Relativistic Quantum Mechanics

PHYS 5000.03

York University

W. A. van Wijngaarden

Textbooks.

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*10. Introduction To Elementary Particles
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* 11. Quantum Field Theory
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by C.C. Gerry & P.L. Knight, Cambridge Press, 2005.

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VI) Relativistic Quantum Electrodynamics (QED) p.103-138
- Feynman diagrams
- Matrix element determination
- Casimir's Trick
- Cross sections
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I) Schrodinger Equation

Derivation

Hamiltonian $H = E$ Energy

$$\underbrace{\frac{p^2}{2m}}_{\substack{\text{Kinetic}\\ \text{Energy}}} + \underbrace{V}_{\substack{\text{Potential}\\ \text{Energy}}} = E$$

In Quantum Mechanics $\vec{p} \to -i\hbar\nabla$ and $E \to i\hbar\dfrac{\partial}{\partial t}$
i.e. operators acting on the wavefunction ψ.

$$\boxed{\left(-\frac{\hbar^2\nabla^2}{2m} + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}} \qquad (1.1)$$

Interpretation of ψ

$\rho \equiv \psi^*\psi$ is probability density

i.e. $\rho\, d^3r$ is probability particle is in volume d^3r.

Continuity Equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}\left(\psi^* \psi \right)$$

$$= \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}$$

$$= \psi^* \frac{1}{i\hbar}\left(\frac{-\hbar^2 \nabla^2}{2m} + V \right) \psi + \psi \frac{1}{-i\hbar}\left(\frac{-\hbar^2 \nabla^2}{2m} + V \right) \psi^*$$

using (1.1)

$$= \frac{-\hbar}{2mi} \left\{ \psi^* (\nabla^2 \psi) - \psi \nabla^2 \psi^* \right\}$$

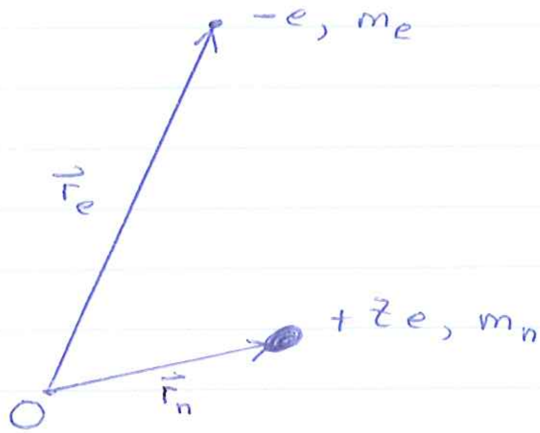
$$= \frac{-\hbar}{2mi} \nabla \cdot \left\{ \psi^* \nabla \psi - \psi \nabla \psi^* \right\}$$

$$\therefore \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0} \quad \text{Continuity Equation} \qquad (1.2)$$

where current density $\vec{J} \equiv \frac{\hbar}{2mi} \left\{ \psi^* \nabla \psi - \psi \nabla \psi^* \right\}$.

The physical significance of the continuity equation is that particles such as electrons are conserved.

Hydrogen Atom



$$V = \frac{-Ze^2}{|\vec{r}_e - \vec{r}_n|} \qquad \text{Coulomb Interaction}$$

Effects Ignored

1. Electron and Nuclear spins $\rightarrow$ magnetism
2. Finite size of nucleus
3. Relativistic Effects
4. Others

Exercise: Rewrite Schrodinger equation for H atom using:

$$\vec{r} \equiv \vec{r}_e - \vec{r}_n \quad , \quad \vec{R} \equiv \frac{m_e \vec{r}_e + m_n \vec{r}_n}{m_e + m_n} \quad \text{(Center of mass)}$$

$$M \equiv m_e + m_n \qquad \mu \equiv \frac{m_e m_n}{m_e + m_n} \quad \text{(Reduced mass)}$$

and set wavefunction $\Phi(\vec{r},\vec{R}) = \psi(\vec{r})\, e^{i\vec{K}\cdot\vec{R}}$ to obtain:

$$\left(\frac{-\hbar^2}{2\mu} \nabla_r^2 + V(r) \right) \psi(\vec{r}) = E\, \psi(\vec{r}) \qquad (1.3)$$

where E is the internal energy of the H atom.

Laplacian in Spherical Coordinates

$$\nabla_r^2 = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \underbrace{\left[\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan\theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right]}_{\equiv L^2 \text{ - square of orbital angular momentum}}$$

Solution of (1.3)

$$\psi = R_{n\ell}(r)\, Y_{\ell m}(\theta, \phi) \qquad (1.4)$$

spherical harmonic

$$R_{n\ell}(r) = A_{n\ell}\, \rho^{\ell} e^{-\rho/2} F_{n\ell}(\rho)$$

where $\rho \equiv \dfrac{2Zr}{a_0 n}$ and $a_0 \equiv \dfrac{\hbar^2}{\mu e^2}$ is Bohr radius.

$$F_{n\ell}(\rho) \equiv \sum_{i=0}^{n-\ell-1} \frac{(-1)^i \left[(n+\ell)!\right]^2 \rho^i}{i!\,(n-\ell-1-i)!\,(2\ell+1+i)!}$$

Associated Laguerre Polynomial

The normalization constant $A_{n\ell}$ is found using

$\int_0^\infty |R_{n\ell}|^2 r^2 dr = 1$ to be:

$$A_{n\ell} = \left(\frac{2Z}{a_0 n}\right)^{3/2} \sqrt{\frac{(n-\ell-1)!}{2n\left[(n+\ell)!\right]^3}}$$

Quantum Numbers

Principal Quant. # $n = 1, 2, 3, \ldots$

Orbital Angular Momentum $\ell = 0, 1, 2 \ldots (n-1)$

Azimuthal Quant. # $m = 0, \pm 1, \pm 2 \ldots \pm \ell$.

Possible Energies

$$\boxed{E_n = -\frac{E_R}{n^2}} \qquad (1.5)$$

where $E_R \equiv +\dfrac{\mu Z^2 e^4}{2\hbar^2}$ is called the Rydberg.

Exercise: Evaluate the wavelength of a photon emitted in a H $2p \to 1s$ transition.

Fine Structure

a) Spin Orbit Interaction

This is the interaction of electron spin (magnetic moment $\mu_B \equiv e\hbar/2m_e c$ or Bohr magneton) with the magnetic field generated by the orbiting nucleus.

$$H_{SO} = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\vec{L}\cdot\vec{S} \text{ where } V = -\frac{Ze^2}{r} \qquad (1.6)$$

Exercise: Derive (1.6)

The energy shift is found using perturbation theory to be:

$$E_{SO} = \langle \psi_{nl} | H_{SO} | \psi_{nl} \rangle$$

$$\frac{E_{SO}}{|E_n|} = \begin{cases} \dfrac{(Z\alpha)^2}{(2l+1)(l+1)n} & j = l + 1/2 \\[2ex] \dfrac{-(Z\alpha)^2}{l(2l+1)n} & j = l - 1/2 \\[2ex] 0 & j = 0 \end{cases} \qquad (1.7)$$

where j is electronic (orbital + spin) angular momentum and

$$\boxed{\alpha, \equiv \frac{e^2}{\hbar c}} \quad \text{Fine Structure Constant}$$

Exercise: a) Evaluate α
b) What is order of magnitude $\frac{E_{SO}}{|E_n|}$?

b) Relativistic Correction

The Kinetic energy is properly given by:

$$K.E. = \sqrt{p^2c^2 + m^2c^4} - mc^2$$

Exercise: Show relativistic correction to Hamiltonian is:

$$H_{rel} = -\frac{p^4}{8m^3c^2} \qquad (1.8)$$

The energy shift is:

$$\frac{E_{rel}}{|E_n|} = -\frac{(Z\alpha)^2}{4n^2}\left(\frac{8n}{2l+1} - 3\right) \quad \forall l \qquad (1.9)$$

c) Darwin Term

This is a relativistic effect introduced by the Dirac equation. It arises because the electron experiences a potential averaged over a region the size of a Compton wavelength $\hbar/mc$.

$$H_D = \frac{\hbar^2}{8m^2c^2} \nabla^2 V(r) \text{ where } V = \frac{-Ze^2}{r} \qquad (1.10)$$

or $$H_D = \frac{\pi Z e^2 \hbar^2}{2 m^2 c^2} \quad \delta(r)$$

delta function

The energy shift is:

$$\frac{E_D}{|E_n|} = \begin{cases} \dfrac{(Z\alpha)^2}{n} & l = 0 \\ 0 & l \neq 0 \end{cases} \qquad (1.11)$$

Exercise: Derive (1.11) from (1.10).

Fine Structure

Using (1.7), (1.9) + (1.11) we get total energy shift

$$\frac{E_F}{|E_n|} = \left(\frac{Z\alpha}{n}\right)^2 \left[\frac{3}{4} - \frac{n}{j + 1/2} \right] \qquad \begin{matrix} \forall\, l \\ j = l \pm 1/2 \end{matrix} \qquad (1.12)$$

Hence the fine structure splits the $2p_{1/2}$ and $2p_{3/2}$ states while the $2s_{1/2}$ and $2p_{1/2}$ states remain degenerate.

H Levels

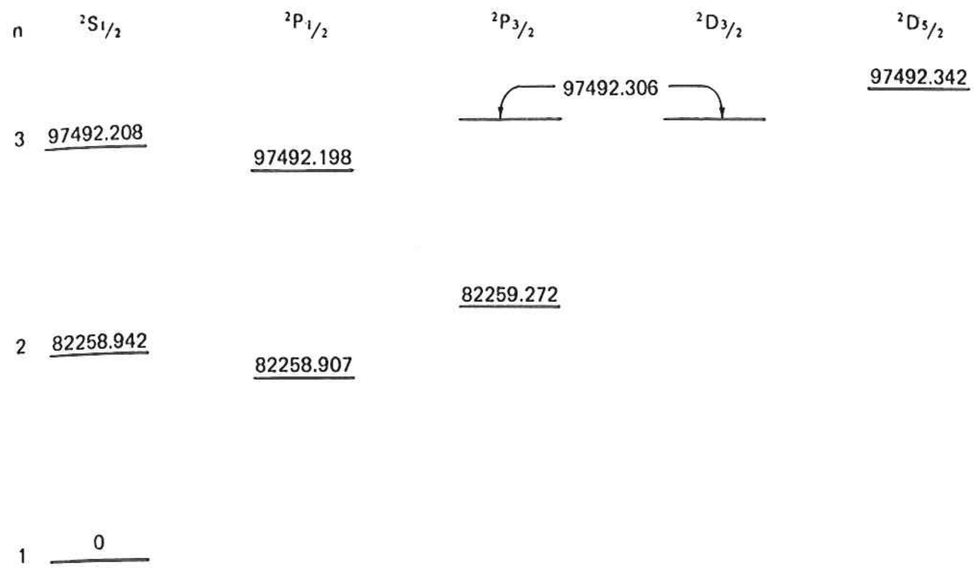


FIG. 16.5 Energy levels of the hydrogen atom for $n = 1, 2$, and 3. The energies are in reciprocal centimeters; the drawing is not to scale.

Hyperfine Interaction

This is the interaction of the nuclear spin $\vec{I}$ with the magnetic field generated by the electron.

$$H_{MD} = \frac{2\mu_B \mu_N g_I}{\hbar^2}\Bigg[\underset{\substack{\text{Orbital}\\ \text{Term}}}{\frac{\vec{I}\cdot\vec{L}}{r^3}} \underbrace{- \frac{\vec{I}\cdot\vec{S}}{r^3} + \frac{3(\vec{I}\cdot\vec{r})(\vec{S}\cdot\vec{r})}{r^5}}_{\substack{\text{Spin - spin}\\ \text{Term}}} + \underset{\substack{\text{Contact}\\ \text{Term}}}{\frac{8\pi}{3}\vec{I}\cdot\vec{S}\,\delta(\vec{r})} \Bigg] \tag{1.13}$$

$$\mu_N \equiv \frac{e\hbar}{2 m_{prot} c}$$ is nuclear magneton

g_I = nuclear g factor

OR. $$H_{MD} = a\,\vec{I}\cdot\vec{J}$$

where $$a = \frac{2\mu_B \mu_N g_I}{\hbar^2}\begin{cases} \dfrac{l(l+1)}{j(j+1)} \langle r^{-3} \rangle & l \neq 0 \\[2ex] \dfrac{8\pi}{3}\,\delta(\vec{r}) & l = 0 \end{cases} \tag{1.14}$$

Exercise: a) Using $\vec{F} = \vec{J} + \vec{I}$ find expressions for the H ground state levels $E_{F=1,0}$.

b) What is the wavelength for a transition between these two levels?

Chapter 1 Assignment

1) Plot the following energies in units of Hz on a log scale.

a) electron rest energy

b) energy separating H 2p and 1s states.

c) fine structure splitting of H $2p_{1/2}$ and $2p_{3/2}$ states

d) hyperfine splitting of H ground state

e) Lamb shift for H $2s_{1/2} - 2p_{1/2}$

f) Splitting of $m_J = \pm 1/2$ sublevels in magnetic field of 1 gauss

g) Splitting of $m_I = \pm 1/2$ sublevels in magnetic field of 1 gauss.

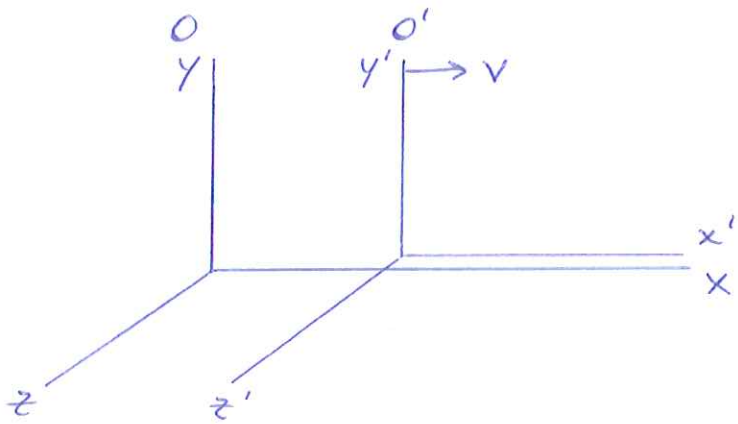
II) Relativity

The position - time coordinates of an event are represented by a so called four vector

$$x^\mu \equiv (x^0, x^1, x^2, x^3)$$

$$\equiv (ct, x, y, z)$$

Two observers O, and O' where O' has velocity v relative to O assign different 4 vectors x^μ and x'^μ to the same event.



x^μ and x'^μ are related by the Lorentz transformation.

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned} \qquad (2.1)$$

where $\beta \equiv \frac{v}{c}$ and $\gamma \equiv (1-\beta^2)^{-1/2}$.

This can be rewritten as:

$$x'^{\mu} = \sum_{\nu=0}^{3} a^{\mu}{}_{\nu}\, x^{\nu} \qquad (2.2)$$

where $a^{\mu}{}_{\nu}$ are elements of the matrix

$$a \equiv \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.3)$$

We now adopt the Einstein summation convention which automatically sums over repeated indices.

$$\therefore (2.2) \Longrightarrow \quad x'^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu} \qquad (2.4)$$

Lorentz Invariant

This is a quantity that is the same for observers O and O'.

eg. $I \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$

$$= (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 \qquad (2.5)$$

Exercise: Verify (2.5)

"I" can be rewritten as

$$I = g_{\mu\nu}\, x^{\mu} x^{\nu} \qquad (2.6)$$

where $g_{\mu\nu}$ are elements of the so called metric

$$g \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (2.7)$$

We now define the covariant 4 vector

$$x_{\mu} \equiv g_{\mu\nu}\, x^{\nu} \qquad (2.8)$$

Similarly $x^{\mu} = g^{\mu\nu} x_{\nu}$ where $g^{\mu\nu} \equiv g_{\mu\nu}$.

Exercise: Show $x_{\mu} = (ct, -x, -y, -z)$
i.e. sign of vector component is reversed

(2.6) then becomes:

$$I = x_{\mu} x^{\mu} \qquad (2.9)$$

Aside: The metric plays an important role in General Relativity where it is modified by gravity $\Rightarrow$ space time curvature.

Contravariant Vector

A contravariant 4 vector A^μ transforms under a Lorentz transformation according to:

$$A'^\nu = a^\nu{}_\mu A^\mu \qquad (2.10)$$

Examples

1) $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$ energy $E = \gamma mc^2$
momentum $\vec{p} = \gamma m \vec{v}$
m = rest mass

2) $J^\mu = (c\rho, \vec{J})$
current density
charge density

3) $A^\mu = (\Phi, \vec{A})$
vector potential
scalar potential

Exercise: Show $\partial^\mu \equiv \frac{\partial}{\partial x_\mu} \equiv \left(\frac{\partial}{\partial ct}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$ is a contravariant vector.

Scalar Product

The scalar product of two 4 vectors is defined by

$$A \cdot B \equiv A_\mu B^\mu \tag{2.11}$$

Exercise: Show $A \cdot B$ is Lorentz invariant.

Examples

1) $p^\mu p_\mu = m^2 c^2$

2) $\partial_\mu J^\mu = 0$ Continuity Equation

3) $\partial_\mu A^\mu = 0$ Lorentz Gauge

Exercise: Verify examples 1–3.

Tensor

A contravariant tensor of rank k has 4^k components that transform under a Lorentz transformation according to:

$$A'^{\alpha_1 \alpha_2 \cdots \alpha_k} = a^{\alpha_1}{}_{\beta_1} a^{\alpha_2}{}_{\beta_2} \cdots a^{\alpha_k}{}_{\beta_k} A^{\beta_1 \beta_2 \cdots \beta_k} \quad (2.12)$$

Hence a scalar (Lor. invariant) is a tensor of rank 0.
" " vector " " 1.

Mixed covariant and contravariant tensors can be constructed with the metric.

i.e. $S^{\mu}{}_{\nu} = g_{\nu\lambda} S^{\mu\lambda}$

Electromagnetic Field Tensor

$$F^{\mu\nu} \equiv \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} \quad (2.13)$$

Exercise: 1) Show $F^{\mu\nu}$ is antisymmetric.
i.e. $F^{\mu\nu} = -F^{\nu\mu}$

2) Evaluate $F^{\mu\nu}$.

3) Verify a) $\partial_{\mu} F^{\mu\nu} = \frac{4\pi}{c} J^{\nu}$

b) $\partial^{\alpha} F^{\mu\nu} + \partial^{\mu} F^{\nu\alpha} + \partial^{\nu} F^{\alpha\mu} = 0$

What is another name for equations a+b?

Example 1

A pion at rest decays into a muon and neutrino.

Before   After

$$\vec{p}_\mu = -\vec{p}_\nu$$

Find the muon momentum.

Solution

Energy Momentum Conservation $\Rightarrow$ $p_\pi = p_\mu + p_\nu$ (1)

particle not 4 vector subscript

where: $p_\pi = (m_\pi c, 0, 0, 0)$ $p_\pi \cdot p_\pi \equiv p_\pi^2 = m_\pi^2 c^2$

$$p_\mu = \left(\frac{E_\mu}{c}, \vec{p}_\mu \right) \qquad p_\mu^2 = m_\mu^2 c^2$$

$$p_\nu = \left(\frac{E_\nu}{c}, \vec{p}_\nu \right) \qquad p_\nu^2 = 0$$ since neutrino is massless.

$$(1) \Rightarrow \quad p_\mu = p_\pi - p_\nu$$

$$p_\mu^2 = p_\pi^2 + p_\nu^2 - 2 p_\pi \cdot p_\nu$$

$$m_\mu^2 c^2 = m_\pi^2 c^2 + 0 - 2 m_\pi E_\nu$$

$$E_\nu = \frac{(m_\pi^2 - m_\mu^2)}{2 m_\pi} c^2$$

But $E_\gamma = c|\vec{p}_\gamma| = c|\vec{p}_\mu|$

$$\therefore |\vec{p}_\mu| = \frac{(m_\pi^2 - m_\mu^2)\,c}{2m_\pi}$$

$m_{\pi^-} = 139.6$ MeV
$m_\mu = 105.7$ MeV.

Example 2

Antiprotons are produced by having a high energy proton strike a proton at rest.

$$p + p \rightarrow p + p + p + \bar{p}$$

What is the minimum energy of the proton for this to occur?

Lab Frame

Before

After

Center of Mass Frame

Before

After (Kinetic energy is 0 for minimum energy incident proton)

In lab frame, before collision

$$p^{\mu}_{TOT} = \left(\frac{E + mc^2}{c} , \ \vec{p} \right)$$

energy of incoming proton

energy of stationary target proton

In center of mass frame, after collision

$$p'^{\mu}_{TOT} = (4mc, 0, 0, 0)$$

Now $p^2 \equiv p_\mu p^\mu$ is invariant. Also energy momentum conservation implies p^μ is conserved.

$$\therefore \quad p^2_{TOT} = p'^2_{TOT}$$

$$\left(\frac{E}{c} + mc\right)^2 - |\vec{p}|^2 = (4mc)^2$$

Using $E^2 = |\vec{p}|^2 c^2 + (mc^2)^2$ we obtain:

$$E = 7mc^2$$

Proton Rest Energy $mc^2 \sim 1$ GeV.

Hence incident proton must have minimum kinetic energy of 6 GeV.

Chapter 2 Assignment

1) Explicitly write components of x_μ and x^μ, and show $x_\mu x^\mu$ is lorentz invariant.

2) Show ∂^μ is a contravariant vector.

3) Electromagnetic Field Tensor

a) Show $F^{\mu\nu}$ is antisymmetric.

b) Evaluate all components of $F^{\mu\nu}$.

c) Show i) $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$

ii) $\partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0$

III) Relativistic Quantum Mechanics

Covariance of Laws of Nature

This refers to the assumption made by special relativity that the laws of physics have the same "form" in different inertial reference frames.

Example

The free particle Schrodinger equation for observer O is

$$-\frac{\hbar^2 \nabla^2}{2m} \psi = i\hbar \frac{\partial \psi}{\partial t} \qquad (3.1)$$

The same form of (3.1) for moving observer O' would be

$$-\frac{\hbar^2 \nabla'^2}{2m} \psi' = i\hbar \frac{\partial \psi'}{\partial t'} \qquad (3.2)$$

where the coordinates are related by a Lorentz transformation

$$x'^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu} \qquad (3.3)$$

Exercise: Show (3.1), (3.2) + (3.3) are mutually inconsistent.

Hence, in order for (3.1) and (3.3) to simultaneously hold, O and O' would experience different physics. Physics would be an impossible subject since the laws of nature would be custom made.

Relativistic Generalization of Quantum Mechanics

To generalize Quantum Mechanics, we begin with the relativistic result

$$E = \sqrt{p^2c^2 + m^2c^4} \qquad (3.4)$$

Replacing E and $\vec{p}$ by the usual operators yields

$$i\hbar \frac{\partial \psi}{\partial t} = \sqrt{-\hbar^2c^2\nabla^2 + m^2c^4}\;\psi \qquad (3.5)$$

The square root of a Laplacian operator is defined by the series expansion

$$i\hbar \frac{\partial \psi}{\partial t} = \left(mc^2 - \frac{\hbar^2\nabla^2}{2m} - \frac{\hbar^4\nabla^4}{8m^3c^2} + \cdots \right)\psi \qquad (3.6)$$

To solve (3.6) for ψ at one space-time spot, an infinite number of derivatives must be evaluated which requires that ψ be known over all space. (3.6) is therefore said to be <u>nonlocal</u>.

A Second Attempt

The complications of the square root in (3.4) can be avoided by squaring.

$$E^2 = p^2c^2 + m^2c^4 \qquad (3.7)$$

Replacing E and $\vec{p}$ by operators yields

$$\boxed{\left(\hbar^2 \frac{\partial^2}{\partial t^2} - \hbar^2 c^2 \nabla^2 + m^2c^4\right)\psi(\vec{x},t) = 0} \qquad (3.8)$$

Klein-Gordon Equation

Defining the so-called D'Alembertian $\Box^2 \equiv \frac{\partial^2}{c^2\partial t^2} - \nabla^2$

(3.8) becomes:

$$\left(\hbar^2\Box^2 + m^2c^2\right)\psi(\vec{x},t) = 0 \qquad (3.9)$$

Exercise: Show $\Box^2$ is a Lorentz scalar.
Hint: Show $\Box^2 = \partial_\mu \partial^\mu$.

Hence (3.9) has the same form in frames O and O' if

$$\psi'(\vec{x}',t') = \psi(\vec{x},t)$$

i.e. the wavefunction transforms as a scalar.

"Problems" with Klein-Gordon

1. To find $\psi(\vec{x},t)$ $\forall t$, we need to know two initial conditions $\psi(\vec{x},0)$ and $\frac{d\psi}{dt}(\vec{x},0)$ since (3.8) contains $\partial^2/\partial t^2$. This is counterintuitive.

2. Consider the plane wave solution of (3.8) i.e. $\psi = \exp\left[i(\vec{p}\cdot\vec{r} - Et)/\hbar\right]$.

$$(3.8) \Longrightarrow \quad E = \pm\sqrt{p^2c^2 + m^2c^4}$$

What do negative energies mean?

3. We wish to define a probability and current densities. Proceeding as with Schrodinger's equation, (3.8) is multiplied on left by ψ^* and we subtract $(3.8)^*$ multiplied on left by ψ.

$$\psi^*\left(\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} - \hbar^2\nabla^2 + m^2c^4\right)\psi - \psi\left(\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} - \hbar^2\nabla^2 + m^2c^4\right)\psi^* = 0 \qquad (3.10)$$

This yields the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0 \qquad (3.11)$$

where
$$\rho \equiv \frac{i\hbar}{2mc^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) \qquad (3.12)$$

$$\vec{J} \equiv \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \qquad (3.13)$$

Exercise: Verify that (3.11) follows from (3.10).

Exercise: Show that for a free particle in the nonrelativistic limit ($|\vec{p}| << m$) that $\rho \to \psi^* \psi$.

Note that (3.12) can be positive or negative. What does a negative probability density mean?

4. The particle spin does not appear in (3.8). Hence the Klein-Gordon equation is only applicable to spin zero particles.

Chapter 3 Assignment

1) Show that the one dimensional Schrodinger equation $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t}$ is not covariant.

2) The Klein Gordon equation can be used to describe exotic atoms where an electron is replaced by a spinless π^- meson. ($m_{\pi^-} = 280\, m_{elect}$) One can show the eigenenergies to be given by:

$$E_{n\ell} = \frac{mc^2}{\left\{1 + (Z\alpha)^2 / \left[n - \ell - \frac{1}{2} + \sqrt{(\ell + 1/2)^2 - (Z\alpha)^2}\,\right]^2\right\}^{1/2}}$$

a) Expand E in terms of α^2 up to and including α^4 term.

b) The $3d \rightarrow 2p$ transition for π^- in ^{59}Co has been measured to be 384.6 ± 1.0 keV. What does theory predict.

