Linear Algebra

Lecture Notes

W. A. van Wijngaarden

Physics Dept., York University

Outline of Math 1025
(Algebra)

1. Polar Coordinates including Graphing + Cylindrical & Spherical Coordinates — 1 week

2. Complex Numbers — 1 week

3. Vectors (n dimensions) — 2-3 weeks
   - includes vector spaces, dot & cross products, unit vectors etc.
   Gram-Schmidt

4. Matrices — 2-3 weeks
   - properties, determinant, inverse
   - transpose, adjoint, Hermitian symmetric, trace etc.

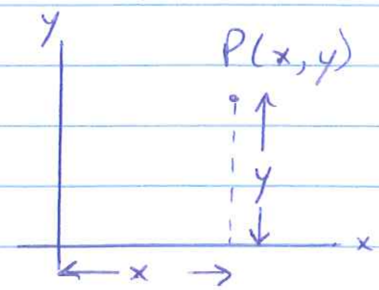
5. Applications of Matrices — 2 weeks
   - solving system of linear equations
   - rotation of a vector
   - coordinate transformation

6. Eigenvalues & Eigenvectors — 2 weeks

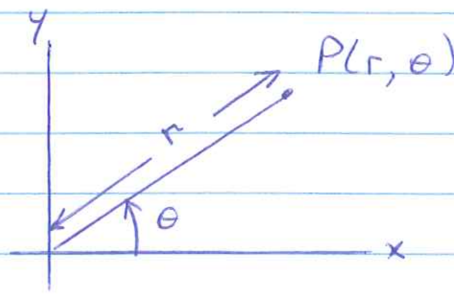
Two Dimensional Coordinate Systems

The location of a point is specified by two numbers.

Cartesian Coordinates



Polar Coordinates



Note that the point specified by (r, θ) is the same as the point $(r, \theta + 2n\pi)$ where $n \in$ integer or $(-r, \theta + (2n+1)\pi)$.

Relation Between Polar & Cartesian Coordinates

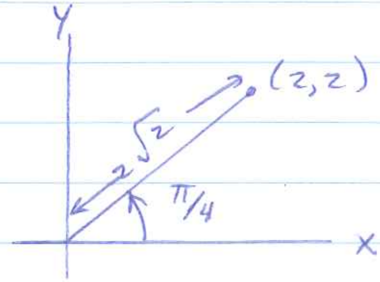
$x = r\cos\theta \qquad (1)$

$y = r\sin\theta \qquad (2)$

$(2) \div (1) \Rightarrow \tan\theta = y/x \quad$ or $\quad \theta = \text{Arctan}(y/x)$

$(1)^2 + (2)^2 \Rightarrow r^2 = x^2 + y^2 \quad$ or $\quad r = \sqrt{x^2+y^2}$

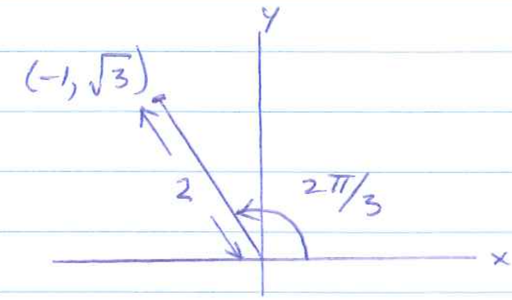
Conversion from Cartesian → Polar



1. $P(x,y) = (2,2)$

$$r = \sqrt{x^2+y^2} = \sqrt{2^2+2^2} = 2\sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{2}{2} = 1 \implies \theta = \frac{\pi}{4}$$

