int_0^{2\pi} \int_0^{\pi} -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{-i\phi} \left(-\sqrt{\frac{15}{8\pi}}\right) \sin\theta\cos\theta \, e^{i\phi} \sin\theta \, d\theta \, d\phi$$

$$= \frac{15}{8\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta.$$

$$= \frac{15}{8\pi} \cdot 2\pi \cdot \int_0^{\pi} (1-\cos^2\theta)\cos^2\theta \sin\theta \, d\theta.$$

Let $u = \cos\theta. \Rightarrow du = -\sin\theta \, d\theta.$

$$= \frac{15}{4} \int_1^{-1} (1-u^2)u^2 \, (-du)$$

$$= \frac{15}{4} \int_{-1}^{1} (u^2 - u^4) \, du.$$

$$= \frac{15}{4} \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_{-1}^{1}$$

$$= \frac{15}{4} \cdot 2 \left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= 1$$

3) $|3\ 3\rangle = |2\ 2\rangle\ |1\ 1\rangle$

$$\underset{J}{3}\quad \underset{M}{3} \qquad \underset{J_1}{2}\quad \underset{M_2}{2} \qquad \underset{J_2}{1}\quad \underset{M_1}{1}$$

$$J_- \equiv J_{1-} + J_{2-}$$

$$J_-|3\,3\rangle = J_{1-}|2\,2\rangle|1\,1\rangle + |2\,2\rangle J_{2-}|1\,1\rangle.$$

Now $J_-|J\,M\rangle = \sqrt{J(J+1)-M(M-1)}\ \hbar\ |J\ M-1\rangle.$

$$\sqrt{3,4-3,2}\ |3\,2\rangle = \sqrt{2,3-2,1}\ |2\,1\rangle|1\,1\rangle + \sqrt{1,2-1,0}\ |2\,2\rangle|1\,0\rangle$$

$$\sqrt{6}\ |3\,2\rangle = \sqrt{4}\ |2\,1\rangle|1\,1\rangle + \sqrt{2}\ |2\,2\rangle|1\,0\rangle$$

$$\boxed{|3\,2\rangle = \sqrt{\frac{2}{3}}\ |2\,1\rangle|1\,1\rangle + \frac{1}{\sqrt{3}}|2\,2\rangle|1\,0\rangle.}$$

$$J_-|3\,2\rangle = \sqrt{\frac{2}{3}}\left(J_{1-}|2\,1\rangle|1\,1\rangle + |2\,1\rangle J_{2-}|1\,1\rangle\right)$$

$$+\frac{1}{\sqrt{3}}\left(J_{1-}|2\,2\rangle|1\,0\rangle + |2\,2\rangle J_{2-}|1\,0\rangle\right)$$

$$\sqrt{3,4-2,1}\ |3\,1\rangle = \sqrt{\frac{2}{3}}\left(\sqrt{2,3-1,0}\ |2\,0\rangle|1\,1\rangle + \sqrt{2}\ |2\,1\rangle|1\,0\rangle\right)$$

$$+\frac{1}{\sqrt{3}}\left(\sqrt{2,3-2}\ |2\,1\rangle|1\,0\rangle + \sqrt{2}\ |2\,2\rangle|1\,-1\rangle\right)$$

$$\sqrt{10}\ |3\,1\rangle = \sqrt{\frac{2}{3}}\left(\sqrt{6}\ |2\,0\rangle|1\,1\rangle + \sqrt{2}\ |2\,1\rangle|1\,0\rangle\right)$$

$$+\frac{1}{\sqrt{3}}\left(2\ |2\,1\rangle|1\,0\rangle + \sqrt{2}\,|2\,2\rangle|1\,-1\rangle\right).$$

$$|31\rangle = \frac{1}{\sqrt{30}}\Big(2\sqrt{3}\ |20\rangle|11\rangle + 2\,|21\rangle|10\rangle + 2\,|21\rangle|10\rangle$$

$$+ \sqrt{2}\ |22\rangle|1{-1}\rangle\Big)$$

$$\boxed{|31\rangle = \frac{1}{\sqrt{15}}\Big(\sqrt{6}\ |20\rangle|11\rangle + 2\sqrt{2}\ |21\rangle|10\rangle + |22\rangle|1{-1}\rangle\Big)}$$

$$J_-|31\rangle = \frac{1}{\sqrt{15}}\Big\{ \sqrt{6}\big(J_{1-}|20\rangle|11\rangle + |20\rangle J_{2-}|11\rangle\big)$$

$$+ 2\sqrt{2}\big(J_{1-}|21\rangle|10\rangle + |21\rangle J_{2-}|10\rangle\big)$$

$$+ J_{1-}|22\rangle|1{-1}\rangle + |22\rangle J_{2-}|1{-1}\rangle\Big\}$$

$$\sqrt{3.4}\ |30\rangle = \sqrt{\frac{6}{15}}\Big(\sqrt{2.3}\ |2{-1}\rangle|11\rangle + \sqrt{2}\ |20\rangle|10\rangle\Big)$$