IV) Dirac Equation

We shall search for an equation to describe a free electron of mass m that is first order in time.

i.e. $$i\hbar \frac{\partial \Psi(\vec{x},t)}{\partial t} = H_0 \Psi(\vec{x},t) \qquad (4.1)$$

We require that the equation have a covariant form. Under a Lorentz transformation $\partial/\partial t$ becomes a linear combination of space & time derivatives.

$$\therefore \quad H_0 = -i\hbar c\left(\alpha_1 \frac{\partial}{\partial x^1} + \alpha_2 \frac{\partial}{\partial x^2} + \alpha_3 \frac{\partial}{\partial x^3}\right) + \beta mc^2 \qquad (4.2)$$

where $\vec{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3)$ and β are dimensionless constants.

$$H_0 = c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2 \qquad (4.3)$$

We shall also require that $E^2 = p^2c^2 + m^2c^4$ hold for the free particle. i.e. Ψ is a solution of the Klein-Gordon equation. Taking the time derivative of (4.1) we obtain:

$$i\hbar \frac{\partial^2 \Psi}{\partial t^2} = (c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2)\frac{\partial \Psi}{\partial t}$$

$$= \left(c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2\right)\frac{1}{i\hbar}\left(c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2\right)\Psi$$

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} = \left[-i\hbar c \sum_j \alpha_j \frac{\partial}{\partial x^j} + \beta mc^2\right]\left[-i\hbar c \sum_k \alpha_k \frac{\partial}{\partial x^k} + \beta mc^2\right]\Psi$$

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} = -\hbar^2 c^2 \sum_{kj} \frac{\alpha_k \alpha_j + \alpha_j \alpha_k}{2} \frac{\partial^2 \Psi}{\partial x^j \partial x^k}$$

$$-imc^3\hbar \sum_k (\alpha_k \beta + \beta \alpha_k) \frac{\partial \Psi}{\partial x^k} + \beta^2 m^2 c^4 \Psi \qquad (4.4)$$

For the right side of (4.4) to be equivalent to the Klein-Gordon equation, we require:

$$\{\alpha_k, \alpha_j\} \equiv \alpha_k \alpha_j + \alpha_j \alpha_k = 2\delta_{kj} \qquad (4.5a)$$

$$\{\alpha_k, \beta\} \equiv \alpha_k \beta + \beta \alpha_k = 0 \qquad (4.5b)$$

$$\beta^2 = 1 \qquad (4.5c)$$

Exercise: Show (4.5) cannot be satisfied if α_k and β are numbers.

$\therefore$ β and α_k and hence Ψ must be matrices.

Properties of α_k and β

Assuming the Hamiltonian H_0 to be hermitian implies:

$$\alpha_k^+ = \alpha_k \qquad (4.6a)$$

$$\beta^+ = \beta \qquad (4.6b)$$

where $^+$ means the conjugate transpose.

Results

a) Eigenvalues of α_k and β are ± 1.

b) $Tr\, \alpha_k = Tr\, \beta = 0$

c) Dimension N of matrices is even.

d) $N \geq 4$.

Proofs.

a) See text on matrix algebra.

b) $\alpha_k \beta = -\beta \alpha_k$ using (4.5b)

$\alpha_k \beta^2 = -\beta \alpha_k \beta$

$\alpha_k = -\beta \alpha_k \beta$ using (4.5c)

$Tr\, \alpha_k = -Tr(\beta \alpha_k \beta)$

$= -Tr(\alpha_k \beta^2)$ using property of Trace

$= -Tr\, \alpha_k$ using (4.5c)

$\therefore Tr\, \alpha_k = 0$

Exercise: Prove $Tr\, \beta = 0$.

c) $$\alpha_k \beta = -\beta \alpha_k \qquad \text{using (4.5b)}$$

$$\det(\alpha_k \beta) = \det(-\beta \alpha_k)$$

$$= \det(-\beta)\det(\alpha_k) \qquad \text{determinant property}$$

$$= (-1)^N \det(\beta)\det(\alpha_k)$$

$$= (-1)^N \det(\alpha_k \beta)$$

$\therefore N$ is even

d) We already showed $N \neq 0$. Now we consider $N = 2$. A complete set of 2×2 matrices is $\{\vec{\sigma}, I\}$. where $\vec{\sigma}$ are the Pauli matrices and I is the identity matrix. However this set does not satisfy (4.5). (Nor does any other set of 2×2 matrices!)

$$\therefore N \geq 4.$$

Standard Representation

The simplest case $N = 4$ is considered. The choice for $\vec{\alpha}$ and β is not unique. The "standard" choice is:

$$\alpha_i \equiv \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\beta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (4.7)$$

where 1 is the 2×2 identity matrix and σ_i are the Pauli spin matrices.

$$\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 \equiv i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore \alpha_2 \equiv \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

The wavefunction $$\Psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \qquad (4.8)$$

Solution of Dirac Equation For Free Particle at Rest

The Dirac equation (4.1) with (4.3) becomes:

$$i\hbar \frac{\partial \Psi}{\partial t} = \beta mc^2 \Psi$$

Using (4.8) $\Rightarrow$ $$i\hbar \begin{pmatrix} \partial\psi_1/\partial t \\ \partial\psi_2/\partial t \\ \partial\psi_3/\partial t \\ \partial\psi_4/\partial t \end{pmatrix} = mc^2 \begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_3 \\ -\psi_4 \end{pmatrix}$$

This yields the following four independent solutions.

$$\Psi_1 = e^{-imc^2t/\hbar}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \Psi_2 = e^{-imc^2t/\hbar}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\Psi_3 = e^{imc^2t/\hbar}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad \Psi_4 = e^{imc^2t/\hbar}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \qquad (4.9)$$

Ψ_1 + Ψ_2 correspond to electrons with energy mc^2. There are two types which we identify with spin up Ψ_1 and spin down Ψ_2.

Ψ_3 + Ψ_4 correspond to "electrons" with energy $-mc^2$. This must be the electron antiparticle called the positron.

$T_{4\times4}$

This is a basis of 16 independent 4×4 matrices listed below.

$T_{4\times4}$	Definition	Number
$\vec{\gamma}$	$\beta\vec{\alpha}$	3
γ_0	β	1
I	Identity	1
$\sigma^{\mu\nu}$	$\frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \equiv \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$	6
γ_5	$i\gamma^0\gamma^1\gamma^2\gamma^3$	1
$\gamma_5\gamma^\mu$		4

(4.10)

OR

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(4.11)

$$\sigma^{ij} = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \sigma^{0i} = i\alpha_i = i\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

where i, j, k are cyclic permutation of 1, 2 + 3

Exercise: a) Show $(\sigma^{0k})^2 = -I$ (4.12)

b) Show $\{\gamma^0, \sigma^{0k}\} = 0$ (4.13)

c) Show $\{\gamma^k, \sigma^{0k}\} = 0$ (4.14)

Expressions For Dirac Equation

Using (4.1) and (4.3) we have:

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(c\vec{\alpha}\cdot\vec{p} + \beta mc^2\right)\Psi$$

Multiplying by $\gamma_0 = \beta$ and dividing by c gives:

$$i\hbar\, \gamma_0 \frac{\partial \Psi}{\partial ct} = (\vec{\gamma}\cdot\vec{p} + mc)\Psi$$

Defining $p_0 \equiv i\hbar \frac{\partial}{\partial x^0}$ we get:

$$(\gamma^\mu p_\mu - mc)\Psi = 0 \qquad (4.15)$$

OR $(\not{p} - mc)\Psi = 0$ where $\not{p} \equiv \gamma^\mu p_\mu$ (4.16)

Covariance of Dirac Equation

Observers O and O' write the following Dirac eqns.

$$i\hbar\, \gamma^\mu \frac{\partial \Psi(x)}{\partial x^\mu} = mc\Psi(x) \tag{4.17a}$$

$$i\hbar\, \gamma'^\mu \frac{\partial \Psi'(x')}{\partial x'^\mu} = mc\Psi'(x') \tag{4.17b}$$

where $x'^\nu = a^\nu{}_\mu x^\mu$ is the Lorentz transformation.

If γ^μ were a 4-vector, then ψ would be invariant and covariance requires $\psi'(x') = \psi(x)$. However, the conventional choice is $\gamma^\mu = \gamma'^\mu$ which in turn complicates the transformation of the wavefunction under a Lorentz transformation. We shall find L such that

$$\psi'(x') = L\, \psi(x) \tag{4.18}$$

The inverse of (4.18) is:

$$\psi(x) = L^{-1}\, \psi'(x') \tag{4.19}$$

Exercise: Show $\dfrac{\partial}{\partial x^\mu} = a^\nu{}_\mu \dfrac{\partial}{\partial x'^\nu}$ (4.20)

Using (4.19) + (4.20), (4.17a) becomes:

$$i\hbar \gamma^{\mu} a^{\nu}{}_{\mu} \frac{\partial}{\partial x'^{\nu}} L^{-1} \psi'(x') = mc L^{-1} \psi'(x')$$

Multiplying by L we obtain:

$$i\hbar a^{\nu}{}_{\mu} \; L \gamma^{\mu} L^{-1} \frac{\partial \psi'(x')}{\partial x'^{\nu}} = mc\psi'(x') \qquad (4.21)$$

Comparing (4.21) with (4.17b) we get:

$$a^{\nu}{}_{\mu} \; L \gamma^{\mu} L^{-1} = \gamma^{\nu}$$

OR $$a^{\nu}{}_{\mu} \gamma^{\mu} = L^{-1} \gamma^{\nu} L \qquad (4.22)$$

i.e. We need to find a matrix operator L that satisfies (4.22).

<u>Solution</u>

For simplicity we consider observer O' moving in the x direction.

$$\therefore \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh\lambda & -\sinh\lambda & 0 & 0 \\ -\sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad (4.23)$$

where $\cosh\lambda \equiv \gamma = (1-\beta^2)^{-1/2}$

<u>Exercise</u>: Show (4.23) is equivalent to $x'^{\nu} = a^{\nu}{}_{\mu} x^{\mu}$.

We shall now show that $L = \exp\left\{ i \frac{\lambda}{2} \sigma^{01} \right\}$ (4.24)

Exercise: Show $L = \cosh\frac{\lambda}{2} + i\sigma^{01} \sinh\frac{\lambda}{2}$ (4.25)

Verification of (4.22) with $\nu = 0$

$$L^{-1}\gamma^0 L = e^{-i\frac{\lambda}{2}\sigma^{01}} \gamma^0 e^{i\frac{\lambda}{2}\sigma^{01}}$$

$$= e^{-i\frac{\lambda}{2}\sigma^{01}} \gamma^0 \left(\cosh\frac{\lambda}{2} + i\sigma^{01} \sinh\frac{\lambda}{2} \right) \text{ using (4.25)}$$

$$= e^{-i\frac{\lambda}{2}\sigma^{01}} \left(\cosh\frac{\lambda}{2} - i\sigma^{01} \sinh\frac{\lambda}{2} \right) \gamma^0 \text{ using (4.13)}$$

$$= e^{-i\lambda\sigma^{01}} \gamma^0 \quad \text{using (4.24) + (4.25)}$$

$$= \left(\cosh\lambda - i\sigma^{01} \sinh\lambda \right) \gamma^0$$

$$= \cosh\lambda \, \gamma^0 - \sinh\lambda \, \gamma^1 \quad \text{using } \sigma^{01}\gamma^0 = -i\gamma^1$$

$$= a^0{}_0 \gamma^0 + a^0{}_1 \gamma^1$$

$$\therefore L^{-1}\gamma^0 L = a^0{}_\mu \gamma^\mu$$

Exercise: Show (4.22) holds for $\nu = 1$.

Generalization

For an observer O' moving with velocity $v\hat{n}$ relative to O:

$$L = \exp\left\{ i \frac{\lambda}{2} n_k \sigma^{0k} \right\} \tag{4.26}$$

Exercise: a) Show $L^{\dagger} = L$ (4.27)

b) Show $L^{-1} = \gamma_0 L^{\dagger} \gamma_0$ (4.28)

Probability and Current

We shall consider $\Psi^\dagger (4.1) - (4.1)^\dagger \Psi$.

$$\Psi^\dagger\left(i\hbar\frac{\partial\Psi}{\partial t}\right) - \left(-i\hbar\frac{\partial\Psi^\dagger}{\partial t}\right)\Psi = \Psi^\dagger\left(-i\hbar c\,\vec{\alpha}\cdot\nabla + \beta mc\right)\Psi$$
$$- \left[\left(-i\hbar c\,\vec{\alpha}\cdot\nabla + \beta mc\right)\Psi\right]^\dagger\Psi$$

$$i\hbar\frac{\partial(\Psi^\dagger\Psi)}{\partial t} = -i\hbar\,\Psi^\dagger c\,\vec{\alpha}\cdot(\nabla\Psi) + mc\,\Psi^\dagger\beta\Psi$$
$$-i\hbar(\nabla\Psi^\dagger)\cdot c\,\vec{\alpha}^\dagger\Psi - mc\,\Psi^\dagger\beta^\dagger\Psi$$

Using $\vec{\alpha}^\dagger = \vec{\alpha}$, $\beta^\dagger = \beta$ this becomes:

$$\frac{\partial(\Psi^\dagger\Psi)}{\partial t} = -\nabla\cdot\left(\Psi^\dagger c\,\vec{\alpha}\,\Psi\right) \qquad (4.29)$$

This is a continuity equation having:

probability density $\rho \equiv \Psi^\dagger\Psi$ (4.30)

current density $\vec{J} \equiv \Psi^\dagger c\,\vec{\alpha}\,\Psi$ (4.31)

Exercise: a) Show $\rho = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \geq 0$

b) Write $\vec{J}$ in terms of components of Ψ.

Dirac Adjoint

This is defined as $\bar{\Psi} \equiv \Psi^{\dagger}\beta$ (4.32)

Exercise: a) Show $\rho = \bar{\Psi}\, \gamma_0\, \Psi$ (4.33)

b) Show $\vec{J} = c\, \bar{\Psi}\, \vec{\gamma}\, \Psi$ (4.34)

Hence
$$J^{\mu} \equiv (c\rho, \vec{J}) = c\, \bar{\Psi}\, \gamma^{\mu}\, \Psi \quad (4.35)$$

Lorentz Transformation of J^{μ}

$$J'^{\mu}(x') = c\, \bar{\Psi}'(x')\, \gamma^{\mu}\, \Psi'(x')$$

$$= c\Psi'(x')^{\dagger}\, \gamma^{0}\, \gamma^{\mu}\, \Psi'(x') \qquad \text{using (4.32)}$$

$$= c\, \Psi^{\dagger}(x)\, L^{\dagger}\, \gamma^{0}\, \gamma^{\mu}\, L\, \Psi(x) \qquad \text{using (4.18)}$$

$$= c\Psi^{\dagger}(x)\, \gamma^{0}(\gamma^{0} L^{\dagger} \gamma^{0})\gamma^{\mu} L\, \Psi(x) \qquad \text{using } (\gamma^{0})^{2} = I$$

$$= c\Psi^{\dagger}(x)\, \gamma^{0}(L^{-1}\, \gamma^{\mu} L)\Psi(x) \qquad \text{using (4.28)}$$

$$= c\, \Psi^{\dagger}(x)\, \gamma^{0}\, a^{\mu}{}_{\nu}\, \gamma^{\nu}\, \Psi(x) \qquad \text{using (4.22)}$$

$$= a^{\mu}{}_{\nu}\, c\, \bar{\Psi}(x)\, \gamma^{\nu}\, \Psi(x) \qquad \text{using (4.32)}$$

$$\therefore\ J'^{\mu}(x') = a^{\mu}{}_{\nu}\, J^{\nu}(x).$$

Hence J^{μ} is a 4 vector.

Exercise: Show the following.

a) $\bar{\Psi}' \Psi' = \bar{\Psi} \Psi$ (4.36a)

b) $\bar{\Psi}' \gamma_5 \Psi' = \bar{\Psi} \gamma_5 \Psi$ (4.36b)

c) $\bar{\Psi}' \gamma_5 \gamma^\mu \Psi' = a^\mu{}_\nu \bar{\Psi} \gamma_5 \gamma^\nu \Psi$ (4.36c)

d) $\bar{\Psi}' \sigma^{\mu\nu} \Psi' = a^\mu{}_\alpha a^\nu{}_\beta \bar{\Psi} \sigma^{\alpha\beta} \Psi$ (4.36d)

Parity Transformation

The parity operation inverts the sign of the space coordinates while leaving the time component unaffected.

$$\text{i.e. } x'^{\nu} = b^{\nu}{}_{\mu} x^{\mu} \tag{4.37}$$

where
$$b^{\nu}{}_{\mu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We shall assume that the Dirac equation has the same form for observer O and parity transformed observer O':

Exercise: Following the steps in the Lorentz transformation, show if $\Psi'(x') = P\,\Psi(x)$ that:

$$P^{-1}\gamma^{\nu}P = b^{\nu}{}_{\mu}\gamma^{\mu} \tag{4.38}$$

We now shall show that $P = \gamma^0$ satisfies (4.38).

Exercise: Show $P^{-1} = \gamma^0$.

Case 1: $\nu = 0$ Trivial

Case 2: $\nu = k$. $P^{-1}\gamma^k P = \gamma^0\gamma^k\gamma^0$
$= -\gamma^k$ using (4.11)

$$\therefore P^{-1}\gamma^k P = b^{k}{}_{\mu}\gamma^{\mu}$$

Effect on Ψ

$$P\,\Psi = \gamma^0 \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_3 \\ -\psi_4 \end{pmatrix}$$

Hence particles and antiparticles have opposite parity.

Parity Transformation of $T_{4\times 4}$

Exercise: Show $P^{-1}\gamma^5 P = -\gamma^5$

i.e. γ^5 transforms as a scalar under Lorentz trans. but not as a scalar under parity.
$\therefore \gamma^5$ is called a pseudoscalar.

Exercise: Show $P^{-1}\gamma^5\vec{\gamma}\,P = \gamma^5\vec{\gamma}$

i.e. $\gamma^5\vec{\gamma}$ transforms as a vector under Lorentz trans. but as a scalar under parity.
$\therefore \gamma^5\vec{\gamma}$ is called a pseudovector.

Summary of Transformation Properties (Parity + Lorentz)

$T_{4\times 4}$	
$\vec{\gamma}$	spacelike vector
γ_0	timelike vector
I	scalar
$\sigma^{\mu\nu}$	tensor
γ_5	pseudoscalar
$\gamma_5\gamma^\mu$	pseudovector

Plane Wave Solutions of Dirac Equation For A Free Particle

Particle At Rest

Previously, we solved the Dirac equation

$$\left(i\hbar\gamma^\mu\partial_\mu - mc\right)\Psi = 0 \qquad (4.39)$$

for a free particle at rest. There were two independent solutions with energy mc^2;

$$\Psi_1 = e^{-imc^2t/\hbar}\,u_1(mc^2,0) \qquad (4.40a)$$

$$\Psi_2 = e^{-imc^2t/\hbar}\,u_2(mc^2,0) \qquad (4.40b)$$

and two solutions with energy $-mc^2$.

$$\Psi_3 = e^{imc^2t/\hbar}\,u_3(-mc^2,0) \qquad (4.40c)$$

$$\Psi_4 = e^{imc^2t/\hbar}\,u_4(-mc^2,0) \qquad (4.40d)$$

The spinors are defined as follows.

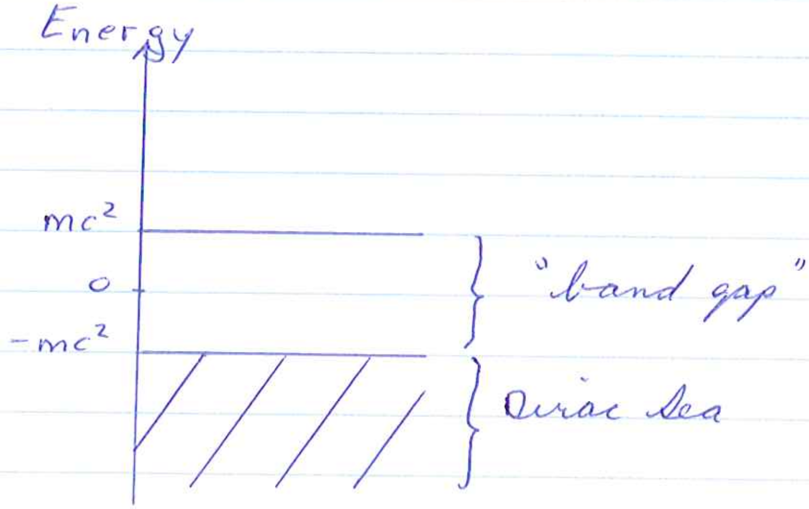
$$u_1(mc^2,0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad u_2(mc^2,0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad u_3(-mc^2,0)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad u_4(-mc^2,0)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$(4.41)$$

Interpretation.

The question arises why particles in states with positive energy don't decay to negative energy states. Dirac postulated that all negative energy states from $-\infty$ to $-mc^2$ are filled by a "sea of negative energy electrons." Hence, by the Pauli Exclusion Principle, electrons occupying positive energy states can't decay.

Exercise: What is the Pauli Exclusion Principle?



A negative energy electron can be freed by absorbing a photon having energy $2mc^2$. This creates a hole in the Dirac sea which acts as a positive charge $+e$ and has momentum opposite to the freed electron.

Exercise: Read chapter in Kittel's Solid State Physics about semiconductors. Note the close analogy with the material here.

The process whereby a photon produces an $e^- e^+$ pair is denoted by:

$$2\gamma \rightarrow e^- + e^+$$

Exercise: What is the order of magnitude of the photon wavelength? What is such a photon called?

The inverse process is called pair annihilation:

i.e. $e^- + e^+ \rightarrow 2\gamma$

Exercise: Why must at least two photons be produced in pair annihilation?

Dirac's prediction of the existence of positively charged electrons or positrons was confirmed in 1933 by Anderson using a cosmic ray cloud chamber.

Solution For Moving Particles

We consider the plane wave solution

$$\Psi(\vec{r},t) = e^{-i(Et-\vec{p}\cdot\vec{r})/\hbar}\, u(E,\vec{p}) \tag{4.42}$$

which can be written using 4 vector notation, giving

$$\Psi(r) = e^{-ix^{\mu}p_{\mu}/\hbar}\, u(p) \tag{4.43}$$

Substituting (4.43) into (4.39) gives:

$$(\gamma^{\mu}p_{\mu} - mc)\, u = 0 \tag{4.44}$$

This is called the momentum space Dirac equation. Note that p_{μ} is now a number rather than an operator.

Now $\gamma^{\mu}p_{\mu} = \gamma^{0}p^{0} - \vec{\gamma}\cdot\vec{p}$

$$= \frac{E}{c}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \vec{p}\cdot\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} E/c & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E/c \end{pmatrix} \tag{4.45}$$

Let $u \equiv \begin{pmatrix} \psi_u \\ \psi_L \end{pmatrix}$ where the upper ψ_u and lower ψ_L components are 2×1 spinors.

Equation (4.44) then becomes:

$$\begin{pmatrix} \frac{E}{c} - mc & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -\frac{E}{c} - mc \end{pmatrix} \begin{pmatrix} \psi_u \\ \psi_L \end{pmatrix} = 0$$

OR $$\left(\frac{E}{c} - mc\right)\psi_u - \vec{p}\cdot\vec{\sigma}\,\psi_L = 0$$

$$\vec{p}\cdot\vec{\sigma}\,\psi_u - \left(\frac{E}{c} + mc\right)\psi_L = 0$$

OR $$\psi_u = \frac{c}{E - mc^2}\,\vec{p}\cdot\vec{\sigma}\,\psi_L \qquad (4.46a)$$

$$\psi_L = \frac{c}{E + mc^2}\,\vec{p}\cdot\vec{\sigma}\,\psi_u \qquad (4.46b)$$

Substituting (4.46b) into (4.46a) gives:

$$\psi_u = \frac{c^2}{E^2 - m^2c^4}\,(\vec{p}\cdot\vec{\sigma})^2\,\psi_u$$

Exercise: Show $(\vec{p}\cdot\vec{\sigma})^2 = \vec{p}^{\,2}$

$$\therefore\ E^2 - m^2c^4 = \vec{p}^{\,2}c^2$$

$$E = \pm\sqrt{m^2c^4 + \vec{p}^{\,2}c^2} \qquad (4.47)$$

Two independent positive energy solutions are found as follows. ($E = +\sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$)

$$\psi_u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xRightarrow{(4.46b)} \psi_L = \frac{c}{E+mc^2}\begin{pmatrix} p_z \\ p_+ \end{pmatrix} \qquad (4.48a)$$

$$\psi_u = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xRightarrow{(4.46b)} \psi_L = \frac{c}{E+mc^2}\begin{pmatrix} p_- \\ -p_z \end{pmatrix} \qquad (4.48b)$$

Similarly two independent negative energy solutions are found as follows. ($E = -\sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$)

$$\psi_L = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xRightarrow{(4.46a)} \psi_u = \frac{c}{E-mc^2}\begin{pmatrix} p_z \\ p_+ \end{pmatrix} \qquad (4.48c)$$

$$\psi_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xRightarrow{(4.46a)} \psi_u = \frac{c}{E-mc^2}\begin{pmatrix} p_- \\ -p_z \end{pmatrix} \qquad (4.48d)$$

where $p_\pm \equiv p_x \pm i p_y$. (4.49)

Choosing the normalization condition $u^\dagger u = 2|E|/c$, the solutions become:

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{cp_+}{E+mc^2} \end{pmatrix} \qquad u_2 = N\begin{pmatrix} 0 \\ 1 \\ \frac{cp_-}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix} \qquad (4.50a)$$

where $E = \sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$

$$u_3 = N \begin{pmatrix} \frac{cp_z}{E-mc^2} \\ \frac{cp_+}{E-mc^2} \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = N \begin{pmatrix} \frac{cp_-}{E-mc^2} \\ \frac{-cp_z}{E-mc^2} \\ 0 \\ 1 \end{pmatrix} \tag{4.50b}$$

where $E = -\sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$ and $N \equiv \sqrt{\frac{|E|}{c} + mc}$.

Spin

The spin operator $\vec{S} \equiv \frac{\hbar}{2}\vec{\Sigma}$ where $\vec{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ (4.51)

Exercise: If $\vec{p} \parallel \hat{z}$, show u_{1-4} are eigenspinors of S_z with u_1 & u_3 having spin up and u_2 & u_4 having spin down.

Positron States

We define $v_1(E, \vec{p}) \equiv u_4(-E, -\vec{p})$ (4.52a)

$$= N \begin{pmatrix} \frac{cp_-}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

and $v_2(E,\vec{p}) \equiv -u_3(-E,-\vec{p})$ (4.52b)

$$= -N \begin{pmatrix} \frac{cp_z}{E+mc^2} \\ \frac{cp_+}{E+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

Note that v_1 & v_2 have a positive energy. We interpret v_1 as a positron having spin down and v_2 as a positron with spin up.