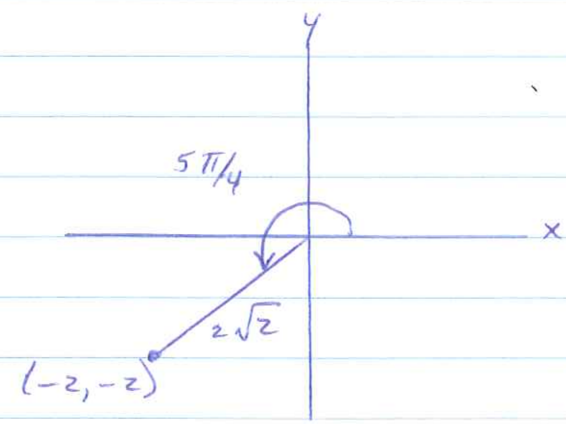


2. $P(x,y) = (-1, \sqrt{3})$

$$r = \sqrt{(-1)^2+3} = 2$$

$$\tan\theta = -\sqrt{3}$$

$$\theta = \frac{2\pi}{3}$$



3. $P(x,y) = (-2,-2)$

$$r = \sqrt{(-2)^2+(-2)^2} = 2\sqrt{2}$$

$$\tan\theta = \frac{-2}{-2} = 1$$

$$\theta = \frac{5\pi}{4}$$

Note that $\tan\frac{\pi}{4} = 1$ as well, but from diagram we see is incorrect. Hence one should always make a diagram.

Conversion Polar → Cartesian

1. $P(r,\theta) = (3, -4\pi/3)$

$x = r\cos\theta = 3\cos\left(\frac{-4\pi}{3}\right) = \frac{-3}{2}$

$y = r\sin\theta = 3\sin\left(\frac{-4\pi}{3}\right) = +\frac{3\sqrt{3}}{2}$

2. $P(r,\theta) = \left(3, \frac{2\pi}{3}\right)$

$x = 3\cos\frac{2\pi}{3} = -\frac{3}{2}$

$y = 3\sin\frac{2\pi}{3} = \frac{3\sqrt{3}}{2}$

Hence $\left(3, -\frac{4\pi}{3}\right)$ is same point as $\left(3, \frac{2\pi}{3}\right)$.

3. $P(r,\theta) = (0, 76^\circ)$

$$\left.\begin{aligned} x &= 0\cos 76^\circ = 0 \\ y &= 0\sin 76^\circ = 0 \end{aligned}\right\} \text{ i.e. origin}$$