$$+ \sqrt{\frac{8}{15}}\Big(\sqrt{2.3}\ |20\rangle|10\rangle + \sqrt{2}\ |21\rangle|1{-1}\rangle\Big)$$

$$+ \frac{1}{\sqrt{15}}\sqrt{2.3-2}\ |21\rangle|1{-1}\rangle$$

$$|30\rangle = \frac{1}{\sqrt{30}}\Big(\sqrt{6}\ |2{-1}\rangle|11\rangle + \sqrt{2}\ |20\rangle|10\rangle\Big)$$

$$+ \sqrt{\frac{2}{45}}\Big(\sqrt{6}\ |20\rangle|10\rangle + \sqrt{2}\ |21\rangle|1{-1}\rangle\Big)$$

$$+ \frac{1}{\sqrt{180}}\sqrt{4}\ |21\rangle|1{-1}\rangle.$$

$$\boxed{|3\,0\rangle = \frac{1}{\sqrt{15}}\left(\sqrt{3}\,|2\,{-1}\rangle|1\,1\rangle + 3\,|2\,0\rangle|1\,0\rangle + \sqrt{3}\,|2\,1\rangle|1\,{-1}\rangle\right)}$$

Similarly applying J_- to $|3\,0\rangle$ yields:

$$\boxed{|3\,{-1}\rangle = \frac{1}{\sqrt{15}}\left(|2\,{-2}\rangle|1\,1\rangle + 2\sqrt{2}\,|2\,{-1}\rangle|1\,0\rangle + \sqrt{6}\,|2\,0\rangle|1\,{-1}\rangle\right)}$$

Note symmetry with $|3\,1\rangle$ & $|3\,{-1}\rangle$.

Similarly applying J_- to $|3\,{-1}\rangle$ & $|3\,{-2}\rangle$ yields:

$$\boxed{\begin{aligned} |3\,{-2}\rangle &= \frac{1}{\sqrt{3}}\left(\sqrt{2}\,|2\,{-1}\rangle|1\,{-1}\rangle + |2\,{-2}\rangle|1\,0\rangle\right) \\ |3\,{-3}\rangle &= |2\,{-2}\rangle|1\,{-1}\rangle. \end{aligned}}$$

4) For spin 1/2 particle $[S^2] = \frac{3\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$$[S^2, S_x] = S^2 S_x - S_x S^2$$

$$= \frac{3\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{3\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$= \underset{\sim}{0}$$

$$[S_x, S_y] = \left(\frac{\hbar}{2}\right)^2 \left\{ \sigma_x \sigma_y - \sigma_y \sigma_x \right\}$$

$$= \left(\frac{\hbar}{2}\right)^2 \left\{ \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \right\}$$

$$= \left(\frac{\hbar}{2}\right)^2 \left\{ \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix} \right\}$$

$$= 2i\left(\frac{\hbar}{2}\right)^2 \sigma_z$$

$$\therefore [S_x, S_y] = i\hbar S_z.$$

5) $$\vec{\mu}_J = \vec{\mu}_L + \vec{\mu}_S$$

$$g_J \vec{J} = g_L \vec{L} + g_S \vec{S}$$

$$g_J \vec{J} = \vec{L} + 2\vec{S}$$

$$g_J \vec{J}^2 = \vec{L}.\vec{J} + 2\vec{S}.\vec{J} \qquad (1)$$

Now $\vec{J}^2 |j m l s\rangle = j(j+1)\hbar^2 |j m l s\rangle$.

$\Rightarrow$ need $\vec{L}.\vec{J} |j m l s\rangle$ + $\vec{S}.\vec{J} |j m l s\rangle$.

Now $$\vec{J} = \vec{L} + \vec{S}$$

$$\vec{S} = \vec{J} - \vec{L}$$

$$\vec{S}.\vec{S} = (\vec{J} - \vec{L}).(\vec{J} - \vec{L})$$
$$= \vec{J}^2 - 2\vec{L}.\vec{J} + \vec{L}^2$$
$$\therefore \vec{L}.\vec{J} = \frac{\vec{J}^2 + \vec{L}^2 - \vec{S}^2}{2}$$

Similarly $$\vec{S}.\vec{J} = \frac{\vec{J}^2 + \vec{S}^2 - \vec{L}^2}{2}$$

Substitute $\vec{L}.\vec{J}$ + $\vec{S}.\vec{J}$ in (1) + operate on $|j m l s\rangle$

gives: $$g_J j(j+1) = \frac{j(j+1) + l(l+1) - s(s+1)}{2}$$
$$+ 2\left[\frac{j(j+1) + s(s+1) - l(l+1)}{2}\right]$$

$$\Rightarrow g_J = 1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}$$