Exercise: Show v_1 & v_2 satisfy

$$\left(\gamma^\mu p_\mu + mc\right) v = 0 \qquad (4.53)$$

Feynman-Stueckelberg Picture

(4.53) can be viewed as a Dirac equation where where the sign of space-time is reversed. Hence antiparticles can be viewed as particles travelling backwards in space-time.

Completeness Relation

$$\sum_{s=1,2} \left[u_s(p)\,\bar{u}_s(p) - v_s(p)\,\bar{v}_s(p)\right] = 2mc\,\underset{\sim}{1} \qquad (4.54)$$

where $\bar{u} \equiv u^+ \gamma^0$ is Dirac adjoint and $\underset{\sim}{1}$ is unit matrix.
Exercise: Show (4.54) is true.

Projection Operators

$$\Lambda_{\pm} \equiv \frac{\pm \not{p} + mc}{2mc} \tag{4.55}$$

Properties

1) $\Lambda_+ \, u_s(p) = u_s(p) \qquad \Lambda_- \, u_s(p) = 0 \qquad (4.56)$

$\Lambda_+ \, v_s(p) = 0 \qquad \Lambda_- \, v_s(p) = v_s(p)$

i.e. Λ_+ projects out particles.
Λ_- " " antiparticles.

2) $\Lambda_{\pm}^2 = \Lambda_{\pm} \qquad (4.57)$

3) $\Lambda_+ \Lambda_- = \Lambda_- \Lambda_+ = 0 \qquad (4.58)$

4) $\Lambda_+ + \Lambda_- = 1 \qquad (4.59)$

Proofs.

1) $$\Lambda_+ u_s(p) = \frac{\not{p} + mc}{2mc} u_s(p)$$

$$= \frac{\not{p} \, u_s(p) + mc u_s(p)}{2mc}$$

$$= \frac{mc \, u_s(p) + mc \, u_s(p)}{2mc} \quad \text{using } (4.44)$$

$$= u_s(p)$$

$$\Lambda_+ v_s(p) = \frac{\not{p} + mc}{2mc} v_s(p)$$

$$= \frac{\not{p} v_s(p) + mc\, v_s(p)}{2mc}$$

$$= \frac{-mc\, v_s(p) + mc\, v_s(p)}{2mc} \quad \text{using (4.53)}$$

$$= 0$$

Λ_- properties follow similarly.

2) $\Lambda_+^2 u_s(p) = \Lambda_+ u_s(p) = u_s(p)$ using (4.56)

$\Lambda_+^2 v_s(p) = \Lambda_+ 0 = 0$ " "

$\therefore \Lambda_+^2 = \Lambda_+$

Exercise: Prove properties 3 + 4.

Interactions and the Dirac Equation

We consider an electron interacting with the electromagnetic field $A^\mu \equiv (\Phi, \vec{A})$. This is described by the Dirac equation where $p_\mu \to p_\mu - \frac{q}{c} A_\mu$ (electron charge $q = -e$)

giving: $$\left[\gamma^\mu \left(p_\mu - \frac{q}{c} A_\mu\right) - mc\right] \Psi = 0 \qquad (4.60)$$

Exercise: Suggest an additional term coupling Ψ and the EM field that preserves covariance. Hint: What about $F_{\mu\nu}$?

(4.60) is said to be obtained using minimal coupling. It can be written in the following way.

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[c\vec{\alpha}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right) + \beta mc^2 + q\,\Phi\right]\Psi \qquad (4.61)$$

Exercise: Show (4.61) follows from (4.60).

We let $$\Psi(\vec{x},t) = e^{-imc^2t/\hbar}\begin{pmatrix}\psi_u(\vec{x},t)\\ \psi_L(\vec{x},t)\end{pmatrix} \qquad (4.62)$$

Using (4.62) in (4.61), we get:

$$i\hbar \frac{\partial \psi_u}{\partial t} = c\,\vec{\sigma}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right)\psi_L + q\,\Phi\,\psi_u \qquad (4.63a)$$

$$i\hbar \frac{\partial \psi_L}{\partial t} = c\,\vec{\sigma}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right)\psi_u + q\,\Phi\,\psi_L - 2mc^2\psi_L \qquad (4.63b)$$

For small fields and low momentum, (4.63b) becomes:

$$i\hbar \frac{\partial \psi_L}{\partial t} \simeq -2mc^2 \psi_L$$

$\therefore \psi_L$ varies at a frequency $2mc^2/\hbar$. The electron is said to jump back and forth to its negative energy components. This is called Zitterbewegung.

Exercise: Evaluate the Zitterbewegung frequency. How does it compare to optical frequencies?

Solution of (4.63)

Exercise: From (4.50), show $\psi_L \sim \frac{v}{c}\psi_u$ for an electron.

Hence to lowest order in $\frac{v}{c}$, (4.63b) is approximated by:

$$0 \simeq c\vec{\sigma}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right)\psi_u - 2mc^2\psi_L$$

$$\psi_L \simeq \frac{c\vec{\sigma}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right)}{2mc^2}\psi_u \tag{4.64}$$

Substituting (4.64) into (4.63a) gives:

$$i\hbar \frac{\partial \psi_u}{\partial t} = \left\{ \frac{\left[c\vec{\sigma}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right)\right]\left[c\vec{\sigma}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right)\right]}{2mc^2} + q\Phi \right\}\psi_u$$

Using $(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\,\vec{\sigma}\cdot(\vec{A}\times\vec{B})$ (4.65)
we get:

$$i\hbar\frac{\partial\psi_u}{\partial t} = \frac{\left[\vec{p} - \frac{q}{c}\vec{A}\right]^2}{2m}\psi_u - \frac{q\hbar}{2mc}\vec{\sigma}\cdot(\nabla\times\vec{A} + \vec{A}\times\nabla)\psi_u$$

$$+ q\Phi\psi_u$$

Exercise: Show $(\nabla\times\vec{A} + \vec{A}\times\nabla)\psi_u = \vec{B}\psi_u$

$$\therefore\ i\hbar\frac{\partial\psi_u}{\partial t} = \left[\frac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2m} - \frac{q\hbar}{2mc}\vec{\sigma}\cdot\vec{B} + q\Phi\right]\psi_u \quad (4.66)$$

This is called the Pauli equation which is Schrodinger result plus spin interaction with a magnetic field.

Magnetic Moment

$$\vec{\mu} \equiv \frac{q\hbar}{2mc}\vec{\sigma}$$

$$= -\mu_B\vec{\sigma} \quad \text{where } \mu_B \equiv e\hbar/2mc$$

$$\equiv -g\,\mu_B\vec{S} \quad \text{where } \vec{S} \equiv \frac{\vec{\sigma}}{2}$$

and the g factor $g = 2$.

Experimental Data

$g_{electron} = 2(1.00115965 2209 \pm 3.1\times10^{-11})$

$g_{muon} = 2(1.001165923 \pm 9\times10^{-9})$

$g_{proton} = 2(2.7928456)$

$g_{neutron} = 2(-1.913148)$

Exercise: What does this tell you about electron, muon, proton & neutron structure?

Corrections to Pauli Equation

Solving (4.63) by keeping higher order terms i.e. $\left(\frac{v}{c}\right)^2$ gives:

$$i\hbar \frac{\partial \psi_u}{\partial t} = H_o \psi_u$$

where
$$H_o = \left\{ \frac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2m} - \frac{p^4}{8m^3c^2} \right\} - \frac{q\hbar}{2mc}\vec{\sigma}\cdot\vec{B} + q\Phi \tag{4.67}$$
$$\underbrace{-\frac{iq\hbar}{8m^2c^2}\vec{p}\cdot\vec{E}}_{\text{Darwin term}} - \frac{iq}{8m^2}\vec{\sigma}\cdot(\nabla\times\vec{E}) - \frac{q\hbar}{4m^2c^2}\vec{\sigma}\cdot(\vec{E}\times\vec{p})$$

Exercise: a) Show $\nabla\times\vec{E} = 0$ for a spherically symmetric potential.

b) Show
$$H_{SO} \equiv -\frac{q\hbar}{4m^2c^2}\vec{\sigma}\cdot(\vec{E}\times\vec{p})$$
$$= \frac{q\hbar}{4m^2c^2}\frac{1}{r}\frac{\partial V}{\partial r}\vec{\sigma}\cdot\vec{L}, \quad \vec{E} = -\hat{r}\frac{\partial V}{\partial r}$$

Note: H_{SO} includes Thomas precession factor.

Central Force Problem

We shall solve the eigenvalue equation

$$H\,\Psi = E\,\Psi \tag{4.68}$$

where the Hamiltonian

$$H = c\vec{\alpha}\cdot\vec{p} + \beta\mu c^2 + V(r) \tag{4.69}$$

and $\mu \equiv \dfrac{m_e m_{nuc}}{m_e + m_{nuc}}$ is the reduced mass.

Exercise: Evaluate reduced mass for H atom and positronium.

Valid Quantum Numbers

We shall find operators that commute with H. First we consider the orbital angular momentum $\vec{L}$.

$$[H, L_1]\,\Psi = [c\vec{\alpha}\cdot\vec{p},\, x_2 p_3 - x_3 p_2]\,\Psi \qquad \text{using (4.69)}$$

$$= \left\{ c\vec{\alpha}\cdot\vec{p}\,(x_2 p_3 - x_3 p_2) - (x_2 p_3 - x_3 p_2)\,c\vec{\alpha}\cdot\vec{p} \right\}\Psi$$

$$= \left\{ c\,\alpha_2 (-i\hbar)\, p_3 - c\,\alpha_3(-i\hbar)\, p_2 \right\}\Psi$$

$$= -i\hbar c \left\{ \alpha_2 p_3 - \alpha_3 p_2 \right\}\Psi$$

$$= -i\hbar c\, (\vec{\alpha}\times\vec{p})_1\, \Psi$$

$$\therefore [H, \vec{L}] = -i\hbar c\,\vec{\alpha}\times\vec{p} \tag{4.70}$$

Next we consider spin $\vec{S} \equiv \frac{\hbar}{2}\vec{\Sigma}$.

$$[H, S_1]\Psi = [c\vec{\alpha}\cdot\vec{p} + \beta\mu c^2, \frac{\hbar}{2}\Sigma_1]\Psi$$

$$= \Big\{ c\alpha_2 p_2 \frac{\hbar}{2}\Sigma_1 - \frac{\hbar}{2}\Sigma_1 c\alpha_2 p_2$$ using matrix properties

$$+ c\alpha_3 p_3 \frac{\hbar}{2}\Sigma_1 - \frac{\hbar}{2}\Sigma_1 c\alpha_3 p_3 \Big\}\Psi$$

$$= \frac{c\hbar}{2}\Big\{ p_2(\alpha_2\Sigma_1 - \Sigma_1\alpha_2) + p_3(\alpha_3\Sigma_1 - \Sigma_1\alpha_3)\Big\}\Psi$$

Exercise: Show the following.

$$\alpha_2\Sigma_1 = -i\alpha_3 \qquad \Sigma_1\alpha_2 = i\alpha_3$$
$$\alpha_3\Sigma_1 = i\alpha_2 \qquad \Sigma_1\alpha_3 = -i\alpha_2$$

$$\therefore [H, S_1]\Psi = \frac{c\hbar}{2}\Big\{ p_2(-i\alpha_3 - i\alpha_3) + p_3(i\alpha_2 + i\alpha_2)\Big\}\Psi$$

$$= ic\hbar\,\{-p_2\alpha_3 + p_3\alpha_2\}\Psi$$

$$= ic\hbar\,(\vec{\alpha}\times\vec{p})_1\Psi$$

$$\therefore [H, \vec{S}] = ic\hbar\,(\vec{\alpha}\times\vec{p}) \qquad (4.71)$$

Combining (4.70) and (4.71) gives

$$[H, \vec{J}] = 0 \qquad (4.72)$$

where $\vec{J} \equiv \vec{L} + \vec{S}$ is the total electronic angular momentum. Hence the eigenstates Ψ can be labelled by quantum numbers j and m.

$\tilde{K}$ Operator

We define $\tilde{K} \equiv -\beta\left(\vec{\Sigma}\cdot\frac{\vec{L}}{\hbar}+1\right)$ (4.73)

Exercise (Homework): a) Show $[H, \tilde{K}]=0$ (4.74)

b) Eigenvalues are $K = \pm\left(j+\frac{1}{2}\right)$ (4.75)

$\tilde{K}$ describes the alignment of the spin and orbital angular momenta.

Form of Eigenstate Ψ

We let
$$\Psi = \begin{pmatrix} i\,\frac{G(r)}{r}\,\mathcal{Y}_{jm} \\ \frac{F(r)}{r}\,\frac{\vec{\sigma}\cdot\vec{r}}{r}\,\mathcal{Y}_{jm} \end{pmatrix} \qquad (4.76)$$

The purpose of the $\vec{\sigma}\cdot\vec{r}$ term is to ensure the opposite parity of the particle and antiparticle parts of the spinor Ψ.

Angular-Spin Wavefunction $\mathcal{Y}_{jm}$

$\mathcal{Y}_{jm}$ is a coupling of the eigenstates of $\vec{L}$ (spherical harmonics $Y_{\ell m_\ell}(\theta,\phi)$ and the eigenstates of $\vec{S}$ which for a spin $\frac{1}{2}$ particle are $\chi_{s\,m_s}$. $\left\{\chi_{\frac{1}{2}\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix}, \chi_{\frac{1}{2}-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix}\right\}$.

$$\mathcal{Y}_{\substack{jm \\ j=\ell\pm\frac{1}{2}}} = \sum_{m_\ell\, m_s} Y_{\ell m_\ell}(\theta,\phi)\,\chi_{s\,m_s}\,C(\ell, s, m_\ell, m_s; j\,m) \qquad (4.77)$$

where the Clebsch Gordon coefficient is only nonzero if $m = m_\ell + m_s$. One can show that:

$$\mathcal{Y}_{j=\ell+\frac{1}{2},m} = +\sqrt{\frac{\ell+m+1/2}{2\ell+1}}\, Y_{\ell,m-\frac{1}{2}}(\theta,\phi)\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{\ell-m+1/2}{2\ell+1}}\, Y_{\ell,m+1/2}(\theta,\phi)\begin{pmatrix}0\\1\end{pmatrix}$$

$$(4.78)$$

$$\mathcal{Y}_{j=\ell-\frac{1}{2},m} = \sqrt{\frac{\ell-m+1/2}{2\ell+1}}\, Y_{\ell m-1/2}(\theta,\phi)\begin{pmatrix}1\\0\end{pmatrix} - \sqrt{\frac{\ell+m+1/2}{2\ell+1}}\, Y_{\ell m+1/2}(\theta,\phi)\begin{pmatrix}0\\1\end{pmatrix}$$

Solution of (4.68)

We shall now solve the eigenvalue problem:

$$\left[c\begin{pmatrix}0 & \vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & 0\end{pmatrix} + \mu c^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + V(r)\,1\right]\Psi = E\,\Psi$$

Substituting (4.76) gives:

$$c\,\vec{\sigma}\cdot\vec{p}\,\frac{F}{r}\,\frac{\vec{\sigma}\cdot\vec{r}}{r}\,\mathcal{Y}_{jm} + \mu c^2\, i\frac{G}{r}\,\mathcal{Y}_{jm} + V i\frac{G}{r}\,\mathcal{Y}_{jm} = E\, i\frac{G}{r}\,\mathcal{Y}_{jm} \qquad (4.79a)$$

$$c\,\vec{\sigma}\cdot\vec{p}\; i\frac{G}{r}\,\mathcal{Y}_{jm} - \mu c^2\frac{F}{r}\,\frac{\vec{\sigma}\cdot\vec{r}}{r}\,\mathcal{Y}_{jm} + V\frac{F}{r}\,\frac{\vec{\sigma}\cdot\vec{r}}{r}\,\mathcal{Y}_{jm} = E\frac{F}{r}\,\frac{\vec{\sigma}\cdot\vec{r}}{r}\,\mathcal{Y}_{jm} \qquad (4.79b)$$

Exercise: a) Show $(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\,\vec{\sigma}\cdot(\vec{A}\times\vec{B})$ (4.80)

b) Show $(\sigma\cdot\vec{r})(\vec{\sigma}\cdot\vec{r}) = \vec{r}\cdot\vec{r} = r^2$

Exercise: Show $\vec{\sigma}\cdot\vec{L}\,\mathcal{Y}_{jm} = -(1+\chi)\hbar\,\mathcal{Y}_{jm}$ (4.81)

i.e. $\vec{\sigma}\cdot\vec{L} = \begin{cases} (j-1/2)\hbar & \text{if } j=\ell+1/2\\ -(j+3/2)\hbar & \text{"} \;\; j=\ell-1/2\end{cases}$

To simplify (4.79a) we consider:

$$\vec{\sigma}\cdot\vec{p}\,\frac{F}{r^2}\,\vec{\sigma}\cdot\vec{r}\,\mathcal{Y}_{jm} = (\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{r})\frac{F}{r^2}\,\mathcal{Y}_{jm}$$

$$= \left[\vec{p}\cdot\vec{r} + i\,\vec{\sigma}\cdot(\vec{p}\times\vec{r})\right]\frac{F}{r^2}\,\mathcal{Y}_{jm} \quad \text{using (4.80)}$$

$$= \left[-i\hbar\left(r\frac{\partial}{\partial r} + 3\right) - i\,\vec{\sigma}\cdot\vec{L}\right]\frac{F}{r^2}\,\mathcal{Y}_{jm}$$

$$= -i\hbar\left\{r\frac{\partial}{\partial r} + 3 - (1+\kappa)\right\}\frac{F}{r^2}\,\mathcal{Y}_{jm} \quad \text{using (4.81)}$$

$$\therefore\ \vec{\sigma}\cdot\vec{p}\,\frac{F}{r^2}\,\vec{\sigma}\cdot\vec{r}\,\mathcal{Y}_{jm} = -i\hbar\left\{\frac{1}{r}\frac{dF}{dr} - \kappa\frac{F}{r^2}\right\}\mathcal{Y}_{jm}$$

Using the above result in (4.79a) we get:

$$-i\hbar c\left\{\frac{1}{r}\frac{dF}{dr} - \kappa\frac{F}{r^2}\right\}\mathcal{Y}_{jm} + i\mu c^2\frac{G}{r}\mathcal{Y}_{jm} + iV\frac{G}{r}\mathcal{Y}_{jm} = iE\frac{G}{r}\mathcal{Y}_{jm}$$

$$-\hbar c\left[\frac{dF}{dr} - \frac{\kappa F}{r}\right] + \mu c^2 G + VG = EG$$

$$\frac{dF}{dr} - \frac{\kappa F}{r} = \frac{(\mu c^2 - E + V)G}{\hbar c} \qquad (4.82a)$$

Exercise: Show $c\,\vec{\sigma}\cdot\vec{p}\,\frac{iG}{r}\mathcal{Y}_{jm} = \hbar c\,\frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left(\frac{dG}{dr} + \frac{\kappa G}{r}\right)\mathcal{Y}_{jm}$

Hint: Write $\vec{\sigma}\cdot\vec{p} = \dfrac{(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{p})}{r^2}$

(4,79b) then becomes:

$$\hbar c\left(\frac{dG}{dr}+\frac{\chi G}{r}\right)\frac{\vec{\sigma}\cdot\vec{r}}{r^2}\mathcal{Y}_{jm}-\mu c^2 F\frac{\vec{\sigma}\cdot\vec{r}}{r^2}\mathcal{Y}_{jm}+VF\frac{\vec{\sigma}\cdot\vec{r}}{r^2}\mathcal{Y}_{jm}$$

$$=EF\frac{\vec{\sigma}\cdot\vec{r}}{r^2}\mathcal{Y}_{jm}$$

$$\hbar c\left(\frac{dG}{dr}+\frac{\chi G}{r}\right)-\mu c^2F+VF=EF$$

$$\frac{dG}{dr}+\frac{\chi G}{r}-\frac{(\mu c^2+E-V)}{\hbar c}F=0 \qquad (4.82b)$$

Hydrogenic Atom

The Coulomb potential $V(r)=-\frac{Ze^2}{r}$

To solve (4,82) we introduce dimensionless variables.

$$\epsilon_1\equiv\frac{\mu c^2+E}{\hbar c}\qquad \epsilon_2\equiv\frac{\mu c^2-E}{\hbar c} \qquad (4,83)$$

$$\rho\equiv\sqrt{\epsilon_1\epsilon_2}\,r$$

Note: $V=-\hbar c\,\frac{Z\alpha}{r}$ where $\alpha\equiv e^2/\hbar c$.

Equations (4.82 a,b) become:

$$\frac{dF}{d\rho}-\frac{\chi}{\rho}F-\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}-\frac{Z\alpha}{\rho}\right)G=0 \qquad (4.84a)$$

$$\frac{dG}{d\rho}+\frac{\chi}{\rho}G-\left(\sqrt{\frac{\epsilon_1}{\epsilon_2}}+\frac{Z\alpha}{\rho}\right)F=0 \qquad (4.84b)$$

Exercise: Show (4.82) $\Longrightarrow$ (4.84).

Asymptotic Solutions of (4.84)

These are useful for finding the general solution valid for all ρ.

$\rho \to \infty$ Region

(4.84) become:
$$\frac{dF}{d\rho} - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\, G = 0 \qquad (4.85a)$$

$$\frac{dG}{d\rho} - \sqrt{\frac{\epsilon_1}{\epsilon_2}}\, F = 0 \qquad (4.85b)$$

(4.85) $\Longrightarrow$
$$\frac{d^2G}{d\rho^2} = G \quad , \quad \frac{d^2F}{d\rho^2} = F$$

These have solutions $G, F \sim e^{-\rho}$.

Exercise: Why is $e^{+\rho}$ not considered?

$\rho \to 0$ Region

(4.84) become:
$$\frac{dF}{d\rho} - \frac{\kappa}{\rho} F + \frac{Z\alpha}{\rho} G = 0 \qquad (4.86a)$$

$$\frac{dG}{d\rho} + \frac{\kappa}{\rho} G - \frac{Z\alpha}{\rho} F = 0 \qquad (4.86b)$$

Let $F = a_0 \rho^s$, $G = b_0 \rho^s$.

$(4.86) \Longrightarrow$
$$s a_0 - \kappa a_0 + Z\alpha b_0 = 0$$
$$s b_0 + \kappa b_0 - Z\alpha a_0 = 0.$$

$$\begin{pmatrix} s-\kappa & Z\alpha \\ -Z\alpha & s+\kappa \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Exercise: a) Show $a_0 = \dfrac{-Z\alpha}{s-\kappa} b_0$ (4.87)

b) Show $s = \pm\sqrt{\kappa^2 - (Z\alpha)^2}$ (4.88)

Exercise: Show that $s>0$ in order for a finite probability density near the origin.

General Solution

Let $F = e^{-\rho} \rho^s \sum_{m=0}^{\infty} a_m \rho^m$ (4.89a)

$G = e^{-\rho} \rho^s \sum_{m=0}^{\infty} b_m \rho^m$ (4.89b)

Exercise: Show (4.89) & (4.84) yield the recursion relations:

$$(s+m-\kappa)\, a_m - a_{m-1} + Z\alpha\, b_m - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\, b_{m-1} = 0 \qquad (4.90a)$$

$$(s+m+\kappa)\, b_m - b_{m-1} - Z\alpha\, a_m - \sqrt{\frac{\epsilon_1}{\epsilon_2}}\, a_{m-1} = 0 \qquad (4.90b)$$

In the limit of large m, (4.90) become:

$$a_m \sim \frac{a_{m-1}}{m} \qquad b_m \sim \frac{b_{m-1}}{m}$$

Exercise: a) If $a_m = \frac{a_{m-1}}{m}$ show $\sum_{m=0}^{\infty} a_m \rho^m = a_0 e^{\rho}$.

b) Show Ψ is then not normalizable.
i.e. $\int \Psi^{+} \Psi \, dV = \infty$.

We shall assume there exists a bound state such that the series in ρ is finite.

$$\text{i.e.} \quad a_{n'+1} = b_{n'+1} = 0 \tag{4.91}$$

Setting $m = n'+1$ in (4.90) gives:

$$a_{n'} = -\sqrt{\frac{\epsilon_2}{\epsilon_1}}\, b_{n'} \tag{4.92}$$

Eigenenergy Determination

Consider $(4.90a) - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\,(4.90b)$.

$$(s+m-K)\,a_m + Z\alpha b_m - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\,(s+m+K)\,b_m + Z\alpha\sqrt{\frac{\epsilon_2}{\epsilon_1}}\,a_m = 0$$

Next we set $m = n'$ and use (4.92).

$$-(s+n'-K)\sqrt{\frac{\epsilon_2}{\epsilon_1}} + Z\alpha - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\,(s+n'+K) - Z\alpha\,\frac{\epsilon_2}{\epsilon_1} = 0$$

$$2(s+n') + Z\alpha\sqrt{\frac{\epsilon_1}{\epsilon_2}}\left(\frac{\epsilon_2}{\epsilon_1} - 1\right) = 0$$

or $\quad 2(s+n') + Z\alpha \dfrac{\epsilon_2 - \epsilon_1}{\sqrt{\epsilon_1 \epsilon_2}} = 0$

Substituting for s, ϵ_1 & ϵ_2 gives:

$$2\left(\sqrt{\chi^2 - (Z\alpha)^2} + n'\right) + Z\alpha \frac{\mu c^2 - E - \mu c^2 - E}{\sqrt{(\mu c^2 + E)(\mu c^2 - E)}} = 0$$

Solving for the energy and defining $n \equiv n' + j + 1/2$ gives:

$$\boxed{E = \frac{\mu c^2}{\sqrt{1 + \left(\dfrac{Z\alpha}{n - (j+1/2) + \sqrt{(j+1/2)^2 - (Z\alpha)^2}}\right)^2}}} \qquad (4.93)$$

Expanding (4.93) using $Z\alpha \ll 1$ gives:

$$E = \mu c^2 \left\{ 1 - \frac{(Z\alpha)^2}{2n^2}\left[1 + \frac{(Z\alpha)^2}{n}\left(\frac{1}{j+1/2} - \frac{3}{4n}\right)\right] + \dots \right\} \quad (4.94)$$

Exercise: Identify the energies in (4.94).