Polar Plots

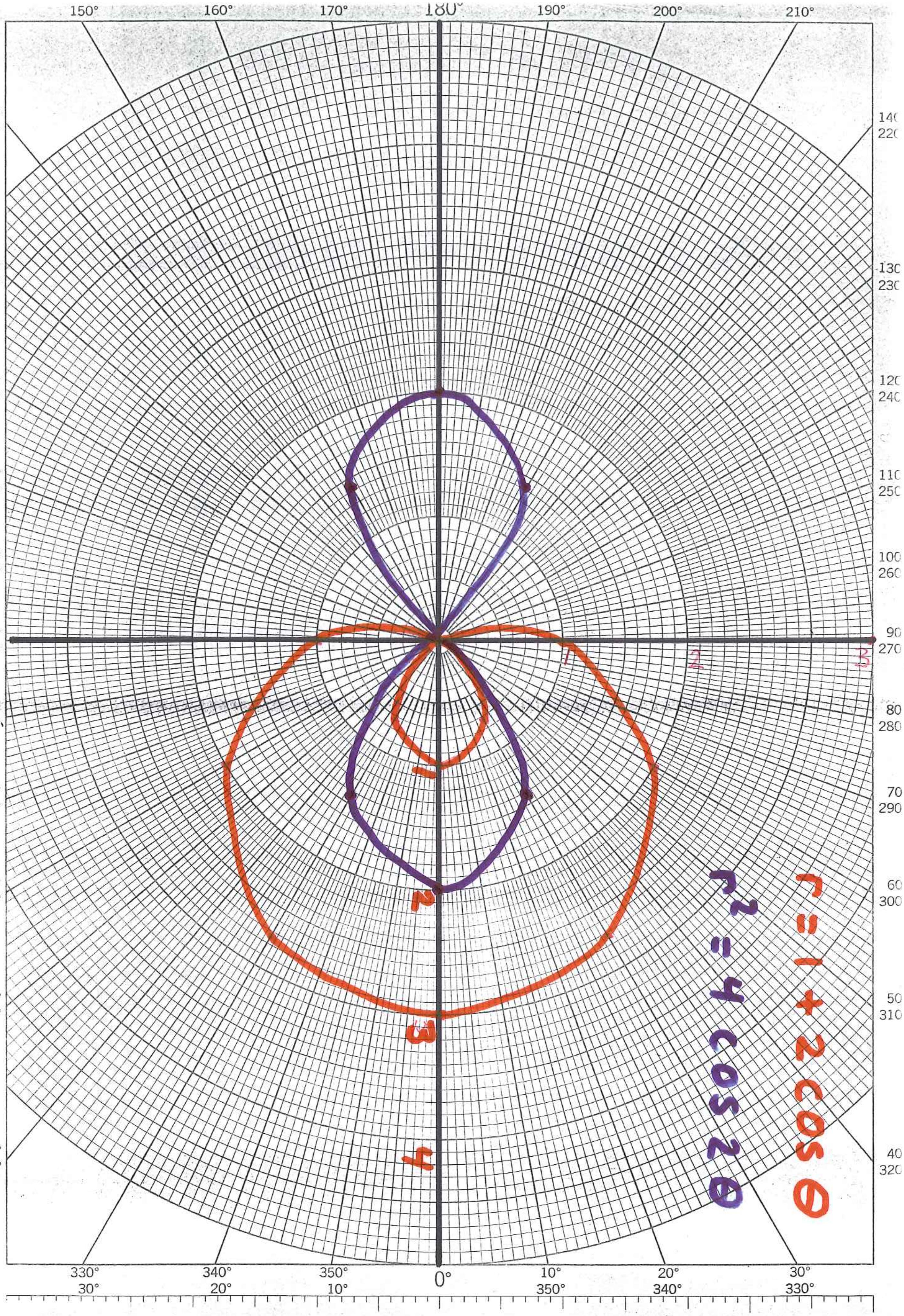
1. $r = 1 + 2\cos\theta$

Note that $r(\theta) = r(-\theta)$.
i.e. function is symmetric

θ	r
0	3
$\pi/6$	$1+\sqrt{3}$
$2\pi/6 = \pi/3$	2
$3\pi/6 = \pi/2$	1
$4\pi/6 = 2\pi/3$	0
$5\pi/6$	$1-\sqrt{3}$
$6\pi/6 = \pi$	$-\phi$
$7\pi/6$	$1-\sqrt{3}$
$8\pi/6 = 4\pi/3$	0
$9\pi/6 = 3\pi/2$	1
$10\pi/6 = 5\pi/3$	2
$11\pi/6$	$1+\sqrt{3}$
$12\pi/6 = 2\pi$	3

2. $r^2 = 4\cos 2\theta$ Note $r(\theta) = r(-\theta)$.

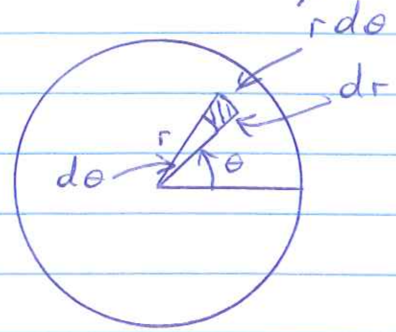
θ	r^2	r
0	4	± 2
$\pi/6$	2	$\pm\sqrt{2}$
$\pi/4$	0	0
$\pi/3$	-2	?



$r = 1 + 2\cos\theta$
$r^2 = 4\cos 2\theta$
1
2
3
4

Application of Polar Coordinates

Find the area of a circle having radius R.



Consider an infinitesimal section of circle between r & $r+dr$ and θ & $\theta + d\theta$.

