Assignment 2

1a) $[x, p_x]\psi = (xp_x - p_x x)\psi$

$$= -i\hbar\left(x\frac{\partial\psi}{\partial x} - \frac{\partial}{\partial x}(x\psi)\right)$$

$$= -i\hbar\left(x\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial x} - \psi\right)$$

$$= i\hbar\psi$$

$\therefore [x, p_x] = i\hbar$

Similarly one can show b + c.

2a) $\vec{L} = \vec{r} \times \vec{p}$

$$= (yp_z - zp_y, -xp_z + zp_x, xp_y - yp_x)$$

$$= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

b) $[L_x, L_y]\psi = (L_x L_y - L_y L_x)\psi$

$$= (-i\hbar)^2\left\{\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(-x\frac{\partial\psi}{\partial z} + z\frac{\partial\psi}{\partial x}\right)\right.$$

$$\left. - \left(-x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right)\left(y\frac{\partial\psi}{\partial z} - z\frac{\partial\psi}{\partial y}\right)\right\}$$

$$= -\hbar^2\left\{-xy\frac{\partial^2\psi}{\partial z^2} + y\frac{\partial\psi}{\partial x} + yz\frac{\partial^2\psi}{\partial x\partial z} + xz\frac{\partial^2\psi}{\partial y\partial z} - z^2\frac{\partial^2\psi}{\partial x\partial y}\right.$$

$$\left. + xy\frac{\partial^2\psi}{\partial z^2} - x\frac{\partial\psi}{\partial y} - xz\frac{\partial^2\psi}{\partial y\partial z} - yz\frac{\partial^2\psi}{\partial x\partial z} + z^2\frac{\partial^2\psi}{\partial x\partial y}\right\}$$

$$[L_x, L_y] = -\hbar^2\left(y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right) = i\hbar L_z\psi$$

$$\Rightarrow [L_x, L_y] = i\hbar L_z$$

c) $[L_x, L^2] = [L_x, L_x^2 + L_y^2 + L_z^2]$

$$= [L_x, L_y^2] + [L_x, L_z^2]$$

$$= L_x L_y^2 - L_y^2 L_x + L_x L_z^2 - L_z^2 L_x$$

$$= L_x L_y^2 - L_y L_x L_y + L_y L_x L_y - L_y^2 L_x$$

$$+ L_x L_z^2 - L_z L_x L_z + L_z L_x L_z - L_z^2 L_x$$

$$= [L_x, L_y]L_y + L_y[L_x, L_y] + [L_x, L_z]L_z + L_z[L_x, L_z]$$

$$= i\hbar L_z L_y + i\hbar L_y L_z - i\hbar L_y L_z - i\hbar L_z L_y$$

$$= 0$$

d) An eigenstate is only an eigenfct. simultaneously of L^2 + one cartesian component since the cartesian components L_x, L_y + L_z do not commute with each other.

3a) $[J_z, J_+] = [J_z, J_x + iJ_y]$

$= [J_z, J_x] + i[J_z, J_y]$

$= i\hbar J_y + i(-i\hbar J_x)$

$= \hbar(J_x + iJ_y)$

$= \hbar J_+$

Similarly $[J_z, J_-] = -\hbar J_-$

b) $[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y]$

$= [J_x, J_x] - i[J_x, J_y] + i[J_y, J_x] + i[J_y, J_y]$

$= -i(i\hbar J_z) + i(-i\hbar J_z)$

$= 2\hbar J_z$

c) $[J^2, J_+] = [J^2, J_x + iJ_y]$

$= [J^2, J_x] + i[J^2, J_y]$

$= 0$

d) $J^2 = J_x^2 + J_y^2 + J_z^2$

$= \frac{1}{2}(J_x + iJ_y)(J_x - iJ_y) + \frac{1}{2}(J_x - iJ_y)(J_x + iJ_y) + J_z^2$

$= \frac{1}{2}\left(J_+ J_- + J_- J_+\right) + J_z^2$

4. See notes for similar problem for $J_+ |j \; m=+j\rangle = 0$ etc.

5a) Basis $= \{ |11\rangle, |10\rangle, |1-1\rangle \} = \{ |j=1, m\rangle \}$.

$$[J_z] = \begin{pmatrix} \langle 11|J_z|11\rangle & \langle 11|J_z|10\rangle & \langle 11|J_z|1-1\rangle \\ \langle 10|J_z|11\rangle & \langle 10|J_z|10\rangle & \langle 10|J_z|1-1\rangle \\ \langle 1-1|J_z|11\rangle & \langle 1-1|J_z|10\rangle & \langle 1-1|J_z|1-1\rangle \end{pmatrix}$$

$$= \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

For J_+, recall $\langle jm|J_+|j\;m'\rangle = \hbar\sqrt{(j-m)(j+m+1)}\;\delta_{m\,m'+1}$

$$\Rightarrow [J_+] = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly $[J_-] = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$

$$[J_x] = \frac{[J_+] + [J_-]}{2} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$[J_y] = \frac{[J_+] - [J_-]}{2i} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle$$

$$\therefore [J^2] = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Alternatively one can add matrices for J_x^2, J_y^2 + J_z^2.