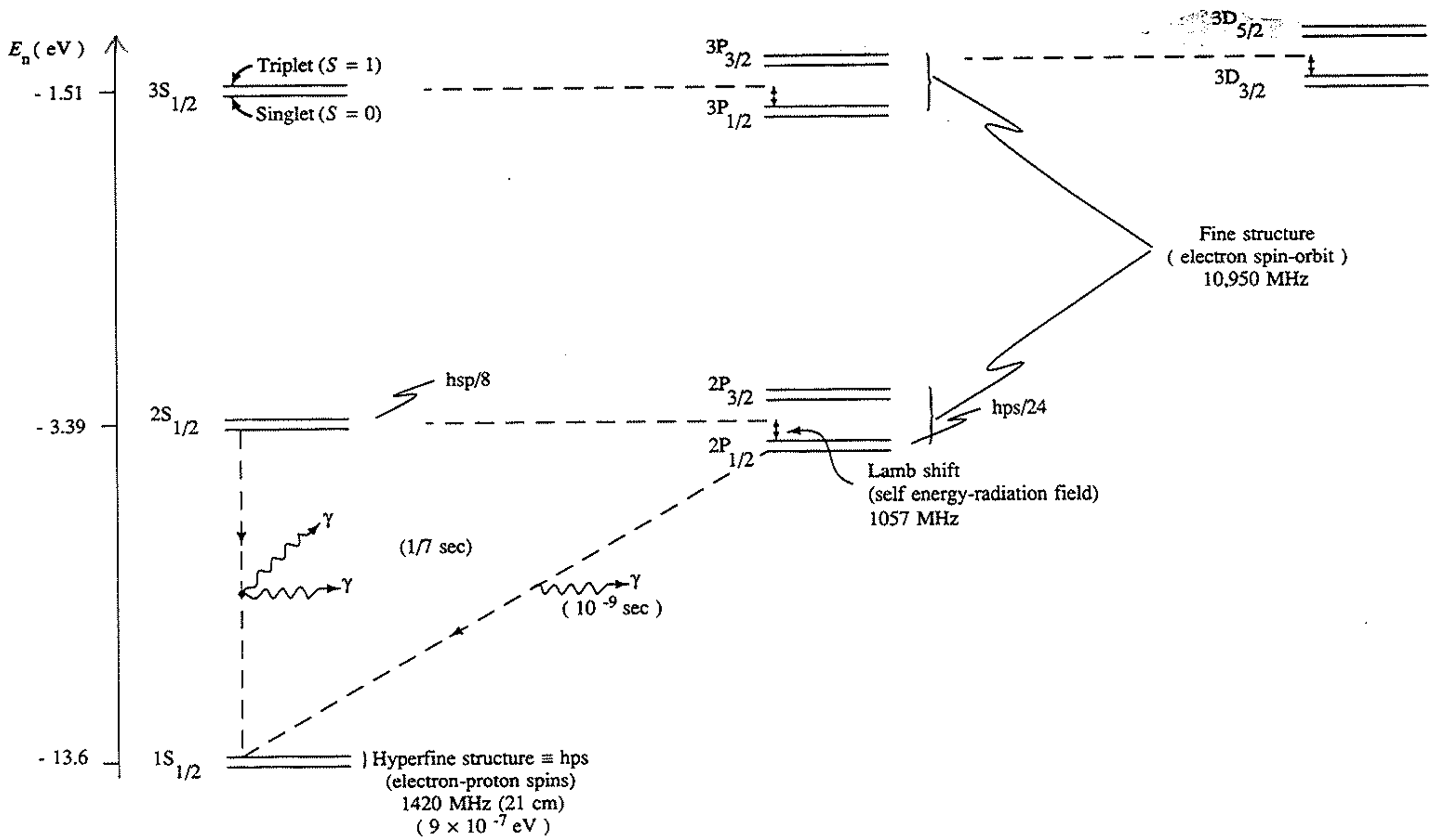
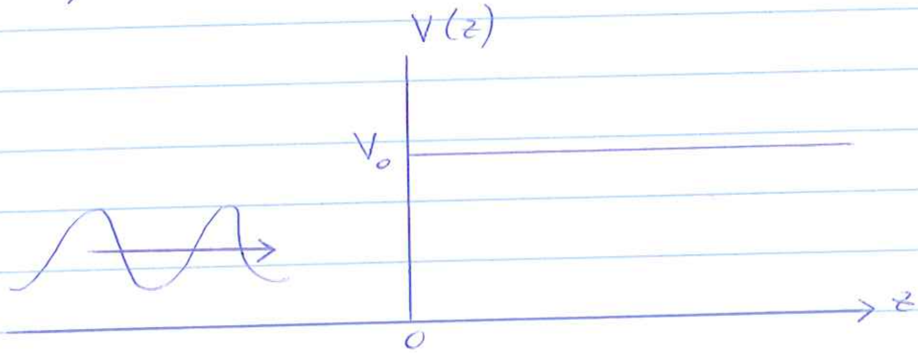


Fig. 16.1 The energy levels of atomic hydrogen labeled by the quantum numbers of the large component of the wave function. Not to scale, adapted from Bjorken and Drell (1964).

Klein Paradox

Consider a free electron wave of energy E striking a potential barrier at $z=0$.



To the left of the barrier ($z<0$), the wavefunction consists of incident and reflected waves.

$$\psi_L = a\, e^{i(p_1 z - Et)/\hbar} \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_1}{E+mc^2} \\ 0 \end{pmatrix} + b\, e^{-i(p_1 z + Et)/\hbar} \begin{pmatrix} 1 \\ 0 \\ \dfrac{-cp_1}{E+mc^2} \\ 0 \end{pmatrix} \tag{4.95}$$

where $p_1^2 c^2 = E^2 - m^2c^4$.

To the right of the barrier ($z>0$), there is only the transmitted wave.

$$\psi_R = d\, e^{i(p_2 z - Et)/\hbar} \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_2}{E-V_0+mc^2} \\ 0 \end{pmatrix} \tag{4.96}$$

where $p_2^2 c^2 = (E-V_0)^2 - m^2c^4$.

Case 1: $|E - V_0| < mc^2$

OR: $V_0 - mc^2 < E < V_0 + mc^2$

In this case p_2 is imaginary and the transmitted wave decays exponentially. Hence the entire incident wave is reflected.

Case 2: $|E - V_0| > mc^2$

OR: a) $E - V_0 > mc^2$
$E > V_0 + mc^2$

b) $E - V_0 < -mc^2$
$E < V_0 - mc^2$
$V_0 > E + mc^2$

In each of cases 2a + 2b, p_2 is real and part of the incident wave is transmitted. We shall now determine the reflection and transmission coefficients.

At the boundary ($z=0$), the wavefunction is continuous.

Exercise: Show $\psi_L = \psi_R$ using (4.95) + (4.96) yields:

$$a + b = d \tag{4.97a}$$

$$a - b = r d \quad \text{where } r \equiv \frac{p_2}{p_1} \frac{E + mc^2}{E - V_0 + mc^2} \tag{4.97b}$$

Exercise: Show (4.97) yields:

$$a = \frac{1+r}{2} d \qquad b = \frac{1-r}{2} d \tag{4.98}$$

Current Density

$$J^\mu = c\,\psi^\dagger \gamma^0 \gamma^\mu \psi$$

$$\therefore \vec{J} = c\,\psi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \psi$$

$$= c\,\psi^\dagger \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \psi .$$

Exercise: Show current density corresponding to incident wave is:

$$\vec{J}_{inc} = 2a^2 \frac{c^2 p_1}{E + mc^2} \hat{z} \qquad (4.99a)$$

Similarly: $$\vec{J}_{ref} = -2b^2 \frac{c^2 p_1}{E + mc^2} \hat{z} \qquad (4.99b)$$

$$\vec{J}_{trans} = 2d^2 \frac{c^2 p_2}{E - V_0 + mc^2} \hat{z} \qquad (4.99c)$$

Reflection Coefficient $R \equiv \dfrac{J_{ref}}{J_{inc}}$

$$= \frac{b^2}{a^2}$$

Using (4.98) we get: $$R = \left(\frac{1-r}{1+r}\right)^2 \qquad (4.100)$$

Exercise: Show transmission coefficient

$$T \equiv \frac{J_{trans}}{J_{inc}} = \frac{4r}{(1+r)^2} \qquad (4.101)$$

Exercise: Show $R + T = 1$.

Let us now examine the implications of (4.100) and (4.101) in cases 2a + 2b.

Case 2a: $E > V_0 + mc^2$

The energy of the incident wave is above the barrier. $r > 0$ and hence (4.100) + (4.101) imply part of the wave is reflected and the rest transmitted.

Case 2b: $V_0 > E + mc^2$

From (4.97b) we see that $r < 0$.

$$\Rightarrow R > 1 \text{ and } T < 0 \text{ !!}$$

This is called Klein's Paradox. More particles are reflected than were incident on the barrier! The reason is that the potential is so strong that e^+e^- pairs are created out of the vacuum. The e^- go to left making $R > 1$ while e^+ go right making $T < 0$. This shows the limitations of the Dirac equation in studying strong fields and the need for a "quantized field theory."

Chapter 4 Assignment

1) Explicitly write out the 4×4 $\vec{\alpha}$ and β matrices and show they satisfy: a) $\{\alpha_k, \alpha_j\} = 2\delta_{kj}$

b) $\{\alpha_k, \beta\} = 0$

2) Explicitly write out the 4×4 $\vec{\gamma}$, γ^0 & γ^5 matrices.

3) Lorentz transformation: Prove (4.36 a $\to$ d).

4) Show a) $P^{-1}\gamma^5 P = -\gamma^5$

b) $P^{-1}\gamma^5\vec{\gamma}P = \gamma^5\vec{\gamma}$

5) Projection Operators $\Lambda_\pm$: Prove the following.

a) $\Lambda_- u_s(p) = 0$ $\quad$ $\Lambda_- v_s(p) = v_s(p)$

b) $\Lambda_\pm^2 = \Lambda_\pm$

c) $\Lambda_+\Lambda_- = \Lambda_-\Lambda_+ = 0$

d) $\Lambda_+ + \Lambda_- = 1$

6) $H = c\vec{\alpha}\cdot\vec{p} + \beta\mu c^2 + V(r)$

$\tilde{K} = -\beta\left(\vec{\Sigma}\cdot\frac{\vec{L}}{\hbar} + 1\right)$

a) Show $[H, \tilde{K}] = 0$

b) Show $\tilde{K}$ has eigenvalues $\pm(j + 1/2)$.

7) Identify the various terms in (4.94).

V) Second Quantization

A formalism that records the number of particles in the available states is now developed.

Fock Space

Consider an orthonormal set of one electron spin orbitals $\psi_1, \psi_2, \ldots, \psi_N$. A maximum of one electron may occupy a single orbital due to the Pauli Exclusion Principle.

Case a: All orbitals are empty. i.e. Vacuum State

$$\Psi_0 \equiv |0_1, 0_2, 0_3 \ldots\rangle \equiv |0\rangle \qquad (5.1)$$

0_1 means 0 electrons are in ψ_1
0_2 " " ψ_2
0_3 " " ψ_3
etc.

Case b: ψ_m is occupied.

$$\Psi_m \equiv |0_1, 0_2, \ldots 1_m, 0_{m+1} \ldots\rangle \qquad (5.2)$$

1_m means one electron is in ψ_m

Case c: ψ_l & ψ_m are occupied.

$$\Psi_{lm} \equiv |0_1, 0_2 \ldots 1_l \ldots 1_m \ldots\rangle \qquad (5.3)$$

However, the Pauli Exclusion Principle says that $\Psi_{\ell m}$ must be antisymmetric under electron exchange.
$\therefore \Psi_{\ell m}$ is given by the Slater Determinant.

$$\Psi_{\ell m} = \frac{1}{\sqrt{2}} \begin{vmatrix} \psi_\ell(1) & \psi_m(1) \\ \psi_\ell(2) & \psi_m(2) \end{vmatrix} \tag{5.4}$$

$$\therefore \quad \Psi_{\ell m} = -\Psi_{m \ell}$$

or $|0_1, 0_2, \ldots 1_\ell \ldots 1_m \ldots\rangle = -|0_1, 0_2 \ldots 1_m \ldots 1_\ell \ldots\rangle$

Case d: ψ_k, ψ_ℓ & ψ_m are occupied.

$$\Psi_{k\ell m} \equiv |0_1, 0_2 \ldots 1_k \ldots 1_\ell \ldots 1_m \ldots\rangle \tag{5.5}$$

$$= \frac{1}{\sqrt{3!}} \begin{vmatrix} \psi_k(1) & \psi_\ell(1) & \psi_m(1) \\ \psi_k(2) & \psi_\ell(2) & \psi_m(2) \\ \psi_k(3) & \psi_\ell(3) & \psi_m(3) \end{vmatrix}$$

$\Psi_0, \Psi_m, \Psi_{\ell m}, \Psi_{k\ell m}$ etc. are called the occupation number or Fock Space representations of the states.

General Case

The Fock Space representation for a system of electrons (fermions) is:

$$\Psi = |n_1, n_2, \ldots n_k \ldots\rangle \tag{5.6}$$

where $n_k = 0$ or 1. Ψ is a Slater determinant.

Orthogonality Relation

Since $\psi_1, \psi_2 \ldots \psi_N$ are orthonormal, we have:

$$\langle \psi_A \mid \psi_B \rangle = \delta_{AB} \tag{5.7}$$

i.e. $\langle n_1', n_2' \ldots n_k' \ldots \mid n_1, n_2 \ldots n_k \ldots \rangle = \delta_{n_1 n_1'} \delta_{n_2 n_2'} \ldots$

$$= \prod_{i=1}^{\infty} \delta_{n_i n_i'}$$

Annihilation / Destruction Operator

$c_k \mid 0_1, 0_2, \ldots 1_k \ldots 1_l \, 1_m \ldots \rangle \equiv \mid 0_1 \, 0_2 \ldots 0_k \, 1_l \, 1_m \ldots \rangle$

or $\quad c_k \Psi_{klm} = \Psi_{lm}$ $\quad$ (5.8)

i.e. c_k removes an electron from the kth orbital.

Exercise: Show $c_l \Psi_{klm} = -\Psi_{km}$.

If the annihilation operator acts on a vacant orbital, the result is 0.

i.e. $c_i \Psi_{klm}$

$= c_i \mid 0_1 \, 0_2 \ldots 0_i \, 1_k \, 1_l \, 1_m \ldots \rangle$

$= 0$

Exercise: Show a) $c_l c_k \Psi_{klm} = +\Psi_m$ $\quad$ (5.9a)

b) $c_k c_l \Psi_{klm} = -\Psi_m$ $\quad$ (5.9b)

We now define the anticommutator

$$\{A, B\} \equiv AB + BA$$

$$(5.9a) \; \& \; (5.9b) \Rightarrow \quad \{c_k, c_\ell\} = 0. \qquad (5.10)$$

Exercise: Show $c_k c_k = 0$ and explain what this means.

Creation Operator

c_k^+ creates an electron in the kth orbital.

$$\text{i.e.} \quad c_k^+ \, \Psi_0 = c_k^+ \, |0_1 \, 0_2 \ldots 0_k \ldots\rangle \qquad (5.11)$$

$$= |0_1 \, 0_2 \ldots 1_k \ldots\rangle$$

$$= \Psi_k.$$

If the orbital is already occupied, the result is zero.

i.e. $c_k^+ \, \Psi_{k\ell m} = 0.$

Exercise: Show $\{c_k^+, c_\ell^+\} = 0.$ $\qquad (5.12)$

Products of c_k and c_l^+ $(k \neq l)$

$$c_l^+ c_k \Psi_k = c_l^+ \Psi_o$$

$$= \Psi_l \tag{5.13a}$$

$$c_k c_l^+ \Psi_k = c_k \Psi_{lk}$$

$$= -c_k \Psi_{kl}$$

$$= -\Psi_l \tag{5.13b}$$

$$(5.13\ a + b) \Longrightarrow \quad \{c_k, c_l^+\} = 0 \quad \text{if } k \neq l \tag{5.14}$$

Products of c_k and c_k^+

$$c_k^+ c_k \Psi_o = 0 \qquad c_k c_k^+ \Psi_o = \Psi_o \tag{5.15a}$$

$$c_k^+ c_k \Psi_k = \Psi_k \qquad c_k c_k^+ \Psi_k = 0 \tag{5.15b}$$

$$(5.15\ a + b) \Longrightarrow \quad \{c_k, c_k^+\} = 1 \tag{5.16}$$

From (5.15) we find $N_k \equiv c_k^+ c_k$ gives the number of electrons (0 or 1) occupying state k.

Summary of Fock Space Rules For Fermions

$$\{c_k, c_\ell\} = \{c_k^+, c_\ell^+\} = 0 \qquad (5.17a)$$

$$\{c_k, c_\ell^+\} = \delta_{k\ell} \qquad (5.17b)$$

$$c_k \,|\, \ldots n_k \ldots \rangle = (-1)^{p_k} \sqrt{n_k} \,|\, \ldots 0_k \ldots \rangle \qquad (5.17c)$$

$$c_k^+ \,|\, \ldots n_k \ldots \rangle = (-1)^{p_k} \sqrt{1-n_k} \,|\, \ldots 1_k \ldots \rangle \qquad (5.17d)$$

Here $n_k = 0$ or 1 and p_k is the occupied number of orbitals to the left of k. $(-1)^{p_k}$ is the appropriate antisymmetrization factor.

Fock Space Rules For Bosons

Bosons (photon, graviton, mesons etc.) are not restricted by the Pauli Exclusion Principle.

$\therefore \Psi = |n_1\ n_2 \ldots n_k \ldots\rangle$ where n_k is any positive integer. Annihilation/creation operators are defined as follows.

$$a_k\,|\ldots n_k \ldots\rangle \equiv \sqrt{n_k}\,|\ldots n_k - 1 \ldots\rangle \qquad (5.18a)$$

$$a_k^\dagger\,|\ldots n_k \ldots\rangle \equiv \sqrt{n_k+1}\,|\ldots n_k + 1 \ldots\rangle \qquad (5.18b)$$

Next we define the commutator

$$[A, B] \equiv AB - BA$$

It can then be shown that:

$$[a_k, a_\ell] = [a_k^\dagger, a_\ell^\dagger] = 0 \qquad (5.18c)$$

$$[a_k, a_\ell^\dagger] = \delta_{k\ell} \qquad (5.18d)$$

Exercise: a) Show $N \equiv \sum_i a_i^\dagger a_i$ gives the total number of particles. (5.19)

b) Show $|n_1\ n_2 \ldots\rangle = \ldots \dfrac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} \dfrac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} |0_1\ 0_2 \ldots\rangle$ (5.20)

Exercise: Read about quantized harmonic oscillators.

Quantization of Electromagnetic Field

Maxwell Equations in Vacuum

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$
$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 0 \tag{5.21}$$

(5.21) implies the existence of a vector potential $\vec{A}$.

$$\vec{B} = \nabla \times \vec{A} \tag{5.22}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{5.23}$$

There is also the gauge restriction. We shall use the Coulomb or transverse gauge:

$$\nabla \cdot \vec{A} = 0 \tag{5.24}$$

Exercise: Using (5.21 – 5.24) derive the wave equation

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0 \tag{5.25}$$

This has the general plane wave solution

$$\vec{A} = \sum_{\vec{k}\lambda} \hat{e}_{\vec{k}\lambda} \left\{ A_{\vec{k}\lambda} e^{i(\vec{k}\cdot\vec{r} - \omega_k t)} + A^*_{\vec{k}\lambda} e^{-i(\vec{k}\cdot\vec{r} - \omega_k t)} \right\} \tag{5.26}$$

Wavevector $\vec{k}$

The solution for $\vec{A}$ is considered for a volume V which is customarily taken to be a cube having side length L. Periodic boundary conditions on $\vec{A}$ are assumed.

$$\text{i.e.}\quad \vec{A}(x,y,z) = \vec{A}(x+L, y+L, z+L)$$

$$\Rightarrow e^{ik_x x} = e^{ik_x(x+L)} \qquad e^{ik_y y} = e^{ik_y(y+L)} \qquad e^{ik_z z} = e^{ik_z(z+L)}$$

$$\therefore k_x = \frac{2\pi}{L} N_x \qquad N_x = 0, \pm 1, \pm 2 \ldots$$

$$k_y = \frac{2\pi}{L} N_y \qquad N_y = \quad " \quad "$$

$$k_z = \frac{2\pi}{L} N_z \qquad N_z = \quad " \quad "$$

$$\text{or}\quad \vec{k} = \frac{2\pi}{L}(N_x, N_y, N_z) \tag{5.27}$$

of modes

modes with N_x between N_x + $N_x + \Delta N_x$
N_y " N_y + $N_y + \Delta N_y$
N_z " N_z + $N_z + \Delta N_z$

is: $\Delta N = \Delta N_x \, \Delta N_y \, \Delta N_z$

$$= \left(\frac{L}{2\pi}\right)^3 \Delta k_x \, \Delta k_y \, \Delta k_z$$

$$= \left(\frac{L}{2\pi}\right)^3 k^2 \sin\theta \, dk \, d\theta \, d\varphi$$ using spherical coords. in $\vec{k}$ space

$$\Delta N = \frac{V}{(2\pi)^3} k^2 \, dk \, d\Lambda \qquad (5.28)$$

solid angle element in $\vec{k}$ space

Exercise: a) Substituting (5.26) into (5.25) show that

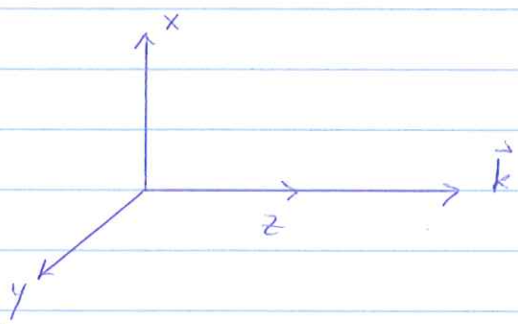
$$\omega_k = |\vec{k}| c \qquad (5.29)$$

b) Show $\nabla \cdot \vec{A} = 0 \Rightarrow \hat{e}_{\vec{k}\lambda} \cdot \vec{k} = 0$ $\qquad (5.30)$

Polarization

There are two independent polarization vectors $\hat{e}_{\vec{k}\lambda}$ for each $\vec{k}$.

Example $\vec{k} \parallel \hat{z}$



Linear Polarization Basis $\hat{e}_{\vec{k}1} = \hat{x} \quad \hat{e}_{\vec{k}2} = \hat{y}$

Circular Polarization Basis

$$\hat{e}_{\vec{k}+1} = \frac{-1}{\sqrt{2}}(\hat{x} + i\hat{y}) \qquad \hat{e}_{\vec{k}-1} = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y}) \qquad (5.31)$$

left circular polarized $\qquad$ right circular polarized

Exercise: Substituting (5.26) into (5.22) + (5.23) derive the following.

$$\vec{E} = \frac{i}{c} \sum_{\vec{k}\lambda} \omega_k \, \hat{e}_{\vec{k}\lambda} \left\{ A_{\vec{k}\lambda} \, e^{i(\vec{k}\cdot\vec{r} - \omega_k t)} - A^*_{\vec{k}\lambda} \, e^{-i(\vec{k}\cdot\vec{r} - \omega_k t)} \right\} \quad (5.32)$$

$$\vec{B} = \frac{i}{c} \sum_{\vec{k}\lambda} \omega_k \left(\hat{k} \times \hat{e}_{\vec{k}\lambda} \right) \left\{ A_{\vec{k}\lambda} \, e^{i(\vec{k}\cdot\vec{r} - \omega_k t)} - A^*_{\vec{k}\lambda} \, e^{-i(\vec{k}\cdot\vec{r} - \omega_k t)} \right\} \quad (5.33)$$

Hamiltonian of Electromagnetic Field

Energy stored in electric and magnetic fields is:

$$W = \frac{1}{8\pi} \int_V \left(E^2 + B^2 \right) dV \quad (5.34)$$

Exercise: Substituting (5.32) + (5.33) into (5.34) show we get:

$$W = \frac{V}{2\pi c^2} \sum_{\vec{k}\lambda} \omega_k^2 \, A_{\vec{k}\lambda} A^*_{\vec{k}\lambda} \quad (5.35)$$

We now define $Q_{\vec{k}\lambda} \equiv \sqrt{\frac{V}{\pi}} \, \frac{1}{2c} \left(A_{\vec{k}\lambda} + A^*_{\vec{k}\lambda} \right)$ (5.36)

$$P_{\vec{k}\lambda} \equiv -i \sqrt{\frac{V}{\pi}} \, \frac{\omega_k}{2c} \left(A_{\vec{k}\lambda} - A^*_{\vec{k}\lambda} \right) \quad (5.37)$$

Exercise: Show $A_{\vec{k}\lambda} = \frac{c}{\omega_k} \sqrt{\frac{\pi}{V}} \left(\omega_k Q_{\vec{k}\lambda} + i P_{\vec{k}\lambda} \right)$ (5.38)

$$A^*_{\vec{k}\lambda} = \frac{c}{\omega_k} \sqrt{\frac{\pi}{V}} \left(\omega_k Q_{\vec{k}\lambda} - i P_{\vec{k}\lambda} \right) \quad (5.39)$$

Substituting (5.38) and (5.39) into (5.35) we get:

$$W = \frac{1}{2} \sum_{\vec{k}\lambda} \left(P^2_{\vec{k}\lambda} + \omega_k^2 Q^2_{\vec{k}\lambda} \right) \tag{5.40}$$

Hence each mode of the radiation field is equivalent to a single harmonic oscillator.

Quantization of W

We quantize using the commutation relations:

$$[Q_{\vec{k}\lambda} , P_{\vec{k}'\lambda'}] = i\hbar\, \delta_{\vec{k}\vec{k}'}\, \delta_{\lambda\lambda'} \tag{5.41a}$$

$$[Q_{\vec{k}\lambda} , Q_{\vec{k}'\lambda'}] = [P_{\vec{k}\lambda} , P_{\vec{k}'\lambda'}] = 0 \tag{5.41b}$$

where Q and P are now operators.

Define
$$a_{\vec{k}\lambda} \equiv \frac{1}{\sqrt{2\hbar\omega_k}} \left(\omega_k Q_{\vec{k}\lambda} + i P_{\vec{k}\lambda} \right) \tag{5.42a}$$

$$a^{\dagger}_{\vec{k}\lambda} \equiv \frac{1}{\sqrt{2\hbar\omega_k}} \left(\omega_k Q_{\vec{k}\lambda} - i P_{\vec{k}\lambda} \right) \tag{5.42b}$$

Exercise: Show (5.41) become:

$$[a_{\vec{k}\lambda} , a^{\dagger}_{\vec{k}'\lambda'}] = \delta_{\vec{k}\vec{k}'}\, \delta_{\lambda\lambda'} \tag{5.43a}$$

$$[a_{\vec{k}\lambda} , a_{\vec{k}'\lambda'}] = [a^{\dagger}_{\vec{k}\lambda} , a^{\dagger}_{\vec{k}'\lambda'}] = 0 \tag{5.43b}$$

Exercise: Show $\hbar\omega_k a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda} = \frac{1}{2}\left(\omega_k^2 Q^2_{\vec{k}\lambda} + P^2_{\vec{k}\lambda}\right) - \frac{1}{2}$ (5.44)

Using (5.44) in (5.40) we find the so called radiation Hamiltonian.

$$H_{rad} = \sum_{\vec{k}\lambda} \hbar\omega_k \left(a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda} + \frac{1}{2}\right) \tag{5.45}$$

$$= \sum_{\vec{k}\lambda} \hbar\omega_k \left(N_{\vec{k}\lambda} + \frac{1}{2}\right)$$

where $N_{\vec{k}\lambda} \equiv a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda}$ is the photon number operator.