Area of infinitesimal patch $= r\,d\theta\,dr$

$$\text{Area of Circle} = \int_0^R \int_0^{2\pi} r\,d\theta\,dr$$

$$= \int_0^R r\,dr \int_0^{2\pi} d\theta$$

$$= \left[\frac{r^2}{2}\right]_0^R \left[\theta\right]_0^{2\pi}$$

$$= \frac{R^2}{2} \cdot 2\pi$$

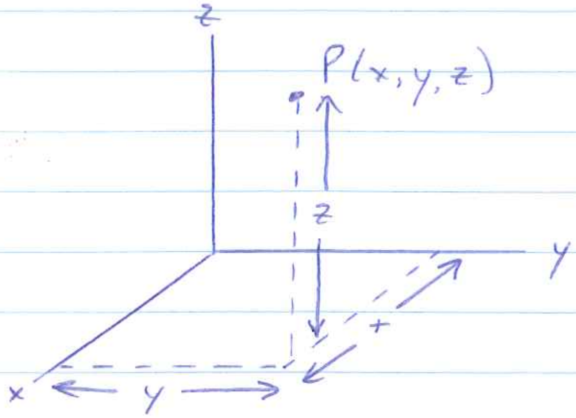
$$= \pi R^2$$

It would be very painful to do this problem in Cartesian coordinates.

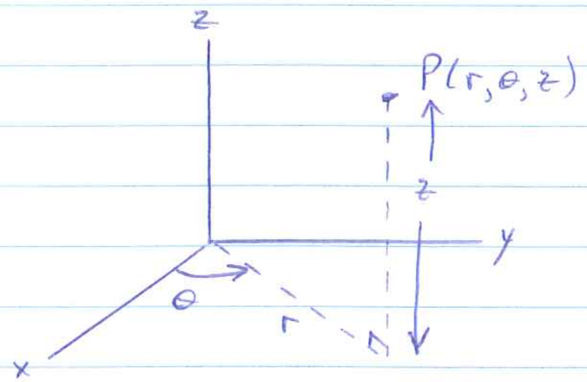
Three Dimensional Coordinate Systems

The location of a point is specified by 3 numbers.

Cartesian Coordinates



Cylindrical Coords. = Polar Coords. + z coordinate



$x = r\cos\theta$

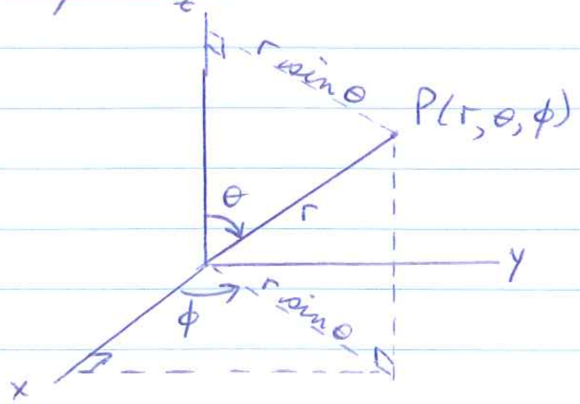
$y = r\sin\theta$

$z = z$

OR $r = \sqrt{x^2+y^2}$

$\theta = \arctan(y/x)$

Spherical Coordinates



$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Note:

1) r & θ in spherical coordinates are different than r & θ in cylindrical coordinates.

2) All points in 3 dimensional space are covered by: $(r \geq 0,\ 0 \leq \theta \leq \pi,\ 0 \leq \phi \leq 2\pi)$.

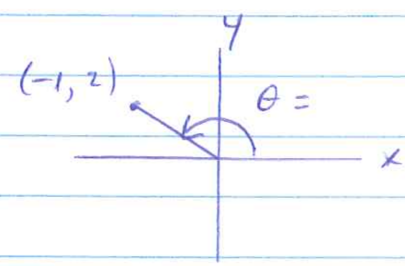
Conversion from Cartesian → Cylindrical

1) $P(x,y,z) = (2,2,2)$

$r = \sqrt{x^2+y^2} = \sqrt{2^2+2^2} = 2\sqrt{2}$

$\tan\theta = \frac{2}{2} = 1 \Rightarrow \theta = \frac{\pi}{4}.$

2) $P(\cancel{x},\cancel{y},\cancel{z}) = (-1,2,3)$



$r = \sqrt{1+4} = \sqrt{5}$

$\tan\theta = -2 \Longrightarrow \theta = 116.6^\circ$

Conversion from Cylindrical → Cartesian

1) $P(r,\theta,z) = (4, -\frac{5\pi}{6}, 2)$

$x = r\cos\theta = 4\cos\left(-\frac{5\pi}{6}\right) = -2\sqrt{3}$

$y = r\sin\theta = 4\sin\left(-\frac{5\pi}{6}\right) = 2$

Conversion from Cartesian → Spherical

1) $P(x, y, z) = (2, 2, 2)$

$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4+4+4} = 2\sqrt{3}$

$\cos\theta = \frac{z}{r} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = 54.7^\circ$