The eigenstates are $|n_{\vec{k}\lambda}\rangle$ which has $n_{\vec{k}\lambda}$ photons in the $\vec{k}\lambda$ mode and corresponds to an energy

$$E = \sum_{\vec{k}\lambda} \hbar\omega_k \left(n_{\vec{k}\lambda} + \frac{1}{2}\right) \tag{5.46}$$

Note that when there aren't any photons i.e $n_{\vec{k}\lambda} = 0$

$$E = \sum_{\vec{k}\lambda} \frac{\hbar\omega_k}{2} = \infty \tag{5.47}$$

This is called the zero point energy.

Field Operators

Using expressions for $a^{+}_{\vec{k}\lambda}$ + $a_{\vec{k}\lambda}$, the vector potential operator is defined by:

$$\vec{A}_H = \sum_{\vec{k}\lambda} \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}}\, \hat{e}_{\vec{k}\lambda} \left\{ a_{\vec{k}\lambda}\, e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a^{+}_{\vec{k}\lambda}\, e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)} \right\}$$

This is a Heisenberg operator since it is time dependent. The corresponding Schrodinger operator is:

$$\vec{A} = \sum_{\vec{k}\lambda} \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}}\, \hat{e}_{\vec{k}\lambda} \left\{ a_{\vec{k}\lambda}\, e^{i\vec{k}\cdot\vec{r}} + a^{+}_{\vec{k}\lambda}\, e^{-i\vec{k}\cdot\vec{r}} \right\} \qquad (5.48)$$

Similarly the field operators are:

$$\vec{E} = \frac{i}{c} \sum_{\vec{k}\lambda} \sqrt{\frac{2\pi\hbar c^2\omega_k}{V}}\, \hat{e}_{\vec{k}\lambda} \left\{ a_{\vec{k}\lambda}\, e^{i\vec{k}\cdot\vec{r}} - a^{+}_{\vec{k}\lambda}\, e^{-i\vec{k}\cdot\vec{r}} \right\} \qquad (5.49)$$

$$\vec{B} = \frac{i}{c} \sum_{\vec{k}\lambda} \sqrt{\frac{2\pi\hbar c^2\omega_k}{V}}\, (\hat{k}\times\hat{e}_{\vec{k}\lambda}) \left\{ a_{\vec{k}\lambda}\, e^{i\vec{k}\cdot\vec{r}} - a^{+}_{\vec{k}\lambda}\, e^{-i\vec{k}\cdot\vec{r}} \right\} \qquad (5.50)$$

Interaction of Radiation and Atoms

Hamiltonian

$$H = H_{rad} + H_{atom} + H_{int} \tag{5.51}$$

where $H_{rad} = \sum_{\vec{k}\lambda} \hbar\omega_k \left(a^{+}_{\vec{k}\lambda} a_{\vec{k}\lambda} + \frac{1}{2} \right)$

$$H_{atom} = \frac{\vec{p}^{\,2}}{2m} + V$$

↖ Coulomb, spin orbit etc.

H_{int} is obtained by replacing $\vec{p} \to \vec{p} + \frac{e}{c}\vec{A}$ in H_{atom} where $-e$ is the electron charge.

Exercise: Show $H_{int} = H_1 + H_2$ (5.52)

where $H_1 \equiv \frac{e}{mc} \vec{p} \cdot \vec{A}$

$$H_2 \equiv \frac{e^2}{2mc^2} A^2$$

Substituting (5.48) we obtain:

$$H_1 = \frac{e}{m} \sum_{\vec{k}\lambda} \sqrt{\frac{2\pi\hbar}{\omega_k V}} \left(\hat{e}_{\vec{k}\lambda} \cdot \vec{p} \right) \left[a_{\vec{k}\lambda} e^{i\vec{k}\cdot\vec{r}} + a^{+}_{\vec{k}\lambda} e^{-i\vec{k}\cdot\vec{r}} \right] \tag{5.53}$$

$$H_2 = \frac{\pi e^2 \hbar}{mV} \sum_{\vec{k}\lambda} \sum_{\vec{k}'\lambda'} \frac{1}{\sqrt{\omega_k \omega_{k'}}} \left(\hat{e}_{\vec{k}\lambda} \cdot \hat{e}_{\vec{k}'\lambda'} \right)$$

$$\cdot \left(a_{\vec{k}\lambda} e^{i\vec{k}\cdot\vec{r}} + a^{+}_{\vec{k}\lambda} e^{-i\vec{k}\cdot\vec{r}} \right) \left(a_{\vec{k}'\lambda'} e^{i\vec{k}'\cdot\vec{r}} + a^{+}_{\vec{k}'\lambda'} e^{-i\vec{k}'\cdot\vec{r}} \right) \tag{5.54}$$

Emission of Radiation

Fermi's Golden Rule

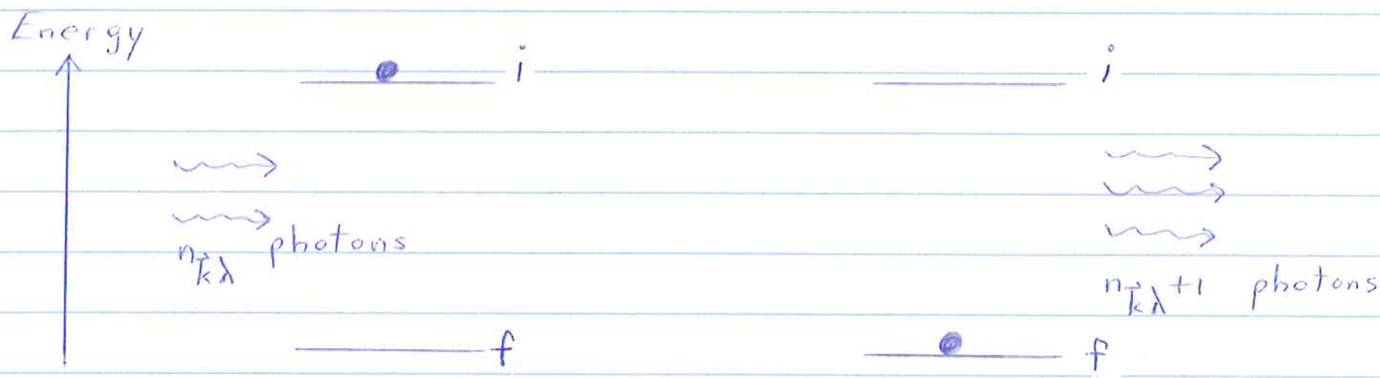
Transition rate from initial state I to final state F is:

$$T_{FI} = \frac{2\pi}{\hbar} |M_{FI}|^2 \, \delta(E_F - E_I) \qquad (5.55)$$

The matrix element is given by the perturbative expansion

$$M_{FI} = \langle F | H_{int} | I \rangle + \sum_n \frac{\langle F | H_{int} | n \rangle \langle n | H_{int} | I \rangle}{E_I - E_n} + \ldots \qquad (5.56)$$

We shall consider an atom that makes a transition from an initial state i to a final state f.



Initial State $|I\rangle = |\psi_i ; n_{\vec{k}\lambda}\rangle$ Final State $|F\rangle = |\psi_f ; n_{\vec{k}\lambda}+1\rangle$

Energy $E_I = E_i + n_{\vec{k}\lambda} \hbar\omega_k$ $\qquad E_F = E_f + (n_{\vec{k}\lambda}+1)\hbar\omega_k$

Exercise: Explain why H_2 does not contribute to T_{fi} in first order perturbation theory.

The only nonzero matrix element is:

$$\langle F | H_{int} | I \rangle = \langle \psi_f ; n_{\vec{k}\lambda}+1 | \frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega_k V}}\ \hat{e}_{\vec{k}\lambda}\cdot\vec{p}\ a^{\dagger}_{\vec{k}\lambda}\, e^{-i\vec{k}\cdot\vec{r}} | \psi_i ; n_{\vec{k}\lambda} \rangle$$

$$= \frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega_k V}}\ \sqrt{n_{\vec{k}\lambda}+1}\ \langle \psi_f | \hat{e}_{\vec{k}\lambda}\cdot\vec{p}\, e^{-i\vec{k}\cdot\vec{r}} | \psi_i \rangle$$

Substituting this in (5.55) yields an emission rate:

$$\Gamma_{emis} = \frac{2\pi}{\hbar}\left(\frac{e}{m}\right)^2 \sum_{\vec{k}} \frac{2\pi\hbar}{V\omega_k}\,(n_{\vec{k}\lambda}+1)\,|\langle \psi_f | \hat{e}_{\vec{k}\lambda}\cdot\vec{p}\, e^{-i\vec{k}\cdot\vec{r}} | \psi_i \rangle|^2$$

$$\cdot\, \delta(E_f + \hbar\omega_k - E_i)$$

where we have summed over all possible photon momentum $\vec{k}$. Using (5.28) this summation can be replaced by an integral:

$$\sum_{\vec{k}} \longrightarrow \int \frac{V}{(2\pi)^3}\, k^2\, dk\, d\Omega$$

$$\therefore\ \Gamma_{emis} = \frac{2\pi}{\hbar}\left(\frac{e}{m}\right)^2 \frac{V}{(2\pi)^3}\int k^2\, dk\, d\Omega\ \frac{2\pi\hbar}{V\omega_k}\,(n_{\vec{k}\lambda}+1)$$

$$\cdot\, |\langle \psi_f | \hat{e}_{\vec{k}\lambda}\cdot\vec{p}\, e^{-i\vec{k}\cdot\vec{r}} | \psi_i \rangle|^2\, \delta(E_f + \hbar\omega_k - E_i)$$

The k integral removes the delta function since $\omega_k = ck$. Hence $\hbar\omega_k = E_i - E_f$ ensuring energy conservation.

$$\therefore \; \Gamma_{emis} = \left(\frac{e}{m}\right)^2 \frac{(n_{\vec{k}\lambda}+1)}{2\pi\hbar c^3}\,\omega \int d\Omega \; |\langle \psi_f | \hat{e}_{\vec{k}\lambda}\cdot\vec{p}\, e^{-i\vec{k}\cdot\vec{r}} |\psi_i\rangle|^2 \qquad (5.57)$$

Now $e^{-i\vec{k}\cdot\vec{r}} = 1 - i\vec{k}\cdot\vec{r} + \ldots$

Exercise: Show for an optical transition in an atom that $\vec{k}\cdot\vec{r} \ll 1$.

Hence we can make the so-called electric dipole approximation $e^{-i\vec{k}\cdot\vec{r}} \simeq 1$. The matrix element then becomes:

$$\hat{e}_{\vec{k}\lambda}\cdot\langle\psi_f|\vec{p}|\psi_i\rangle$$

Exercise: Using $\vec{p} = -\frac{im}{\hbar}[\vec{r}, H_{atom}]$

show $\langle\psi_f|\vec{p}|\psi_i\rangle = -\frac{im}{\hbar}(E_i - E_f)\langle\psi_f|\vec{r}|\psi_i\rangle$

$$= -im\omega\langle\psi_f|\vec{r}|\psi_i\rangle \qquad (5.58)$$

Exercise: Derive the electric dipole selection rules.
$\Delta l = \pm 1$, $\Delta m = 0, \pm 1$
where l is orbital angular momentum and m is its azimuthal component.

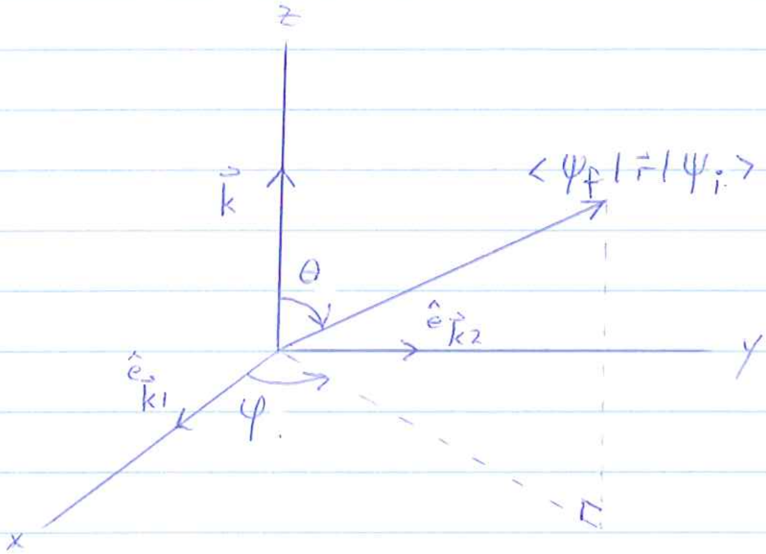
Using (5.58) in (5.57) one gets:

$$\Gamma_{emis} = \frac{\alpha\,(n_{\vec{k}\lambda}+1)\,\omega^3}{2\pi c^2}\int d\Omega \; |\hat{e}_{\vec{k}\lambda}\cdot\langle\psi_f|\vec{r}|\psi_i\rangle|^2 \qquad (5.59)$$

This is the emission rate of a photon having polarization described by $\hat{e}_{\vec{k}\lambda}$.

Sum Over Photon Polarizations

We consider the case where the detector is insensitive to the photon polarization. Letting $\vec{k}$ define the $\hat{z}$ direction, the possible polarization vectors are $\hat{e}_{\vec{k}1} = \hat{x}$ and $\hat{e}_{\vec{k}2} = \hat{y}$.



$$\hat{e}_{\vec{k}1} \cdot \langle \psi_f | \vec{r} | \psi_i \rangle = |\langle \psi_f | \vec{r} | \psi_i \rangle| \sin\theta \cos\varphi$$

$$\hat{e}_{\vec{k}2} \cdot \langle \psi_f | \vec{r} | \psi_i \rangle = |\langle \psi_f | \vec{r} | \psi_i \rangle| \sin\theta \sin\varphi$$

$$\therefore \sum_\lambda |\hat{e}_{\vec{k}\lambda} \cdot \langle \psi_f | \vec{r} | \psi_i \rangle|^2 = |\langle \psi_f | \vec{r} | \psi_i \rangle|^2 \sin^2\theta \qquad (5.60)$$

Exercise: Show angular integral $\int d\Omega \, \sin^2\theta = \frac{8\pi}{3}$ (5.61)

Substituting (5.60) and (5.61) into (5.59) yields:

$$\Gamma_{emis} = \frac{4\alpha\omega^3}{3c^2}(n+1)\,|\langle\psi_f|\vec{r}|\psi_i\rangle|^2 \qquad (5.62)$$

Hence photons can be emitted spontaneously (when no photons are present) or via stimulated emission due to $n \neq 0$.

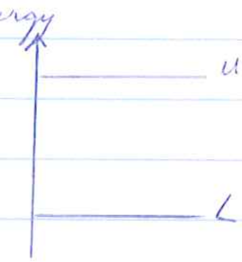
Exercise: Consider the transition $2P \to 1S$ in hydrogen. Show the radiative lifetime $\tau(2P\to 1S) = 1.6$ nsec.

Exercise: Derive the transition rate for photon absorption

$$\Gamma_{abs} = \frac{4\alpha\omega^3}{3c^2}\, n\, |\langle\psi_f|\vec{r}|\psi_i\rangle|^2 \qquad (5.63)$$

Planck Radiation Law

Consider the case where photon absorption and emission processes are in equilibrium.



$$\text{i.e. } \psi_u \rightleftarrows \psi_L + \gamma$$

Let $\begin{Bmatrix} N_u \\ N_L \end{Bmatrix}$ be # atoms in state $\begin{Bmatrix} \psi_u \\ \psi_L \end{Bmatrix}$.

Equilibrium $\Rightarrow$ # photons emitted per sec. = # photons absorbed per sec.

$$N_U \, \Gamma_{emis} = N_L \, \Gamma_{abs}$$

Using (5.62) and (5.63) one gets:

$$\frac{N_u}{N_L} = \frac{n_k}{n_k + 1} \qquad (5.64)$$

But thermal equilibrium gives $\dfrac{N_u}{N_L} = e^{-\hbar\omega/kT}$ (5.65)

Exercise: Using (5.64) + (5.65) show the # photons in mode k is $n_k = \dfrac{1}{e^{\hbar\omega/kT} - 1}$ (5.66)

Energy of photons with wavevector between k + k+dk   dE = photon energy × # photons per mode × # modes

$$dE = \hbar\omega \times \frac{1}{e^{\hbar\omega/kT} - 1} \times 2 \cdot 4\pi k^2 dk \frac{V}{(2\pi)^3}$$

sum over 2 polarizations

Exercise: Show energy per unit frequency per unit volume $U \equiv \frac{dE}{V d\nu}$ is:

$$\boxed{U(\nu) = \frac{8\pi h (\nu/c)^3}{e^{h\nu/kT} - 1} \quad \text{Planck Law}} \qquad (5.67)$$

Higher Order Electromagnetic Interactions

Raman Scattering

An atom absorbs a photon $|\vec{k}\lambda\rangle$ and emits a photon $|\vec{k}'\lambda'\rangle$.

initial state $|I\rangle = |\psi_i ; n_{\vec{k}\lambda}, n_{\vec{k}'\lambda'}\rangle$

final state $|F\rangle = |\psi_f ; n_{\vec{k}\lambda}-1, n_{\vec{k}'\lambda'}+1\rangle$

This process occurs via Hamiltonian H_2 (5.54) acting in first order or by H_1 (5.53) acting in second order.

Exercise: Explain why Raman scattering can't occur by having H_1 act in first order.

Exercise: Derive the following matrix element.

$$\langle F | H_2 | I \rangle = \frac{2\pi e^2 \hbar}{mV} \sqrt{\frac{n_{\vec{k}\lambda}(n_{\vec{k}'\lambda'}+1)}{\omega_k \, \omega_{k'}}} \, (\hat{e}_{\vec{k}\lambda} \cdot \hat{e}_{\vec{k}'\lambda'}) \langle \psi_f | e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} | \psi_i \rangle$$

Using the dipole approximation this becomes:

$$\langle F | H_2 | I \rangle = \frac{2\pi e^2 \hbar}{mV} \sqrt{\frac{n_{\vec{k}\lambda}(n_{\vec{k}'\lambda'}+1)}{\omega_k \, \omega_{k'}}} \, (\hat{e}_{\vec{k}\lambda} \cdot \hat{e}_{\vec{k}'\lambda'}) \, \delta_{fi} \qquad (5.68)$$

Action of H_1 in Second Order

Atom in state ψ_i absorbs (emits) photon $|\vec{k}\lambda\rangle$ ($|\vec{k}'\lambda'\rangle$) and goes to intermediate state ψ_{L_1} (ψ_{L_2}) which emits (absorbs) photon $|\vec{k}'\lambda'\rangle$ ($|\vec{k}\lambda\rangle$) and atom ends up in final state ψ_f.

Illustration

$|I\rangle = |\psi_i ; n_{\vec{k}\lambda}, n_{\vec{k}'\lambda'}\rangle$

$|L_1\rangle = |\psi_{\ell_1} ; n_{\vec{k}\lambda} - 1, n_{\vec{k}'\lambda'}\rangle$

$|L_2\rangle = |\psi_{\ell_2} ; n_{\vec{k}\lambda}, n_{\vec{k}'\lambda'} - 1\rangle$

$|F\rangle = |\psi_f ; n_{\vec{k}\lambda} - 1, n_{\vec{k}\lambda} + 1\rangle$

Exercise: Show the following:

$\langle F| H_1 |L_1\rangle\langle L_1| H_1 |I\rangle$

$$= \frac{2\pi e^2 \hbar}{m^2 V} \sqrt{\frac{n_{\vec{k}\lambda}(n_{\vec{k}'\lambda'}+1)}{\omega_k \, \omega_{k'}}} \langle\psi_f| \hat{e}_{\vec{k}'\lambda'} \cdot \vec{p} \, e^{-i\vec{k}'\cdot\vec{r}} |\psi_{\ell_1}\rangle\langle\psi_{\ell_1}| \hat{e}_{\vec{k}\lambda} \cdot \vec{p} \, e^{i\vec{k}\cdot\vec{r}} |\psi_i\rangle$$

Similarly:

$\langle F | H_1 | L_2 \rangle \langle L_2 | H_1 | I \rangle$

$$= \frac{2\pi e^2 \hbar}{m^2 V} \sqrt{\frac{n_{\vec{k}\lambda}(n_{\vec{k}'\lambda'}+1)}{\omega_k \omega_{k'}}} \langle \psi_f | \hat{e}_{\vec{k}\lambda} \cdot \vec{p} e^{i\vec{k}\cdot\vec{r}} | \psi_{\ell_2} \rangle \langle \psi_{\ell_2} | \hat{e}_{\vec{k}'\lambda'} \cdot \vec{p} e^{-i\vec{k}'\cdot\vec{r}} | \psi_i \rangle$$

$$\therefore \sum_L \frac{\langle F | H_1 | L \rangle \langle L | H_1 | I \rangle}{E_I - E_L} = \frac{2\pi e^2 \hbar}{m^2 V} \sqrt{\frac{n_{\vec{k}\lambda}(n_{\vec{k}'\lambda'}+1)}{\omega_k \omega_{k'}}}$$

$$\cdot \sum_\ell \left[\frac{(\hat{e}_{\vec{k}'\lambda'} \cdot \vec{p}_{f\ell})(\hat{e}_{\vec{k}\lambda} \cdot \vec{p}_{\ell i})}{E_i - E_\ell + \hbar\omega_k} + \frac{(\hat{e}_{\vec{k}\lambda} \cdot \vec{p}_{f\ell})(\hat{e}_{\vec{k}'\lambda'} \cdot \vec{p}_{\ell i})}{E_i - E_\ell - \hbar\omega_{k'}} \right] \tag{5.69}$$

where we have used the electric dipole approximation and defined $\vec{p}_{ij} \equiv \langle \psi_i | \vec{p} | \psi_j \rangle$.

The matrix element for Raman scattering is obtained by adding (5.68) + (5.69).

$$M_{FI} = \frac{2\pi e^2 \hbar}{m V} \sqrt{\frac{n_{\vec{k}\lambda}(n_{\vec{k}'\lambda'}+1)}{\omega_k \omega_{k}'}} \cdot \left\{ (\hat{e}_{\vec{k}\lambda} \cdot \hat{e}_{\vec{k}'\lambda'})\, \delta_{fi} \right.$$

$$\left. + \frac{1}{m} \sum_\ell \left[\frac{(\hat{e}_{\vec{k}'\lambda'} \cdot \vec{p}_{f\ell})(\hat{e}_{\vec{k}\lambda} \cdot \vec{p}_{\ell i})}{E_i - E_\ell + \hbar\omega_k} + \frac{(\hat{e}_{\vec{k}\lambda} \cdot \vec{p}_{f\ell})(\hat{e}_{\vec{k}'\lambda'} \cdot \vec{p}_{\ell i})}{E_i - E_\ell - \hbar\omega_{k'}} \right] \right\} \tag{5.70}$$

The transition rate is found using Fermi's Golden Rule.

$$T_{FI} = \frac{2\pi}{\hbar} \, |M_{FI}|^2 \, \delta(E_I - E_F)$$

Summing over all possible scattered photon momentum $\vec{k}'$ gives:

$$T_{FI} = \frac{2\pi}{\hbar} \sum_{\vec{k}'} |M_{FI}|^2 \, \delta(E_I - E_F)$$

Next, the summation is replaced by an integral.

$$T_{FI} = \frac{2\pi}{\hbar} \frac{V}{(2\pi)^3} \int k'^2 \, dk' \, d\Omega \, |M_{FI}|^2 \, \delta(E_i + \hbar\omega_k - E_f - \hbar\omega_{k'})$$

$$= \frac{2\pi}{\hbar} \frac{V}{(2\pi)^3} \frac{\omega_{k'}^2}{\hbar c^3} \, d\Omega \, |M_{FI}|^2$$

where the delta function yields $\hbar\omega_{k'} = E_i - E_f + \hbar\omega_k$. Substituting (5.70) gives:

$$T_{FI} = r_e^2 \frac{c}{V} \frac{\omega_{k'}}{\omega_k} \, d\Omega \, n_{\vec{k}\lambda}(n_{\vec{k}'\lambda'}+1)$$

$$\cdot \left| (\hat{e}_{\vec{k}\lambda} \cdot \hat{e}_{\vec{k}'\lambda'}) \delta_{fi} + \frac{1}{m} \sum_{\ell} \left[\frac{(\hat{e}_{\vec{k}'\lambda'} \cdot \vec{p}_{f\ell})(\hat{e}_{\vec{k}\lambda} \cdot \vec{p}_{\ell i})}{E_i - E_\ell + \hbar\omega_k} \right. \right. \quad (5.71)$$

$$\left. \left. + \frac{(\hat{e}_{\vec{k}\lambda} \cdot \vec{p}_{f\ell})(\hat{e}_{\vec{k}'\lambda'} \cdot \vec{p}_{\ell i})}{E_i - E_\ell - \hbar\omega_{k'}} \right] \right|^2$$

<u>Kramers-Heisenberg Dispersion Formula.</u>

where $r_e = e^2/mc^2$ is classical electron radius.

Cross Section

$$d\sigma = \frac{\text{rate of photon scatter into } d\Omega}{\text{incoming photon flux}}$$

$$d\sigma = \frac{\Gamma_{FI}}{n_{\vec{k}\lambda}\frac{c}{V}}$$

Using (5.71) we get:

$$\frac{d\sigma}{d\Omega} = r_e^2 \frac{\omega_{k'}}{\omega_k}(n_{\vec{k}'\lambda'}+1)\Bigg|(\hat{e}_{\vec{k}\lambda}\cdot\hat{e}_{\vec{k}'\lambda'})\,\delta_{fi} \tag{5.72}$$

$$+\frac{1}{m}\sum_{\ell}\left[\frac{(\hat{e}_{\vec{k}'\lambda'}\cdot\vec{p}_{f\ell})(\hat{e}_{\vec{k}\lambda}\cdot\vec{p}_{\ell i})}{E_i-E_\ell+\hbar\omega_k}+\frac{(\hat{e}_{\vec{k}\lambda}\cdot\vec{p}_{f\ell})(\hat{e}_{\vec{k}'\lambda'}\cdot\vec{p}_{\ell i})}{E_i-E_\ell-\hbar\omega_{k'}}\right]\Bigg|^2$$

Note that there is a stimulated term ($\propto n_{\vec{k}'\lambda'}$) and a "spontaneous" part of Raman scattering. Before continuing our simplification of (5.72) we shall consider the special case of Thomson scattering.

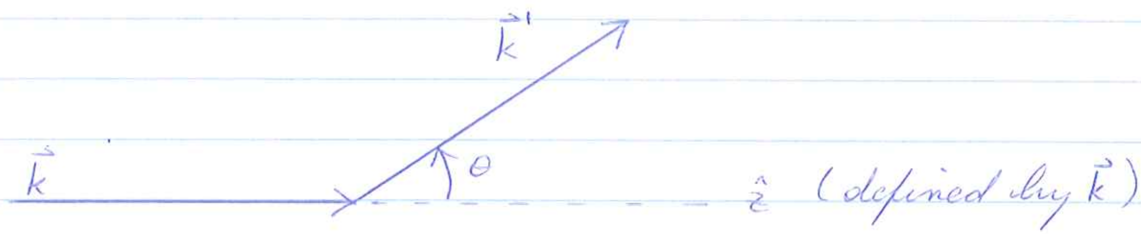
Thomson Scattering

We consider elastic scattering ($\omega_k = \omega_{k'}$) of high energy photons $\hbar\omega \gg |E_i - E_\ell|$. (5.72) then simplifies to:

$$\frac{d\sigma}{d\Omega} = r_e^2\,\delta_{fi}\,(\hat{e}_{\vec{k}\lambda}\cdot\hat{e}_{\vec{k}'\lambda'})^2 \tag{5.73}$$

Note that the atom is now unaffected. i.e. the process looks like elastic photon scattering off free electrons.

We shall find the cross section for the case when the incoming photons are unpolarized and the detector is insensitive to the polarization of the scattered photon.



Incident Photon

$\hat{k} = (0, 0, 1)$
$\hat{e}_{k1} = (1, 0, 0)$
$\hat{e}_{k2} = (0, 1, 0)$

Scattered Photon

$\hat{k}' = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$
$\hat{e}_{k'1} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$

$\hat{e}_{k'2} = (\sin\varphi, -\cos\varphi, 0)$

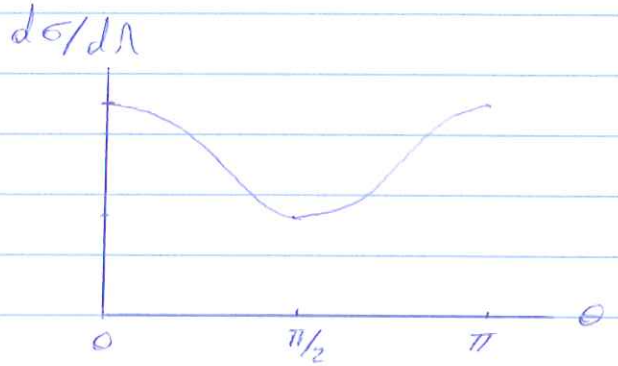
Exercise: Verify that $\hat{e}_{k'\lambda} \cdot \hat{k}' = 0$.

The cross section is found by averaging (5.73) over the incident photon polarizations and summing over the scattered photon polarizations.

i.e. $$\frac{d\sigma}{d\Omega} = \frac{1}{2} \sum_{\lambda\lambda'} r_e^2 \left(\hat{e}_{k\lambda} \cdot \hat{e}_{k'\lambda'}\right)^2$$

$$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2}\Big\{ \cos^2\theta\cos^2\varphi + \sin^2\varphi$$
$$+ \cos^2\theta\sin^2\varphi + \cos^2\varphi \Big\}$$

$$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2}\left(1 + \cos^2\theta\right)$$



The cross section is maximum for scattering in the forward or backward directions.

Exercise: a) Show total cross section is

$$\sigma = \frac{8\pi}{3} r_e^2$$ Thomson Scattering Cross Section

b) Evaluate σ in cm^2.

Raman Scattering Cross Section

We shall rewrite (5.72) using the following simplifying notation.

$$\hat{e} \equiv \hat{e}_{\vec{k}\lambda} \qquad \hat{e}' \equiv \hat{e}_{\vec{k}'\lambda'} \qquad \omega \equiv \omega_k \qquad \omega' \equiv \omega_{k'} \qquad n' \equiv n_{\vec{k}'\lambda'}$$

$$\frac{d\sigma}{d\Omega} = r_e^2 \frac{\omega'}{\omega}(n'+1) \left| (\hat{e}\cdot\hat{e}')^2 \delta_{fi} + \frac{1}{m}\sum_l \left[\frac{(\hat{e}'\cdot\vec{p}_{fl})(\hat{e}\cdot\vec{p}_{li})}{E_i - E_l + \hbar\omega} + \frac{(\hat{e}\cdot\vec{p}_{fl})(\hat{e}'\cdot\vec{p}_{li})}{E_i - E_l - \hbar\omega'} \right] \right|^2$$

We shall rewrite this using:

1) $\vec{p}_{ij} = \frac{im}{\hbar}(E_i - E_j)\langle i|\vec{r}|j\rangle$

2) $\hat{e}\cdot\hat{e}'\delta_{fi} = \frac{m}{\hbar^2}\sum_l \Big[(E_l - E_i + \hbar\omega')\langle f|\hat{e}'\cdot\vec{r}|l\rangle\langle l|\hat{e}\cdot\vec{r}|i\rangle - (E_f - E_l + \hbar\omega')\langle f|\hat{e}\cdot\vec{r}|l\rangle\langle l|\hat{e}'\cdot\vec{r}|i\rangle \Big]$

This will be shown in the homework.

Using these results, the cross section becomes:

$$\frac{d\sigma}{d\Omega} = r_e^2 \frac{\omega'}{\omega}(n'+1)\frac{m^2}{\hbar^4} \cdot \Bigg| \sum_l \Bigg\{ \langle f|\hat{e}'\cdot\vec{r}|l\rangle\langle l|\hat{e}\cdot\vec{r}|i\rangle \left[(E_l - E_i + \hbar\omega') - \frac{(E_f - E_l)(E_l - E_i)}{E_i - E_l + \hbar\omega} \right]$$

$$- \langle f|\hat{e}\cdot\vec{r}|l\rangle\langle l|\hat{e}'\cdot\vec{r}|i\rangle \left[(E_f - E_l + \hbar\omega') - \frac{(E_f - E_l)(E_l - E_i)}{E_i - E_l - \hbar\omega'} \right] \Bigg\} \Bigg|^2$$

This can be simplified to give:

$$\frac{d\sigma}{d\Omega} = r_e^2 \; \omega \omega'^3 \; (n'+1) \; m^2 \qquad (5.74)$$

$$\cdot \left| \sum_l \left\{ \frac{\langle f|\hat{e}'\cdot\vec{r}|l\rangle\langle l|\hat{e}\cdot\vec{r}|i\rangle}{E_i - E_l + \hbar\omega} + \frac{\langle f|\hat{e}\cdot\vec{r}|l\rangle\langle l|\hat{e}'\cdot\vec{r}|i\rangle}{E_i - E_l - \hbar\omega'} \right\} \right|^2$$

If $\omega' < \omega$ the scattered photon is said to be Stokes shifted. Similarly when $\omega' > \omega$, it is said to be anti-Stokes shifted.

Rayleigh Scattering

This occurs when $\omega = \omega'$ (i.e. $i = f$) for low-energy photons such that $\hbar\omega \ll |E_i - E_l|$. (5.74) then becomes:

$$\frac{d\sigma}{d\Omega} = r_e^2 m^2 \omega^4 \left| \sum_l \left[\frac{\langle i|\hat{e}'\cdot\vec{r}|l\rangle\langle l|\hat{e}\cdot\vec{r}|i\rangle}{E_i - E_l} + \frac{\langle i|\hat{e}\cdot\vec{r}|l\rangle\langle l|\hat{e}'\cdot\vec{r}|i\rangle}{E_i - E_l} \right] \right|^2 \qquad (5.75)$$

Note $\frac{d\sigma}{d\Omega} \propto \omega^4$ which is responsible for a blue sky and red sunset.

Chapter 5 Assignment

1a) Show $N = \sum_i a_i^+ a_i$ gives the total number of particles when operating on $\Psi = |n_1\, n_2 \cdots n_k\rangle$.

b) Show $|n_1\, n_2 \cdots\rangle = \cdots \frac{(a_2^+)^{n_2}}{\sqrt{n_2!}} \frac{(a_1^+)^{n_1}}{\sqrt{n_1!}} |0_1, 0_2 \cdots\rangle$ is normalized.

2a) Evaluate $\langle n_k | \vec{E} | n_k \rangle$.

b) How do nonzero electric fields arise?

3) $H_1 = \frac{e}{mc} \vec{p} \cdot \vec{A} \qquad H_2 = \frac{e^2}{2mc^2} A^2$

Estimate the laser power such that $H_1 \sim H_2$.

4) Derive the following expression for the $2p \rightarrow 1s$ transition in a hydrogenic atom.

$$\Gamma = \left(\frac{2}{3}\right)^8 \frac{mc^2}{\hbar} \alpha (Z\alpha)^4$$

For H, show $\tau_{2p} = 1.596$ nsec. (Experimental result is 1.600 ± 0.004 nsec.)

5) Derive the expression for absorption rate given by (5.63)

6) Thomson Cross Section $\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2}(1 + \cos^2\theta)$

Integrate $\frac{d\sigma}{d\Omega}$ to get the total cross section + evaluate it.

VI Relativistic Quantum Electrodynamics (QED)

This chapter describes the relativistic theory describing the interaction of photons and matter. The derivations are not always rigorous due to time constraints.

Nonrelativistic Theory

The coupling of radiation with electrons is given by:

$$H = \frac{e\hbar}{mc} \vec{A} \cdot \vec{p} \tag{6.1}$$

where the electron has mass m and charge $-e$.
For a quantized radiation field, the vector potential $\vec{A}$ was found to be:

$$\vec{A} = \sum_{\vec{k}\lambda} \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}}\, \hat{e}_{\vec{k}\lambda} \left\{ a_{\vec{k}\lambda} e^{ikx} + a^{+}_{\vec{k}\lambda} e^{-ikx} \right\} \tag{6.2}$$

where $kx \equiv k^\mu x_\mu$.

Relativistic Generalization

The obvious generalization of (6.1) is:

$$H_\gamma = \frac{e\hbar}{mc} A^\mu p_\mu \tag{6.3}$$

where $A^\mu = (\Phi, \vec{A})$. Unlike $\vec{A}$ given in (6.2), A^μ includes effects of static Coulomb fields that do not radiate.

The Hamiltonian describing quantized electrons and photons is given by:

$$H_{e\gamma} = \int d^4x \;\bar{\psi}(x)\; \frac{e\hbar}{mc}\; A^\mu p_\mu \;\psi(x) \qquad (6.4)$$

where $\psi(x)$ is the general plane wave solution of the Dirac equation. It will be shown in the homework to be given by:

$$\psi(\vec{x},t) = \sum_{ps} \sqrt{\frac{m}{VE_p}} \left\{ e^{ipx} u_s(p)\, b_{ps} + e^{-ipx} v_s(p)\, d^\dagger_{ps} \right\} \qquad (6.5a)$$

The adjoint is:

$$\psi^\dagger(\vec{x},t) = \sum_{ps} \sqrt{\frac{m}{VE_p}} \left\{ e^{-ipx} u^\dagger_s(p)\, b^\dagger_{ps} + e^{ipx} v^\dagger_s(p)\, d_{ps} \right\} \qquad (6.5b)$$

b_{ps} ($b^\dagger_{ps}$) destroys (creates) an electron of momentum p + spin s.

d_{ps} ($d^\dagger_{ps}$) " " positron "

The operators satisfy the fermion commutators

$$\left\{ b^\dagger_{ps},\, b_{p's'} \right\} = \delta_{ss'}\,\delta_{pp'}$$

$$\left\{ d^\dagger_{ps},\, d_{p's'} \right\} = \delta_{ss'}\,\delta_{pp'} \qquad (6.6)$$

Mixed commutators of b and d such as $\left\{ d_{ps},\, b^\dagger_{p's'} \right\} = 0$.

We now substitute $\bar{\psi}$, A^μ, ψ into (6.4). For simplicity we only write the exponential factors and Fock operators.

$$H_{e\gamma} \propto \int d^4x \sum_{p'} \left(e^{-ip'x} b^+_{p'} + e^{ip'x} d_{p'} \right) \cdot \sum_k \left(e^{ikx} a_k + e^{-ikx} a^+_k \right) \cdot \sum_p \left(e^{ipx} b_p + e^{-ipx} d^+_p \right) \tag{6.7}$$

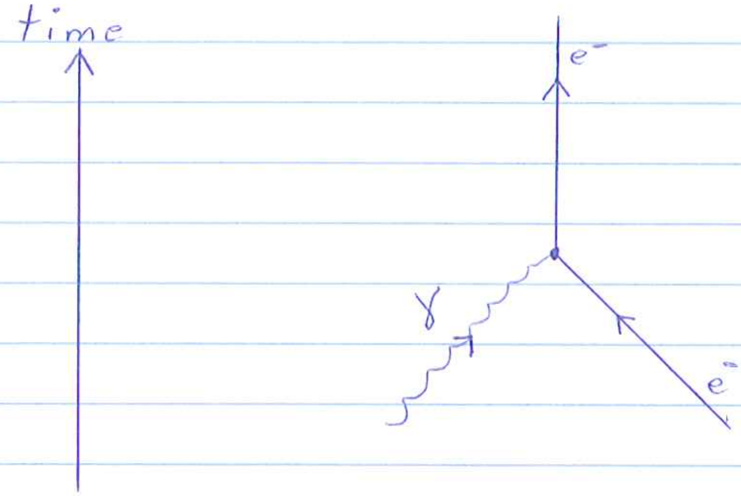
There are 8 terms in (6.7) such as:

$$\int d^4x \; e^{i(-p'+k+p)x} \; b^+_{p'} \, a_k \, b_p \tag{6.8}$$

Here: b_p destroys an electron having energy momentum p
a_k " photon " " k
$b^+_{p'}$ creates an electron " " p'

The integral is zero unless $p' = k+p$ i.e. energy and momentum are conserved.

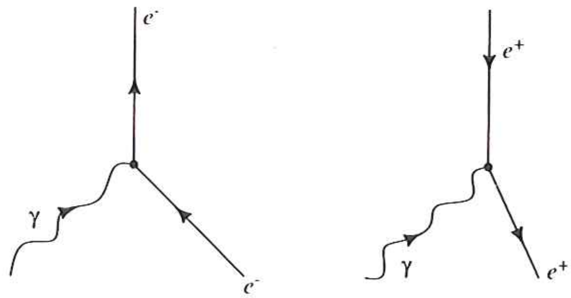
Schematic Representation of $b^\dagger b a$



This so called Feynman diagram neatly summarizes all the algebra contained in terms of $H_{e\gamma}$. The point where the 3 lines (2 electron and 1 photon) intersect, is called a vertex. Positrons are represented as electrons with the arrow reversed since they can be viewed as "electrons travelling backwards in time." The Feynman diagrams corresponding to the eight terms of (6.7) are shown on the next page.

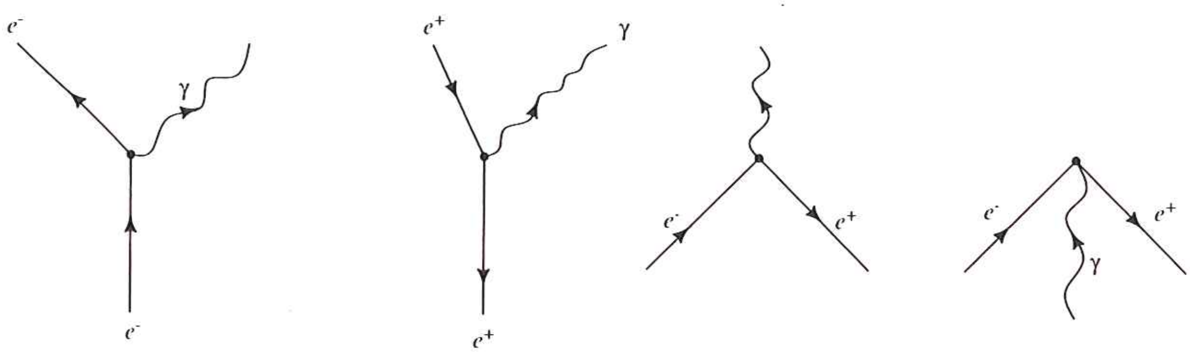
First Order Feynman Diagrams

$b^\dagger b a$ $\quad$ $d d^\dagger a$ $\quad$ $b^\dagger d^\dagger a$ $\quad$ $b^\dagger d^\dagger a^\dagger$



Hermitian adjoint :

$b b^\dagger a^\dagger$ $\quad$ $d^\dagger d a^\dagger$ $\quad$ $d b a^\dagger$ $\quad$ $d b a$



Let us examine the diagram for $dba^\dagger$ in the e^-e^+ center of mass frame.

e^- has energy momentum $\left(\frac{E}{c}, \vec{p}\right)$

e^+ " " $\left(\frac{E}{c}, -\vec{p}\right)$

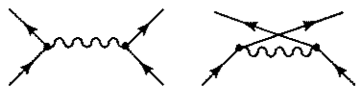
$\therefore$ γ has energy momentum $\left(\frac{2E}{c}, 0\right)$, i.e. γ is a photon without momentum! γ therefore cannot be a "real" photon that can be detected by a photomultiplier and is said to be a virtual particle. Hence the above first order diagrams don't occur by themselves. However two or more first order diagrams can be combined into higher order diagrams that correspond to observed processes.

CATALOG OF BASIC QUANTUM ELECTRODYNAMIC PROCESSES

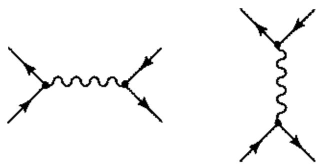
Second-order processes

Elastic

Electron–muon scattering ($e + \mu \rightarrow e + \mu$)
(Mott scattering ($M \gg m$) $\Rightarrow$ Rutherford scattering ($v \ll c$))

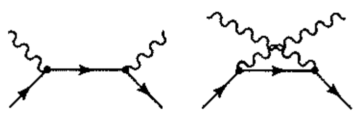


Electron–electron scattering ($e^- + e^- \rightarrow e^- + e^-$)
(Møller scattering)



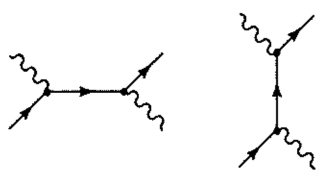
Electron–positron scattering ($e^- + e^+ \rightarrow e^- + e^+$)
(Bhabha scattering)

Inelastic



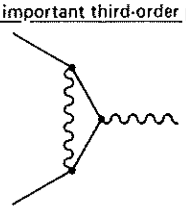
Pair annihilation ($e^- + e^+ \rightarrow \gamma + \gamma$)

Pair production ($\gamma + \gamma \rightarrow e^- + e^+$)



Compton scattering ($\gamma + e^- \rightarrow \gamma + e^-$)

Most important third-order process



$\Rightarrow$ Anomalous magnetic moment of electron

Transition Rate, Cross Section & Matrix Element

The transition rate from initial state i to final state f is given by Fermi's Golden Rule:

$$T_{fi} = \frac{2\pi}{\hbar} |M_{fi}|^2 \cdot (\text{phase space}) \qquad (6.9)$$

where the matrix element is given by:

$$M_{fi} = \langle f | H_{e\gamma} | i \rangle + \sum_n \frac{\langle f | H_{e\gamma} | n \rangle \langle n | H_{e\gamma} | i \rangle}{E_i - E_n} + \dots \qquad (6.10)$$

Relativistic Generalization

Golden Rule
For Decays

$$1 \longrightarrow 2 + 3 + 4 + \dots + n$$

$$d\Gamma = |M|^2 \frac{S}{2\hbar m_1} \left[\left(\frac{c\, d^3\vec{p}_2}{(2\pi)^3 2E_2} \right) \left(\frac{c\, d^3\vec{p}_3}{(2\pi)^3 2E_3} \right) \dots \left(\frac{c\, d^3\vec{p}_n}{(2\pi)^3 2E_n} \right) \right]$$
$$\cdot (2\pi)^4 \delta^4(p_1 - p_2 - p_3 - \dots - p_n) \qquad (6.11)$$

S = product of factors $1/j!$ for each group of j identical particles in final state.

For the special case of a 2 particle decay $1 \rightarrow 2 + 3$, we get:

Total Decay Rate $\Gamma = \dfrac{S |\vec{p}|\, |M|^2}{8\pi \hbar m_1^2 c}$ where $|\vec{p}| \equiv |\vec{p}_3| = |\vec{p}_2|$ (6.12)

Golden Rule For Scattering

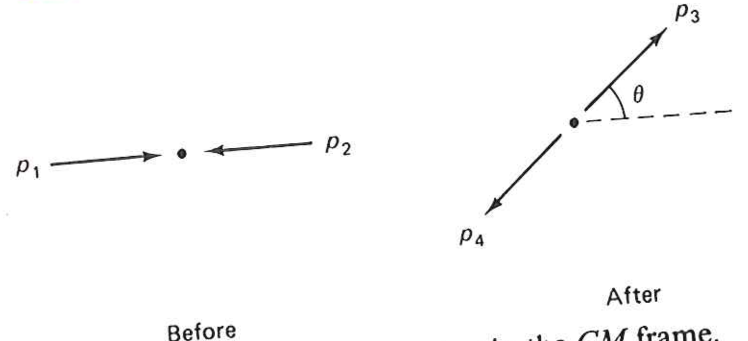
$$1 + 2 \rightarrow 3 + 4 + \cdots + n$$

Cross Section

$$d\sigma = |M|^2 \frac{\hbar^2 S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \left[\left(\frac{c\, d^3\vec{p}_3}{(2\pi)^3 2E_3}\right)\left(\frac{c\, d^3\vec{p}_4}{(2\pi)^3 2E_4}\right)\cdots\left(\frac{c\, d^3\vec{p}_n}{(2\pi)^3 2E_n}\right)\right] \quad (6.13)$$

$$\cdot (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 \cdots - p_n)$$

Case 1: Two Body Scattering $1 + 2 \rightarrow 3 + 4$ **in Center of Mass Frame**



Two-body scattering in the *CM* frame.

(6.13) yields:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S |M|^2}{(E_1 + E_2)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} \quad (6.14$$

where $|\vec{p}_f| \equiv |\vec{p}_3| = |\vec{p}_4|$ and $|\vec{p}_i| \equiv |\vec{p}_1| = |\vec{p}_2|$.

Case 2: Elastic Scattering in Lab Frame $1 + 2 \rightarrow 1 + 2$ where target (2) is heavy $m_2 c^2 \gg E_1$ and is at

$$\left(\frac{d\sigma}{d\Omega}\right)_{Lab} = \left(\frac{\hbar}{8\pi m_2 c}\right)^2 |M|^2 \quad (6.1$$

Feynman Rules For Finding M

1. *Notation.* Label the incoming and outgoing four-momenta $p_1, p_2, \ldots, p_n$, and the corresponding spins $s_1, s_2, \ldots, s_n$; label the internal four-momenta $q_1, q_2, \ldots$. Assign arrows to the lines as follows: the arrows on *external* fermion lines indicate whether it is an electron or a positron; arrows on *internal* fermion lines are assigned so that the "direction of the flow" through the diagram is preserved (i.e., every vertex must have one arrow entering and one arrow leaving). The arrows on external photon lines point "forward"; for internal photon lines the choice is arbitrary. (See Fig. 7.1.)

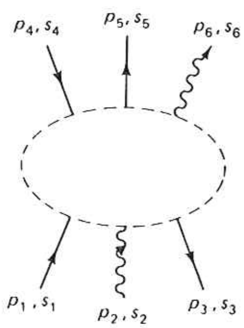


Figure 7.1 A typical QED diagram, with external lines labeled. (Internal lines not shown.)

2. *External Lines.* External lines contribute factors as follows:

Electrons:	Incoming ( ): u
	Outgoing ( ): $\bar{u}$
Positrons:	Incoming ( ): $\bar{v}$
	Outgoing ( ): v
Photons:	Incoming ( ): ϵ^{μ}
	Outgoing ( ): $\epsilon^{\mu *}$

3. *Vertex Factors.* Each vertex contributes a factor

$$ig_e\gamma^{\mu}$$

The dimensionless coupling constant g_e is related to the charge of the positron: $g_e = e\sqrt{4\pi/\hbar c} = \sqrt{4\pi\alpha}$.*

* In Heaviside–Lorentz units, with $\hbar$ and c set equal to 1, g_e *is* the charge of the positron, and hence is written "e" in most texts. In this book I use Gaussian units, and keep all factors of $\hbar$ and c. The easiest way to avoid trouble over units is to express all results in terms of the universal dimensionless quantity α. In writing the Feynman rules for QED I assume we are dealing with electrons and positrons. In *general*, the QED coupling constant is $-q\sqrt{4\pi/\hbar c}$, where q is the charge of the *particle* (as opposed to the *anti*particle). For electrons, $q = -e$, but for "up" quarks, say, $q = \frac{2}{3}e$.

4. *Propagators.* Each internal line contributes a factor as follows:

$$\text{\textit{Electrons and positrons:}}\quad \frac{i(\gamma^{\mu}q_{\mu} + mc)}{q^2 - m^2c^2}$$

$$\text{\textit{Photons:}}\quad \frac{-ig_{\mu\nu}}{q^2}$$

5. *Conservation of Energy and Momentum.* For each vertex, write a delta function of the form

$$(2\pi)^4\, \delta^4(k_1 + k_2 + k_3)$$

where the k's are the three four-momenta coming *into* the vertex (if an arrow leads *outward,* then k is *minus* the four-momentum of that line, except for external positrons*). This factor enforces conservation of energy and momentum at the vertex.

6. *Integrate Over Internal Momenta.* For each internal momentum q, write a factor

$$\frac{d^4q}{(2\pi)^4}$$

and integrate.

7. *Cancel the Delta Function.* The result will include a factor

$$(2\pi)^4\delta^4(p_1 + p_2 + \cdots - p_n)$$

corresponding to overall energy-momentum conservation. Cancel this factor, and what remains is $-i\mathcal{M}$.