$\tan\phi = \frac{y}{x} = \frac{2}{2} = 1 \Rightarrow \phi = 45^\circ$

2) $P(x, y, z) = (-1, 2, 3)$

$r = \sqrt{1+4+9} = \sqrt{14}$

$\cos\theta = \frac{3}{\sqrt{14}} \Rightarrow \theta = 36.7^\circ$

$\tan\phi = \frac{2}{-1} = -2 \Rightarrow \phi = 116.6^\circ$

Conversion from Spherical → Cartesian

1) $P(r, \theta, \phi) = \left(3, \frac{\pi}{6}, \frac{\pi}{4}\right)$

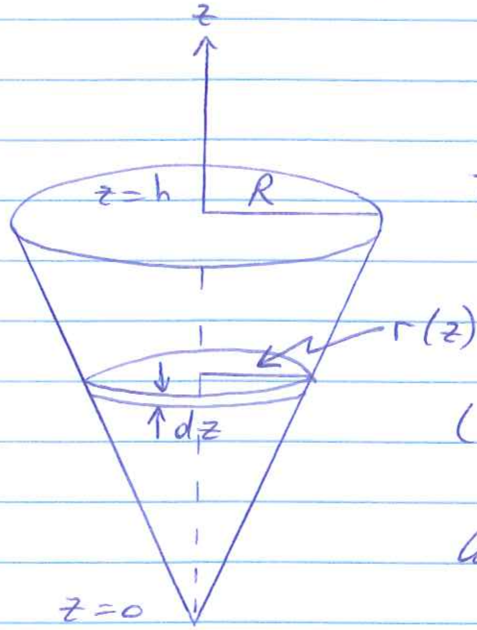
$x = r\sin\theta\cos\phi = 3\sin\frac{\pi}{6}\cos\frac{\pi}{4} = 3 \cdot \frac{1}{2}\frac{1}{\sqrt{2}} = \frac{3}{2\sqrt{2}}$

$y = r\sin\theta\sin\phi = 3\sin\frac{\pi}{6}\sin\frac{\pi}{4} = 3 \cdot \frac{1}{2}\frac{1}{\sqrt{2}} = \frac{3}{2\sqrt{2}}$

$z = r\cos\theta = 3\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2}$

Application of Cylindrical Coordinates

Find the volume of a cone having radius R & height h.



Consider disk at height z having thickness dz.

radius of disk $r(z) = \frac{z R}{h}$

(Check: $r(0) = 0$, $r(h) = R$)

Area of disk is $\pi r^2(z)$

Vol. of disk is $\pi r^2(z)\, dz$

$$\text{Volume of Cone} = \int_{z=0}^{z=h} \pi r^2(z)\, dz$$

$$= \int_0^h \pi \left(\frac{zR}{h}\right)^2 dz$$

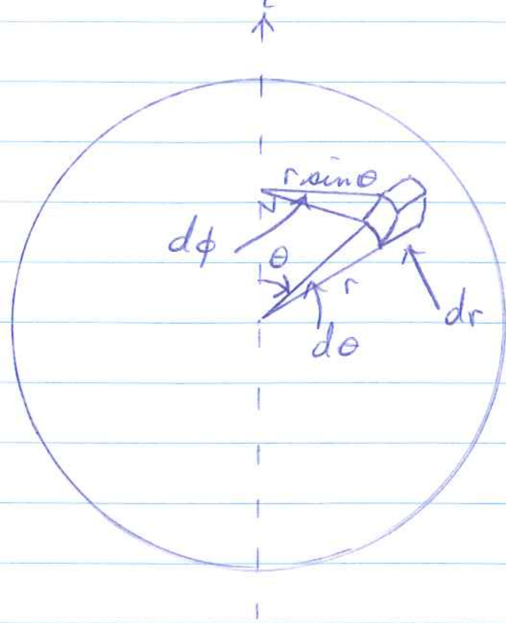
$$= \frac{\pi R^2}{h^2} \int_0^h z^2\, dz$$

$$= \frac{\pi R^2}{h^2} \left. \frac{z^3}{3} \right|_0^h$$

$$= \frac{\pi R^2 h}{3}$$

Application of Spherical Coordinates

Find volume of a sphere having radius R.