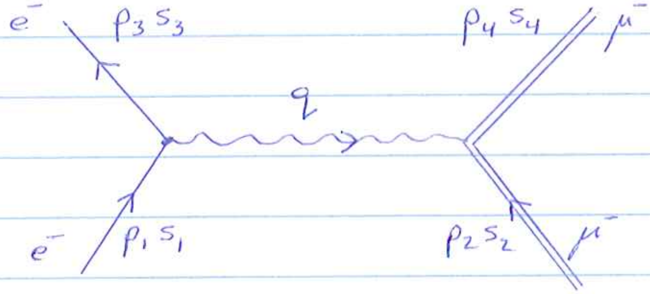
As before, the procedure is to write down all diagrams contributing to the process in question (up to the desired order), calculate the amplitude ($\mathcal{M}$) for each one, and add them up to get the *total* amplitude, which is then inserted into the appropriate formula for the cross section or the lifetime, as the case may be. There's just one new twist, here: the antisymmetrization of fermion wave functions requires that we insert a minus sign in combining amplitudes that differ only in the interchange of two identical external fermions. It doesn't matter *which* diagram you associate the minus sign with, since the total will be squared eventually anyway; but there must be a *relative* minus sign between them.

8. *Antisymmetrization.* Include a minus sign between diagrams that differ only in the interchange of two incoming (or outgoing) electrons (or positrons), or of an incoming electron with an outgoing positron (or vice versa).

Determination of Matrix Elements Using Feynman Rules

1) Electron-muon Scattering

A muon is just a heavy electron. $m_{muon} = 207\, m_{electron}$



Now we apply the Feynman rules proceeding backwards along each fermion line.

$$\int \underbrace{\left[\bar{u}^{s_3}(p_3)\; i g_e \gamma^\mu u^{s_1}(p_1) \right]}_{\text{electron line}} \underbrace{\left(\frac{-i g_{\mu\nu}}{q^2} \right)}_{\substack{\text{photon} \\ \text{propagator}}} \underbrace{\left[\bar{u}^{s_4}(p_4)\; i g_e \gamma^\nu u^{s_2}(p_2) \right]}_{\text{muon line}}$$

$$\cdot (2\pi)^4 \delta^4(p_1 - p_3 - q)(2\pi)^4 \delta^4(p_2 + q - p_4) \frac{d^4 q}{(2\pi)^4}$$

$$= \frac{i g_e^2}{(p_1 - p_3)^2} \left[\bar{u}(3) \gamma^\mu u(1) \right] \left[\bar{u}(4) \gamma_\mu u(2) \right] (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

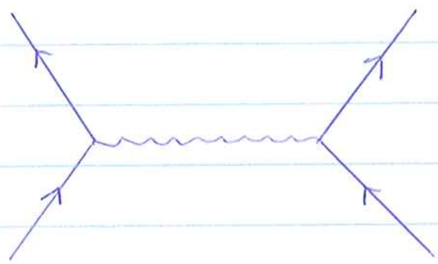
where we have used $u^{s_i}(p_i) \equiv u(i)$

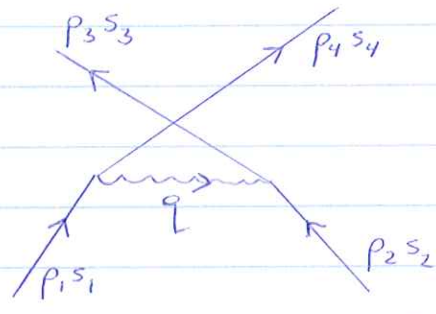
$$\therefore M = \frac{-g_e^2}{(p_1 - p_3)^2} \left[\bar{u}(3) \gamma^\mu u(1) \right] \left[\bar{u}(4) \gamma_\mu u(2) \right] \qquad (6.16)$$

Note: $u(2)$ + $u(4)$ satisfy Dirac equation for muons!

2) Electron-electron Scattering

This is given by two diagrams.

a) 

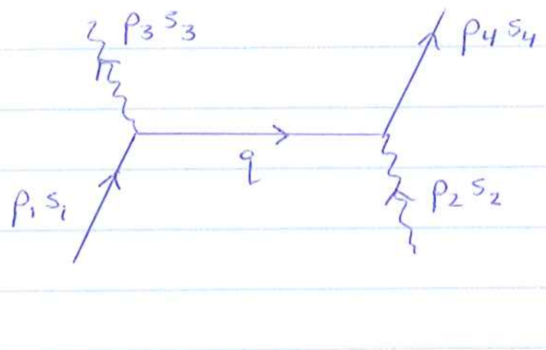
b) 

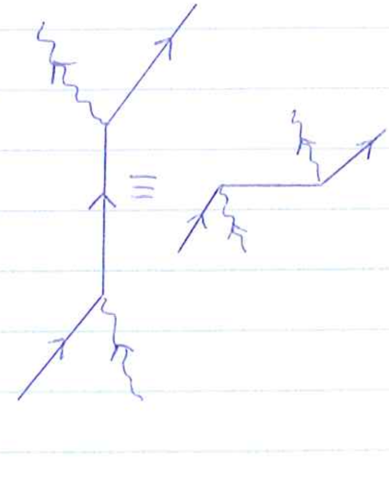
Note that if we exchange the two outgoing electrons in b we get diagram a. Hence, by rule 8, $M = M_a - M_b$. M_a is same as found for $e^-\mu^-$ scattering except that $u(2)$ + $u(4)$ now satisfy Dirac equation for electrons.

Exercise: Show $$M_b = \frac{-g_e^2}{(p_1 - p_4)^2}\left[\bar{u}(4)\gamma^\mu u(1)\right]\left[\bar{u}(3)\gamma_\mu u(2)\right] \quad (6.17)$$

3) Compton Scattering

This is given by two diagrams.

a) 

b) 

Exercise: Explain why $M = M_a + M_b$.

We shall first find M_a.

$$\int \left[\bar{u}(4)\, \epsilon_\mu(2)\, i g_e \gamma^\mu \right] \frac{i\left[\gamma^\alpha q_\alpha + mc\right]}{q^2 - m^2c^2} \left[i g_e \gamma^\nu \epsilon_\nu(3)^* u(1) \right]$$

$$\cdot (2\pi)^4 \delta^4(p_1 - q - p_3)\; (2\pi)^4 \delta^4(p_2 + q - p_4)\; \frac{d^4 q}{(2\pi)^4}$$

$$= -i \frac{g_e^2}{(p_1 - p_3)^2 - m^2c^2} \left[\bar{u}(4) \not{\epsilon}(2) \left[\not{p}_1 - \not{p}_3 + mc \right] \not{\epsilon}(3)^* u(1) \right] \cdot (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

where $\not{a} \equiv a_\mu \gamma^\mu$

$$\therefore M_a = \frac{g_e^2}{(p_1 - p_3)^2 - m^2c^2} \left[\bar{u}(4) \not{\epsilon}(2) \left(\not{p}_1 - \not{p}_3 + mc \right) \not{\epsilon}(3)^* u(1) \right] \quad (6.18)$$

Exercise: Show $M_b = \frac{g_e^2}{(p_1 + p_2)^2 - m^2c^2} \left[\bar{u}(4) \not{\epsilon}(3)^* \left(\not{p}_1 + \not{p}_2 + mc \right) \not{\epsilon}(2)\, u(1) \right]$

(6.19)

Casimir's Trick

The evaluation of the matrix element involves a lot of tedious algebra. A simplification occurs when an experiment is insensitive to spin and/or polarization. To find the cross section, we are then interested in

$\langle |M|^2 \rangle \equiv$ average over initial spins, sum over final spins of $|M|^2$

For the case of electron muon scattering:

$$|M|^2 = \frac{g_e^4}{(p_1 - p_3)^4} \left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]\left[\bar{u}(3)\gamma_\nu u(1)\right]^*\left[\bar{u}(4)\gamma^\nu u(2)\right]^* \quad (6.20)$$

This contains terms of the form

$$G \equiv \left[\bar{u}(a)\, \Gamma_1\, u(b)\right]\left[\bar{u}(a)\, \Gamma_2\, u(b)\right]^* \quad (6.21)$$

We shall first evaluate the complex conjugate term.

$$\left[\bar{u}(a)\Gamma_2 u(b)\right]^* = \left[\bar{u}(a)\Gamma_2 u(b)\right]^\dagger \quad \text{since } [\;] \text{ is a } 1\times 1 \text{ matrix}$$

$$= \left[u(a)^\dagger \gamma^0 \Gamma_2 u(b)\right]^\dagger$$

$$= u(b)^\dagger \Gamma_2^\dagger \gamma^{0\dagger} u(a)$$

$$= u(b)^\dagger \gamma^0\gamma^0 \Gamma_2^\dagger \gamma^0 u(a) \quad \text{since } (\gamma^0)^2 = 1,\ \gamma^{0\dagger} = \gamma^0$$

$$= \bar{u}(b)\, \bar{\Gamma}_2\, u(a) \quad \text{where } \bar{\Gamma}_2 \equiv \gamma^0 \Gamma_2^\dagger \gamma^0.$$

$$\therefore \quad G = \left[\bar{u}(a)\, T_1\, u(b)\right]\left[\bar{u}(b)\, \bar{T}_2\, u(a)\right] \tag{6.22}$$

Exercise: Using (4.54) + (4.55) derive

$$\sum_{s=1,2} u^s \bar{u}^s = \not{p} + mc \tag{6.23a}$$

$$\sum_{s=1,2} v^s \bar{v}^s = \not{p} - mc \tag{6.23b}$$

Hence, summing G over the spins of particle b gives:

$$\sum_{s_b} G = \bar{u}(a)\, T_1 \left\{ \sum_{s_b} u^{s_b}(p_b)\, \bar{u}^{s_b}(p_b) \right\} \bar{T}_2\, u(a)$$

$$= \bar{u}(a)\, T_1\, (\not{p}_b + m_b c)\, \bar{T}_2\, u(a) \qquad \text{using (6.23a)}$$

$$= \bar{u}(a)\, Q\, u(a)$$

where $Q \equiv T_1 (\not{p}_b + m_b c) \bar{T}_2$.

Next we sum over the spins of particle a.

$$\sum_{s_a} \sum_{s_b} G = \sum_{s_a} \bar{u}^{s_a}(p_a)\, Q\, u^{s_a}(p_a)$$

$$= \sum_{s_a} \left[\bar{u}^{s_a}(p_a)\right]_i Q_{ij} \left[u^{s_a}(p_a)\right]_j \quad \text{where we have inserted matrix indices}$$

$$= Q_{ij} \sum_{s_a} \left[u^{s_a}(p_a)\right]_j \left[\bar{u}^{s_a}(p_a)\right]_i$$

$$= Q_{ij} \sum_{s_a} \left[u^{s_a}(p_a)\, \bar{u}^{s_a}(p_a)\right]_{ji}$$

$$\therefore \sum_{s_a} \sum_{s_b} G = Q_{ij} (\not{p}_a + m_a c)_{ji} \qquad \text{using (6.23a)}$$

$$= Tr\left[Q (\not{p}_a + m_a c)\right]$$

$$= Tr\left[T_1 (\not{p}_b + m_b c) \bar{T}_2 (\not{p}_a + m_a c)\right]$$

$$\boxed{\therefore \sum_{s_a s_b} \left[\bar{u}(a) T_1 u(b)\right]\left[\bar{u}(a) T_2 u(b)\right]^* = Tr\left[T_1 (\not{p}_b + m_b c) \bar{T}_2 (\not{p}_a + m_a c)\right] \qquad (6.24)}$$

Hence we now have to evaluate the trace of a series of γ matrices.

Useful Results For Evaluating Traces

Casimir's trick reduces everything down to a problem of calculating the trace of some complicated product of γ matrices. This algebra is facilitated by a number of theorems, which I collect together below (I'll leave the proofs to you—see Problems 7.29 through 7.32). First of all, I should mention three facts about traces in general: if A and B are any two matrices, and α is any number

1. $Tr(A + B) = Tr(A) + Tr(B)$
2. $Tr(\alpha A) = \alpha Tr(A)$
3. $Tr(AB) = Tr(BA)$

It follows from number 3 that $Tr(ABC) = Tr(CAB) = Tr(BCA)$, but this is *not* equal, in general, to the trace of the matrices taken in the other order: $Tr(ACB) = Tr(BAC) = Tr(CBA)$. In this way you can "peel" matrices off the back end of a product and move them around to the front, but you must preserve the ordering. It is useful to note that

4. $g_{\mu\nu}g^{\mu\nu} = 4$

and to recall the fundamental anticommutation relation for the γ matrices (together with an associated rule for "slash" products):

5. $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ 5′. $\not{a}\not{b} + \not{b}\not{a} = 2a \cdot b$

From these there follows a sequence of "contraction theorems":

6. $\gamma_\mu\gamma^\mu = 4$
7. $\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$ 7′. $\gamma_\mu\not{a}\gamma^\mu = -2\not{a}$
8. $\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu = 4g^{\nu\lambda}$ 8′. $\gamma_\mu\not{a}\not{b}\gamma^\mu = 4a \cdot b$
9. $\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\lambda\gamma^\nu$ 9′. $\gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}$

And finally, there is a set of "trace theorems":

10. The trace of the product of an *odd* number of gamma matrices is *zero*
11. $Tr(1) = 4$
12. $Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$ 12′. $Tr(\not{a}\not{b}) = 4a \cdot b$
13. $Tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$ 13′. $Tr(\not{a}\not{b}\not{c}\not{d}) = 4(a \cdot b\, c \cdot d - a \cdot c\, b \cdot d + a \cdot d\, b \cdot c)$

Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the product of an *even* number of γ matrices, it follows from rule 10 that $Tr(\gamma^5\gamma^\mu) = Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda) = 0$. When γ^5 is multiplied by an *even* number of γ's, we find

14. $Tr(\gamma^5) = 0$
15. $Tr(\gamma^5\gamma^\mu\gamma^\nu) = 0$ 15′. $Tr(\gamma^5\not{a}\not{b}) = 0$
16. $Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma}$ 16′. $Tr(\gamma^5\not{a}\not{b}\not{c}\not{d}) = 4i\epsilon^{\mu\nu\lambda\sigma}a_\mu b_\nu c_\lambda d_\sigma$

where

$$\epsilon^{\mu\nu\lambda\sigma} \equiv \begin{cases} -1, & \text{if } \mu\nu\lambda\sigma \text{ is an } even \text{ permutation of } 0123, \\ +1, & \text{if } \mu\nu\lambda\sigma \text{ is an } odd \text{ permutation,} \\ 0, & \text{if any two indices are the same.*} \end{cases}$$

Examples

1) Electron Muon Scattering

Applying Casimir's trick twice in (6.20); once for electrons and a second time for muons we find:

$$\langle |M|^2 \rangle = \frac{g_e^4}{4(p_1-p_3)^4} Tr\left[\gamma^\mu (\not{p}_1 + mc)\gamma^\nu (\not{p}_3 + mc)\right] \cdot Tr\left[\gamma_\mu (\not{p}_2 + Mc)\gamma_\nu (\not{p}_4 + Mc)\right] \quad (6.25)$$

where we have used $\gamma^0 \gamma^{\nu\dagger} \gamma^0 = \gamma^\nu$.

Exercise: Where does factor of 1/4 come from?

$$Tr\left[\gamma^\mu (\not{p}_1 + mc)\gamma^\nu (\not{p}_3 + mc)\right]$$

$$= Tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3) + mc\left[\underbrace{Tr(\gamma^\mu \not{p}_1 \gamma^\nu) + Tr(\gamma^\mu \gamma^\nu \not{p}_3)}_{=0 \text{ by rule 10}}\right]$$

$$+ (mc)^2 \underbrace{Tr(\gamma^\mu \gamma^\nu)}_{=4g^{\mu\nu} \text{ by rule 12}}$$

The first trace can be simplified as follows.

$$Tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3) = (p_1)_\lambda (p_3)_\sigma \, Tr(\gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\sigma)$$

$$= (p_1)_\lambda (p_3)_\sigma \, 4\left[g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\sigma} g^{\lambda\nu}\right] \text{ using rule 13}$$

$$= 4\left[p_1^\mu p_3^\nu - g^{\mu\nu}(p_1 \cdot p_3) + p_3^\mu p_1^\nu\right]$$

$$\therefore \mathrm{Tr}\left[\gamma^\mu (\not{p}_1 + mc)\gamma^\nu(\not{p}_3 + mc)\right]$$
$$= 4\left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}\left((mc)^2 - (p_1\cdot p_3)\right)\right] \quad (6.26)$$

The second trace in (6.25) is the same as the first with $m \to M$, $1 \to 2$ and $3 \to 4$, and has lowered indices.

$$\therefore \mathrm{Tr}\left[\gamma_\mu (\not{p}_2 + Mc)\gamma_\nu(\not{p}_4 + Mc)\right]$$
$$= 4\left[p_{2\mu} p_{4\nu} + p_{4\mu} p_{2\nu} + g_{\mu\nu}\left((Mc)^2 - (p_2\cdot p_4)\right)\right] \quad (6.27)$$

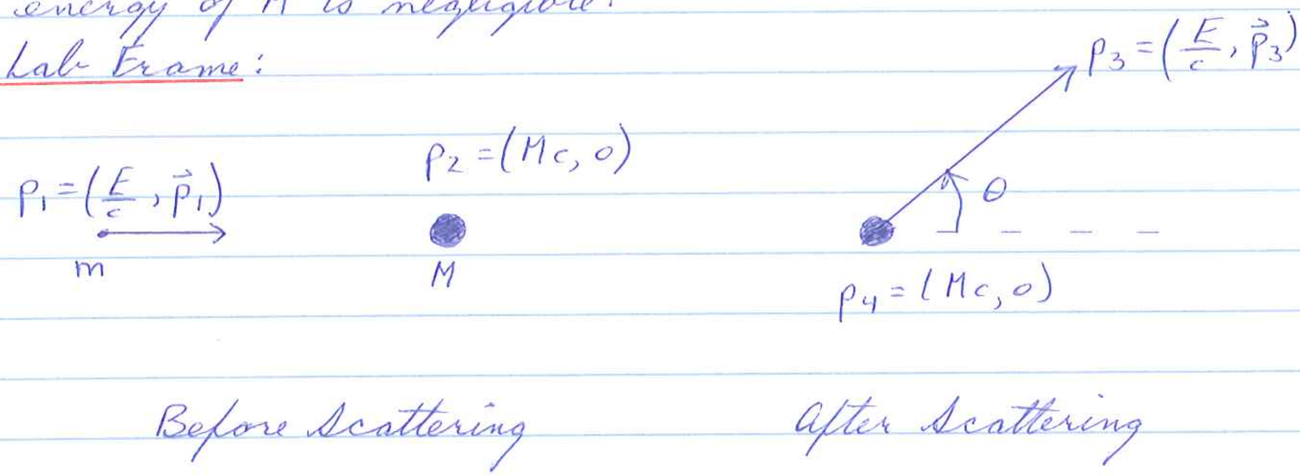
Substituting (6.26) + (6.27) into (6.25) gives:

$$\langle |\mathcal{M}|^2 \rangle = \frac{4 g_e^4}{(p_1 - p_3)^4}\Big\{(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) \quad (6.28)$$
$$- (p_1\cdot p_3)(Mc)^2 - (p_2\cdot p_4)(mc)^2 + 2(mMc^2)^2\Big\}$$

Mott Scattering

We shall consider the case where $M \gg m$ such that kinetic energy of M is negligible.

Lab Frame:



$$(p_1 - p_3)^2 = -(\vec{p}_1 - \vec{p}_3)^2$$

$$= -(\vec{p}_1 \cdot \vec{p}_1 + \vec{p}_3 \cdot \vec{p}_3 - 2\vec{p}_1 \cdot \vec{p}_3)$$

$$= -2|\vec{p}_1|^2 + 2|\vec{p}_1|^2 \cos\theta \quad \text{using } |\vec{p}_1| = |\vec{p}_3|$$

$$= -2|\vec{p}_1|^2(1 - \cos\theta)$$

$$(p_1 - p_3)^2 = -4|\vec{p}_1|^2 \sin^2\frac{\theta}{2}$$

$$p_1 \cdot p_2 = p_3 \cdot p_4 = p_1 \cdot p_4 = p_2 \cdot p_3 = EM$$

$$p_2 \cdot p_4 = (Mc)^2$$

$$p_1 \cdot p_3 = \left(\frac{E}{c}\right)^2 - \vec{p}_1 \cdot \vec{p}_3$$

$$= (mc)^2 + |\vec{p}_1|^2 - |\vec{p}_1|^2 \cos\theta \quad \text{using } E^2 = c^2|\vec{p}_1|^2 + (mc^2)^2$$

$$= (mc)^2 + 2|\vec{p}_1|^2 \sin^2\frac{\theta}{2}$$

Exercise: Substitute the above results into (6.28) and show:

$$\langle |M|^2 \rangle = \left(\frac{g_e^2 \, Mc}{|\vec{p}_1|^2 \sin^2 \frac{\theta}{2}} \right)^2 \left\{ (mc)^2 + |\vec{p}_1|^2 \cos^2 \frac{\theta}{2} \right\} \qquad (6.29)$$

The cross section is given by

$$\left(\frac{d\sigma}{d\Omega} \right)_{Lab} = \left(\frac{\hbar}{8\pi Mc} \right)^2 \langle |M|^2 \rangle$$

$$= \left(\frac{4\pi\alpha\hbar}{8\pi \, |\vec{p}_1|^2 \sin^2 \frac{\theta}{2}} \right)^2 \left\{ (mc)^2 + |\vec{p}_1|^2 \cos^2 \frac{\theta}{2} \right\}$$

$$\boxed{\left(\frac{d\sigma}{d\Omega} \right)_{Lab} = \left(\frac{\alpha\hbar}{2 |\vec{p}_1|^2 \sin^2 \frac{\theta}{2}} \right)^2 \left\{ (mc)^2 + |\vec{p}_1|^2 \cos^2 \frac{\theta}{2} \right\} \qquad (6.30)}$$

Mott Scattering Formula

Exercise: In the nonrelativistic limit, show that

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{2 m v^2 \sin^2 \theta/2} \right)^2$$

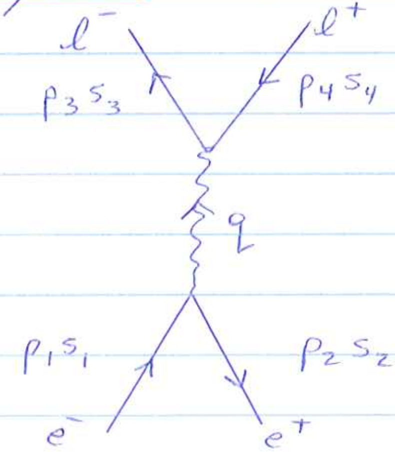
Rutherford Cross Section

2) Lepton Pair Production in $e^- e^+$ Collisions

$$e^- + e^+ \longrightarrow \ell^- + \ell^+ \qquad \begin{array}{ll} \ell = \mu & m_\mu = 207\, m_e \\ \ell = \tau & m_\tau = 3487\, m_e \end{array}$$

This process has helped discover particles of higher mass such as the tau lepton.

Feynman Diagram



$$m_3 = m_4 \equiv M$$
$$m_1 = m_2 \equiv m$$

Exercise: Show the matrix element is given by:

$$M = \frac{-g_e^2}{(p_1+p_2)^2}\left[\bar{u}(3)\,\gamma^\mu\, v(4)\right]\left[\bar{v}(2)\,\gamma_\mu\, u(1)\right]$$

Using Casimir's trick, we find M^2 averaged over the electron + positron spins and summed over ℓ^-, ℓ^+ spins to be:

$$\langle |M|^2 \rangle = \frac{1}{4}\frac{g_e^4}{(p_1+p_2)^4} \cdot Tr\left[\gamma^\mu (\not{p}_4 - Mc)\,\gamma^\nu (\not{p}_3 + Mc)\right]$$

$$\cdot\, Tr\left[\gamma_\mu (\not{p}_1 + mc)\,\gamma_\nu (\not{p}_2 - mc)\right]$$

<u>Exercise</u>: Show $Tr\left[\gamma^\mu(\not{p}_4 - Mc)\gamma^\nu(\not{p}_3 + Mc)\right]$

$$= 4\left[p_4^\mu p_3^\nu + p_3^\mu p_4^\nu - g^{\mu\nu}\left((Mc)^2 + (p_3 \cdot p_4)\right)\right]$$

The other trace is evaluated similarly. Hence we obtain:

$$\langle |M|^2 \rangle = \frac{1}{4}\,\frac{g_e^4}{(p_1+p_2)^4}\cdot 16\left[p_4^\mu p_3^\nu + p_3^\mu p_4^\nu - g^{\mu\nu}\left((Mc)^2 + (p_3\cdot p_4)\right)\right]$$

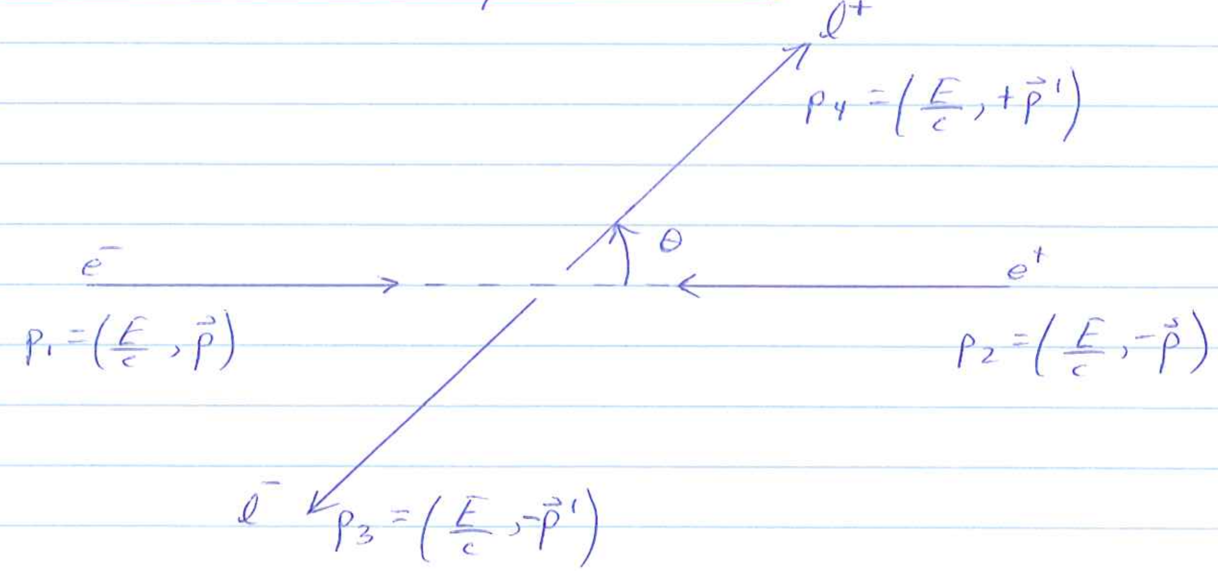
$$\cdot\left[p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - g_{\mu\nu}\left((mc)^2 + (p_1\cdot p_2)\right)\right]$$

After some algebra, this simplifies to:

$$\langle |M|^2 \rangle = \frac{8 g_e^4}{(p_1+p_2)^4}\Big\{(p_1\cdot p_4)(p_2\cdot p_3) + (p_2\cdot p_4)(p_1\cdot p_3) \qquad (6.31)$$

$$+ (p_3\cdot p_4)(mc)^2 + (p_1\cdot p_2)(Mc)^2 + 2(Mc)^2(mc)^2\Big\}$$

<u>Evaluation in Center of Mass Frame</u>.



$$p_1 \cdot p_4 = p_2 \cdot p_3 = \frac{E^2}{c^2} - \vec{p} \cdot \vec{p}\,' = \frac{E^2}{c^2} - |\vec{p}||\vec{p}\,'| \cos\theta$$

$$p_2 \cdot p_4 = p_1 \cdot p_3 = \frac{E^2}{c^2} + \vec{p} \cdot \vec{p}\,' = \frac{E^2}{c^2} + |\vec{p}||\vec{p}\,'| \cos\theta$$

$$p_1 \cdot p_2 = \frac{E^2}{c^2} + |\vec{p}|^2$$

$$p_3 \cdot p_4 = \frac{E^2}{c^2} + |\vec{p}\,'|^2$$

$$(p_1 + p_2)^2 = \left(\frac{2E}{c}\right)^2$$

Equation (6.31) then becomes:

$$\langle |M|^2 \rangle = \frac{8 g_e^4}{\left(\frac{2E}{c}\right)^4} \left\{ \left(\frac{E^2}{c^2} - |\vec{p}||\vec{p}\,'| \cos\theta\right)^2 + \left(\frac{E^2}{c^2} + |\vec{p}||\vec{p}\,'| \cos\theta\right)^2 \right.$$

$$+ \left(\frac{E^2}{c^2} + |\vec{p}\,'|^2\right)(mc)^2 + \left(\frac{E^2}{c^2} + |\vec{p}|^2\right)(Mc)^2$$

$$\left. + 2 (Mc)^2 (mc)^2 \right\}$$

For processes in which muons or tau leptons are created the electron energy $E \simeq |\vec{p}|c$. Also since $m \ll M$, we can neglect the third and fifth terms in the brackets.

$$\therefore \langle |M|^2 \rangle = \frac{8 g_e^4}{\left(\frac{2E}{c}\right)^4} \left\{ \left(\frac{E^2}{c^2} - \frac{E}{c} |\vec{p}\,'| \cos\theta\right)^2 + \left(\frac{E^2}{c^2} + \frac{E}{c} |\vec{p}\,'| \cos\theta\right)^2 \right.$$

$$\left. + 2 \frac{E^2}{c^2} (Mc)^2 \right\}$$

After some algebra, this simplifies to give:

$$\langle |M|^2 \rangle = \frac{g_e^4}{E^2} \left\{ E^2 + |\vec{p}\,'|^2 c^2 \cos^2\theta + (Mc^2)^2 \right\}$$

The center of mass cross section is found using (6.14).

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{1}{(2E)^2} \frac{|\vec{p}\,'|}{E/c} \langle |M|^2 \rangle$$

Exercise: Show this simplifies to:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{e^4}{16E^4} \frac{c|\vec{p}\,'|}{E} \left[E^2 + |\vec{p}\,'|^2 c^2 \cos^2\theta + (Mc^2)^2 \right] \qquad (6.32)$$

Exercise: Integrate (6.32) to find:

$$(\sigma_{TOT})_{CM} = \frac{\pi e^4}{4E^4} \frac{c|\vec{p}\,'|}{E} \left[E^2 + \frac{|\vec{p}\,'|^2 c^2 \cancel{\cos^2\theta}}{3} + (Mc^2)^2 \right] \qquad (6.33)$$

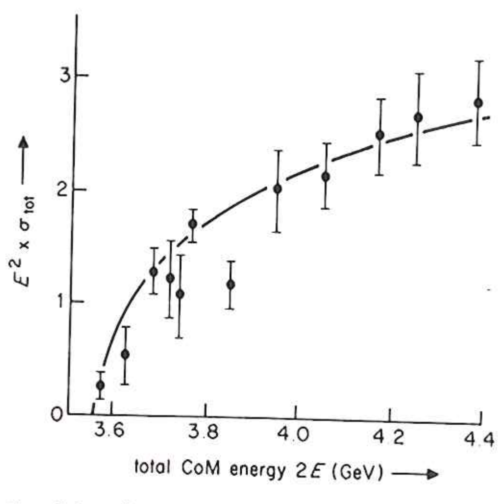


Fig. 8.2. $E^2 \times \sigma_{tot}$ (in arbitrary units) for the process $e^+e^- \to \tau^+\tau^-$ near threshold $2E = 2m_\tau$. [After W. Bacino *et al.*, *Phys. Rev. Lett.* **41** (1978), 13.] Φ: experimental data; curve: fit of cross-section formula (6.33) with $m_\tau = 1782$ MeV.

Exercise: In the limit $E \gg Mc^2$ show (6.32) + (6.33) become:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{e^4}{16E^2}\left(1+\cos^2\theta\right) \qquad (6.34)$$

$$\left(\sigma_{TOT}\right)_{CM} = \frac{\pi}{3}\frac{e^4}{E^2} \qquad (6.35)$$

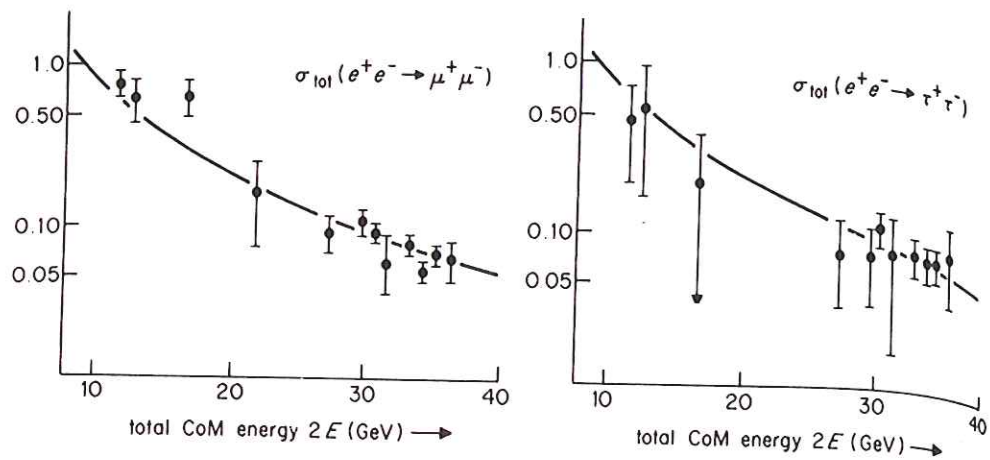
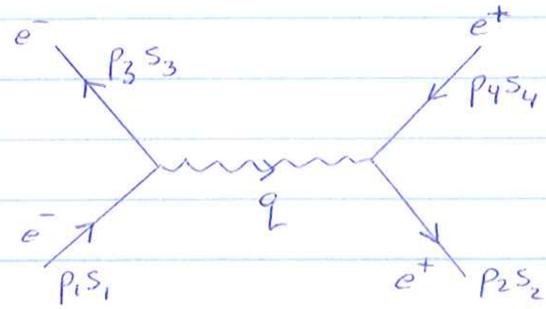


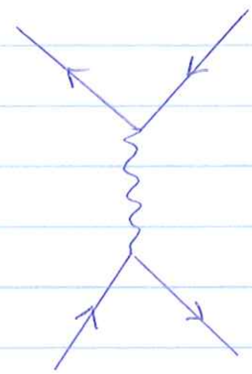
Fig. 8.3. The total cross-sections (in nb) for the processes $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \tau^+\tau^-$ at relativistic energies. [After D. P. Barber *et al.*, *Phys. Rev. Lett.* **43** (1979) 1915.] $\bar{\Phi}$: experimental data; curves: theoretical cross-section formula (6.35).

3) Electron - Positron Scattering $e^+ + e^- \to e^+ + e^-$
(Bhabha)

a)



b)



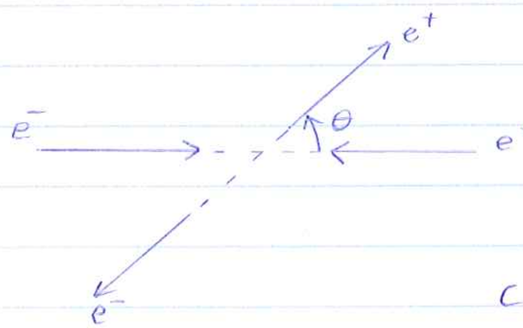
Exercise: Show $M = M_a - M_b$ where

$$M_a = \frac{-g_e^2}{(p_1 - p_3)^2}\left[\bar{u}(3)\,\gamma^\mu u(1)\right]\left[\bar{v}(2)\,\gamma_\mu v(4)\right]$$

$$M_b = \frac{-g_e^2}{(p_1 + p_2)^2}\left[\bar{u}(3)\,\gamma^\mu v(4)\right]\left[\bar{v}(2)\,\gamma_\mu u(1)\right]$$

Exercise: In the limit $E \gg mc^2$ show that

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{e^4}{8E^2}\left\{\frac{1+\cos^4\theta/2}{\sin^4\theta/2} + \frac{1+\cos^2\theta}{2} - \frac{2\cos^4\theta/2}{\sin^2\theta/2}\right\} \quad (16.36)$$



CM Frame

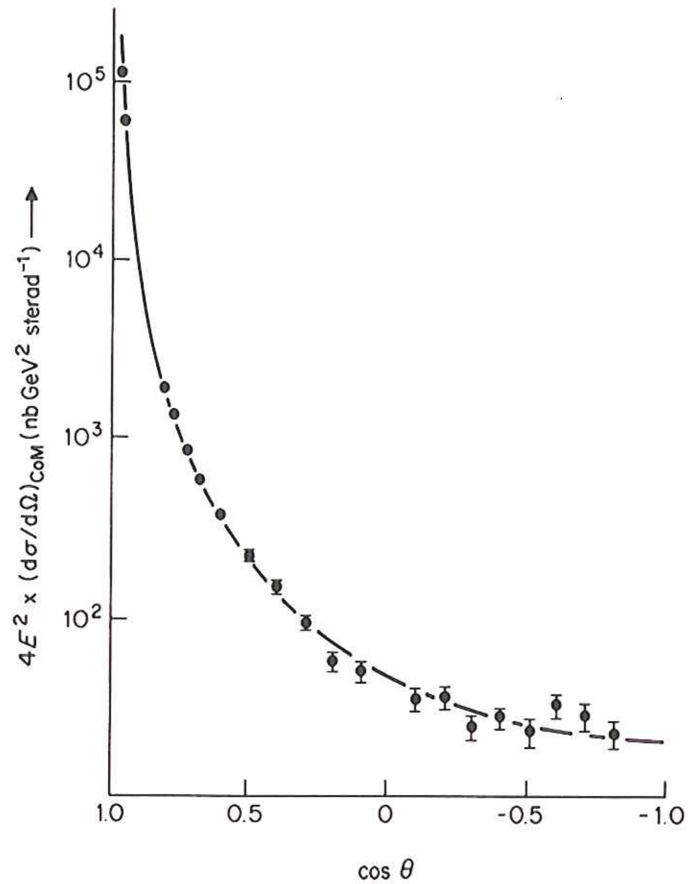


Fig. 8.4. The differential cross-section $(d\sigma/d\Omega)_{CoM}$ for Bhabha scattering, $e^+e^- \to e^+e^-$, at the total CoM energy $2E = 34$ GeV, [After H. J. Behrend *et al.*, *Phys. Lett*, **103B** (1981), 148.] Φ: experimental data; curve: QED cross-section formula (6.36).

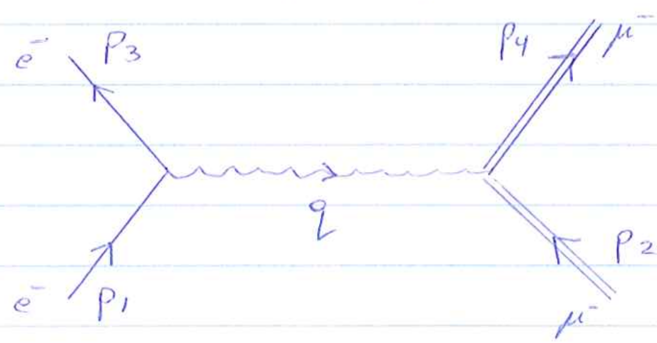
Radiative Corrections

So far, only Feynman diagrams having two vertices have been considered. This yields matrix elements squared $|M|^2 \propto \alpha^2$ where α is the fine structure constant. In general a diagram may have n vertices such that $|M|^2 \propto \alpha^n$.

Exercise: Estimate value of $\dfrac{M^2(4\text{ vertices})}{M^2(2\text{ vertices})}$.

Electron-Muon Scattering

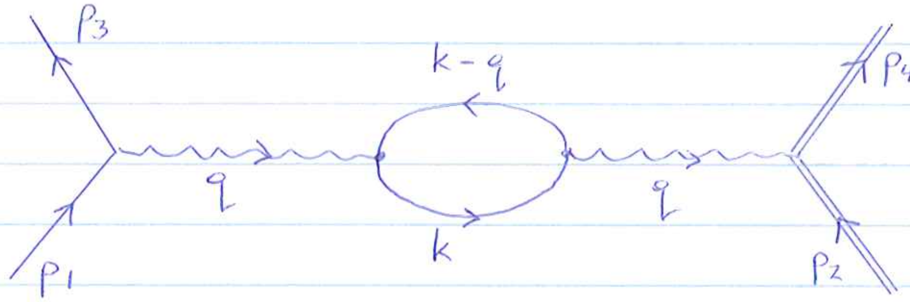
The lowest order diagram for this process is:



$$|M|^2 = -g_e^2 \left[\bar{u}(3)\gamma^\mu u(1)\right] \frac{g_{\mu\nu}}{q^2} \left[\bar{u}(4)\gamma^\nu u(2)\right] \qquad (6.37)$$

where $q \equiv p_1 - p_3$.

An example of a 4th order correction to this process is the so called vacuum polarization diagram which arises when a virtual photon momentarily splits into a $e^+ e^-$ pair.



This diagram has the following amplitude.

$$M = \frac{-i g_e^4}{q^4} \left[\bar{u}(3)\gamma^\mu u(1)\right] \left\{ \int \frac{d^4k}{(2\pi)^4} \frac{Tr\left[\gamma_\mu(\not{k}+mc)\gamma_\nu(\not{q}-\not{k}+mc)\right]}{(k^2-m^2c^2)((q-k)^2-m^2c^2)} \right\} \left[\bar{u}(4)\gamma^\nu u(2)\right] \tag{6.38}$$

The effect of adding (6.38) to (6.37) is to modify the photon propagator

$$\frac{g_{\mu\nu}}{q^2} \longrightarrow \frac{g_{\mu\nu}}{q^2} - \frac{i}{q^4} I_{\mu\nu}$$

where $$I_{\mu\nu} \equiv -g_e^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr\left[\gamma_\mu(\not{k}+mc)\gamma_\nu(\not{q}-\not{k}+mc)\right]}{(k^2-m^2c^2)((q-k)^2-m^2c^2)} \tag{6.39}$$

After integrating over k, the only 4-vector left in (6.39) is q_μ

$$\therefore \ I_{\mu\nu} = -i g_{\mu\nu} q^2 I(q^2) + q_\mu q_\nu J(q^2) \tag{6.40}$$

Exercise: Show that the second term in (6.40) does not contribute to M given by (6.38).

It can be shown that $I(q^2)$ is given as follows.

$$I(q^2) = \frac{g_e^2}{12\pi^2}\left\{\int_{m^2}^{\infty}\frac{dx}{x} - 6\int_0^1 z(1-z)\ln\left(1 - \frac{q^2}{m^2c^2}z(1-z)\right)dz\right\} \qquad (6.41)$$

The first integral diverges logarithmically. We therefore impose a cutoff M_c.

$$\text{i.e. } \int_{m^2}^{\infty}\frac{dx}{x} \longrightarrow \int_{m^2}^{M_c^2}\frac{dx}{x} = \ln\left(\frac{M_c^2}{m^2}\right) \qquad (6.42)$$

The second integral in (6.41)

$$f(x) \equiv 6\int_0^1 z(1-z)\ln\left(1 + x z(1-z)\right)dz \qquad (6.43)$$

where $x \equiv \frac{-q^2}{m^2c^2}$, is finite. Substituting (6.43) & (6.42) into (6.41), we get:

$$I(q^2) = \frac{g_e^2}{12\pi^2}\left\{\ln\left(\frac{M_c^2}{m^2}\right) - f\left(\frac{-q^2}{m^2c^2}\right)\right\} \qquad (6.44)$$

Hence the amplitude for electron muon scattering, including vacuum polarization, is:

$$|M|^2 = -g_e^2\left[\bar{u}(3)\gamma^\mu u(1)\right]\frac{g_{\mu\nu}}{q^2}\left\{1 - \frac{g_e^2}{12\pi^2}\left[\ln\left(\frac{M_c^2}{m^2}\right) - f\left(\frac{-q^2}{m^2c^2}\right)\right]\right\}\left[\bar{u}(4)\gamma^\nu u(2)\right] \qquad (6.45)$$

We now introduce the renormalized coupling constant

$$g_R \equiv g_e \sqrt{1 - \frac{g_e^2}{12\pi^2} \ln\left(\frac{M_c^2}{m^2}\right)} \qquad (6.46)$$

(6.45) then becomes:

$$M = -g_R^2 \left[\bar{u}(3)\gamma^\mu u(1)\right] \frac{g_{\mu\nu}}{q_L^2} \left\{1 + \frac{g_R^2}{12\pi^2} f\left(\frac{-q_L^2}{m^2c^2}\right)\right\} \left[\bar{u}(4)\gamma^\nu u(2)\right] \qquad (6.47)$$

Comments

1) (6.47) has no explicit infinities or dependence on M_c. $g_R = \sqrt{4\pi\alpha_R} = \sqrt{\frac{4\pi e_R^2}{\hbar c}}$ where e_R is interpreted as the electron charge measured in a laboratory experiment. In contrast $g_e = \sqrt{\frac{4\pi e^2}{\hbar c}}$ and e is known as the bare electron charge which is not observable. Earlier work in which we interpreted g_e as due to the physical electron charge remains valid since including higher order effects replaces g_e by g_R.

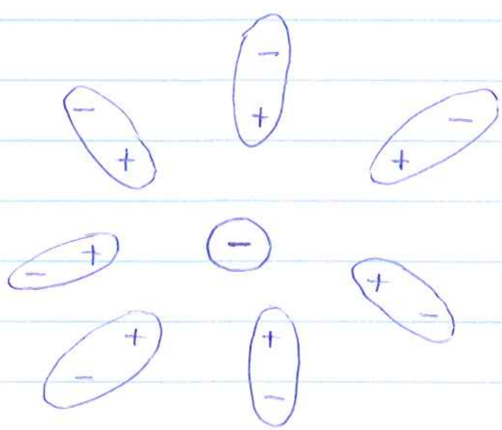
2) Comparing (6.47) and (6.37), we find the effect of the vacuum polarization term is to cause the coupling constant to depend on q^2.

i.e. $$g_R(q_L^2) = g_R(0) \sqrt{1 + \frac{g_R(0)^2}{12\pi^2} f\left(\frac{-q_L^2}{m^2c^2}\right)} \qquad (6.48)$$

or in terms of the fine structure constant

$$\alpha(q^2) = \alpha(0) \sqrt{1 + \frac{\alpha(0)}{3\pi} f\left(\frac{-q^2}{m^2 c^2}\right)} \qquad (6.49)$$

Hence the effective charge of the electron (muon) depends on the momentum transferred in the collision. At higher momentum, the particles come closer and penetrate the vaccuum polarization charge cloud that surrounds the electron (muon). This effective modification of α only becomes significant at high energy.



Electron Surrounded By Polarized Vacuum.

Experimental Tests of QED Radiative Corrections

1) QED can evaluate electron g factor using the following graphs.

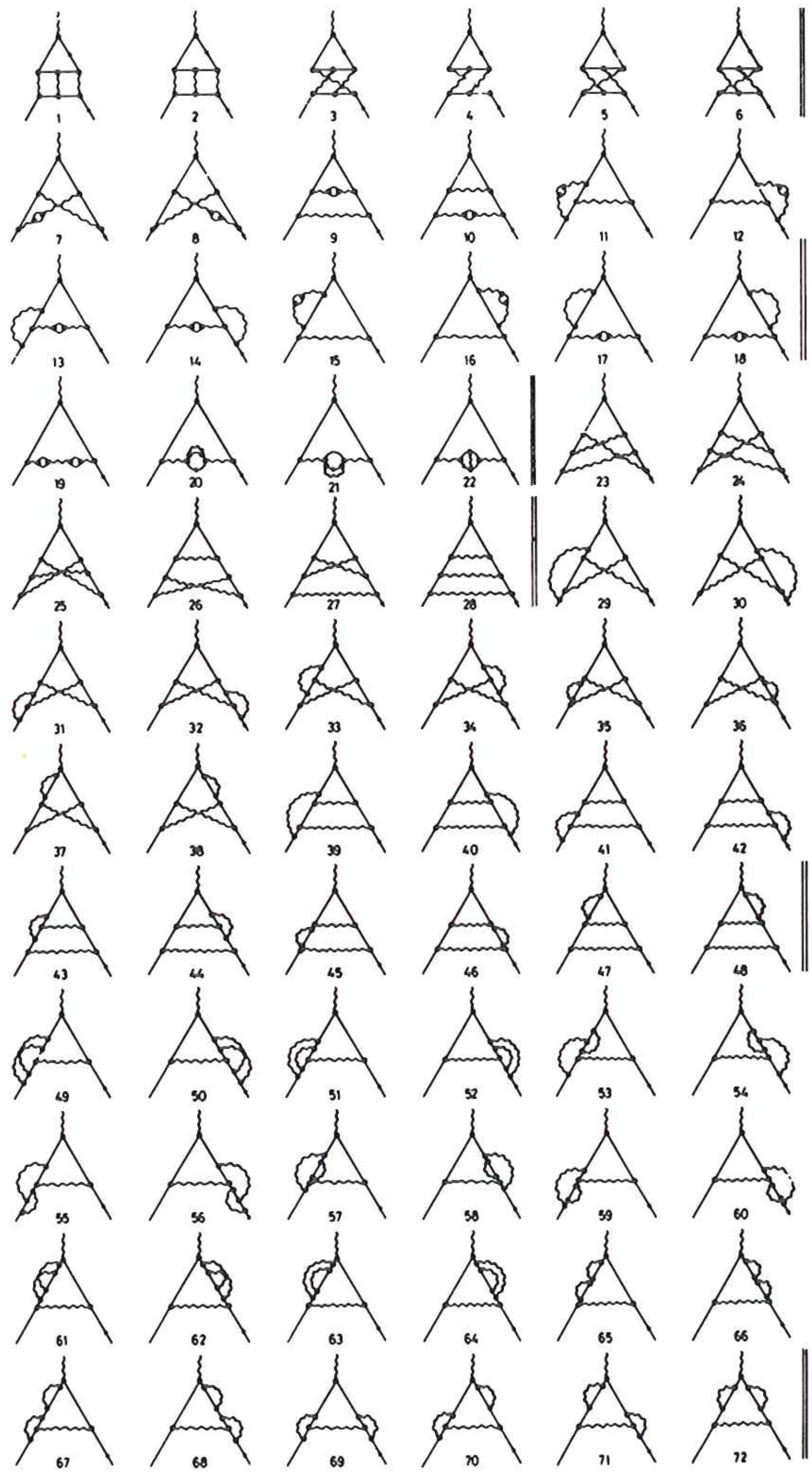


Fig. 8.2 The Feynman graphs which have to be evaluated in computing the α^3 corrections to the lepton magnetic moments (after Lautrup *et al.* 1972).

The theoretical results are:

$$\left(\frac{g-2}{2}\right)_{\text{elect.}} = \frac{\alpha}{2\pi} - 0.32848\left(\frac{\alpha}{\pi}\right)^2 + 1.19\left(\frac{\alpha}{\pi}\right)^3 + \cdots$$

$$= (1{,}159{,}652.4 \pm 0.4) \times 10^{-9}$$

$$\left(\frac{g-2}{2}\right)_{\text{muon}} = \frac{\alpha}{2\pi} + 0.76578\left(\frac{\alpha}{\pi}\right)^2 + 24.45\left(\frac{\alpha}{\pi}\right)^3 + \cdots$$

$$= (1{,}165{,}851.7 \pm 2.3) \times 10^{-9}$$

These agree with the experimental results given earlier.

P.S. I believe theoreticians have computed α^4 terms as well. The new results still agree with experiment!

2) Lamb Shifts

This refers to the energy separating the $2S_{1/2}$ and $2P_{1/2}$ states in H which Dirac predicts are degenerate. Somewhat dated results are given below.

	H	D	He^+
Theory (MHz)	1058.03 ± 0.15	1059.38 ± 0.15	14055 ± 3
Expt. (MHz)	1057.77 ± 0.10	1059.00 ± 0.10	14043 ± 13

P.S. There have been substantially better experimental results recently, but as yet no discrepancy with theory.

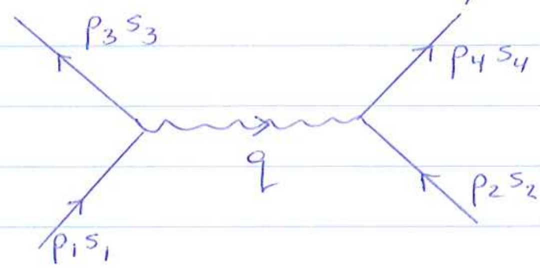
Concluding Remarks

The amazing success of QED has spurred on efforts to generalize the theory. These came to fruition in the 1960's when additional "Feynman rules" were postulated to account for the weak interaction. This electroweak theory predicted the existence of three heavy particles $W^{\pm}$, Z^{0} (masses ~ 80 GeV) which were observed in the 1980's. Work is presently underway to include the strong and eventually the gravitational interactions into a so called Grand Unified Theory (GUT).

Chapter 6 Assignment

1) Cross Sections: Derive (6.14) from (6.13).

2) Write the matrix element for:



3) Show $\sum_{s=1,2} u^s \bar{u}^s = \not{p} + mc$

$$\sum_{s=1,2} v^s \bar{v}^s = \not{p} - mc$$

4) Show $Tr(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$

5) For Bhabha scattering with CM energy $E \gg mc^2$, derive $\left(\frac{d\sigma}{d\Omega}\right)_{CM}$ given by (6.36).

6) Derive the following expression for the Lamb shift

$$E_L = \frac{4}{3\pi} \frac{Z^4 \alpha^5}{n^3} mc^2 \ln\left(\frac{1}{Z\alpha}\right) \delta_{l0}$$

a) Is it easier to measure E_L in Li^{++} or H?
b) " " in excited or ground states?

Hint: See Bethe + Salpeter's book.