Consider infinitesimal volume element between r & r+dr, θ & θ+dθ and φ & φ+dφ.

Infinitesimal cube has volume

$$dV = (r\,d\theta)(r\sin\theta\,d\phi)(dr)$$

$$= r^2\sin\theta\;d\phi\,d\theta\,dr$$

Volume of Sphere $V = \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin\theta\, d\phi\, d\theta\, dr$

Upper limit on θ integral is not 2π since that would count the sphere's volume twice.

$$V = \int_0^R r^2\,dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi$$

$$= \frac{R^3}{3}\left(-\cos\theta\right)_0^\pi \; 2\pi$$

$$= \frac{R^3}{3}\left(-\cos\pi + \cos 0\right) 2\pi$$

$$= \frac{4\pi}{3} R^3$$

Complex Numbers

Imaginary Number

The solutions of $z^2+1=0$ are $z=\pm\sqrt{-1}$.

$\sqrt{-1}$ is not a real number. It is called an imaginary number and denoted by i. $i \equiv \sqrt{-1}$

Complex Number

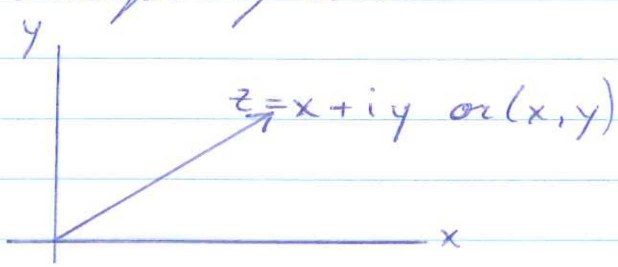
A complex number has the form $z = x+iy$ where $x, y \in R \equiv$ real numbers.

Nomenclature

One says the real part of z or $Re(z) = x$.
" " "imaginary" " " $Im(z) = y$.

Geometrical Interpretation

The complex number $z=x+iy$ can be plotted on a two dimensional graph where the horizontal or x axis is the real axis and the vertical or y axis is the imaginary axis.



Hence, z is a vector in the complex plane.

Definitions

Equality

Two complex numbers are equal if their real and imaginary parts are separately equal.

eg. Solve $a+1-2i = 3bi + 4 \qquad a, b \in R$

Equating real parts $\Rightarrow \quad a+1=4$

$a=3$

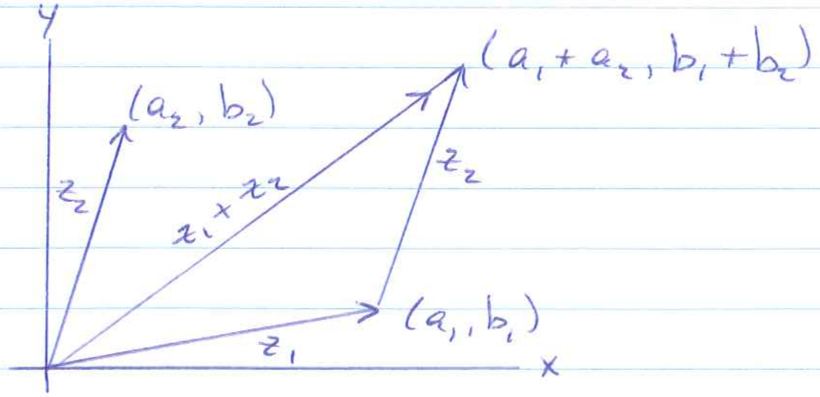
Equating imaginary parts $\Rightarrow \quad -2 = 3b$

$b = \frac{-2}{3}$

Addition / Subtraction

If $z_1 = a_1 + b_1 i$
$z_2 = a_2 + b_2 i$ $\quad (a_1, a_2, b_1, b_2 \in R)$ then $z_1 \pm z_2 = (a_1 \pm a_2) + i(b_1 \pm b_2)$

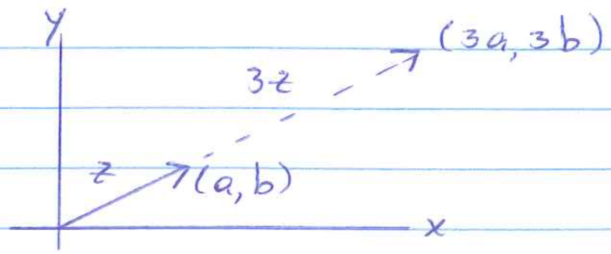
Graphically, addition can be viewed as follows.



Scalar Multiplication

If $k \in R$ then $kz = k(a + bi)$
$= ka + ikb$.

eg.



Multiplication of Two Complex Numbers

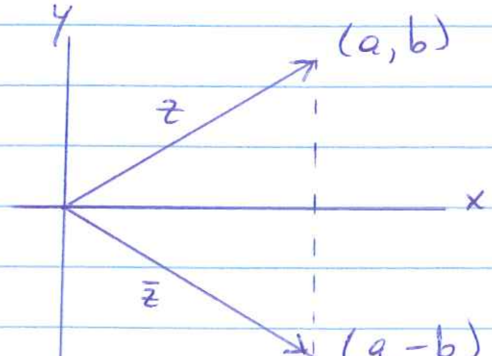
If $z_1 = a_1 + b_1 i$
$z_2 = a_2 + b_2 i$
then $z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i)$

$= a_1 a_2 + i a_1 b_2 + i a_2 b_1 - b_1 b_2$

$= a_1 a_2 - b_1 b_2 + i(a_1 b_2 + a_2 b_1)$

Complex Conjugate

If $z = a + bi$ then $\bar{z} \equiv a - bi$ and is called the conjugate of z.



Note that $z = \bar{z}$ if z is real.

Modulus of a Complex Number

The modulus or absolute value of a complex number $z = a + bi$ is defined as $|z| \equiv \sqrt{a^2 + b^2}$.

Theorem

$z\bar{z} = |z|^2$

Proof: Let $z = a + bi$
$\bar{z} = a - bi$

$$\begin{aligned} L.S. &= z\bar{z} \\ &= (a+bi)(a-bi) \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

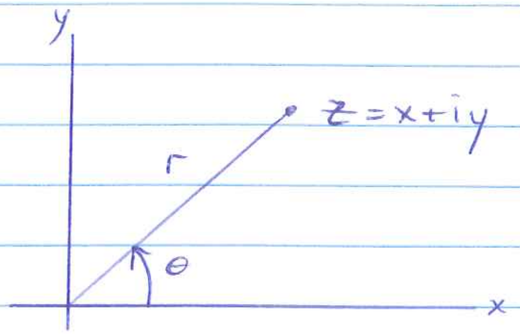
$\therefore L.S. = R.S.$

Example

Write $\dfrac{3+2i}{2+5i}$ in the form $a + bi$.

$$\begin{aligned} \frac{3+2i}{2+5i} &= \frac{3+2i}{2+5i} \cdot \frac{2-5i}{2-5i} \\ &= \frac{6 - 15i + 4i + 10}{4 + 25} \\ &= \frac{16 - 11i}{29} \end{aligned}$$

Polar Form of Complex Numbers



$x = r\cos\theta$

$y = r\sin\theta$

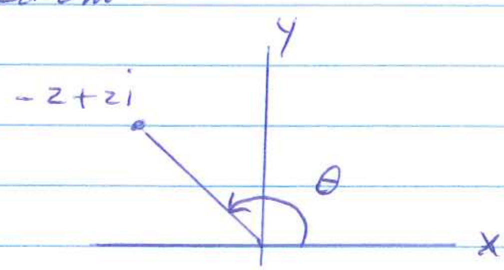
OR: $r = \sqrt{x^2+y^2}$

$\theta = \operatorname{Arctan}(y/x)$

$$\begin{aligned} \therefore\quad z &= x + iy \\ &= r\cos\theta + i\, r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

Example

Write $z = -2+2i$ in polar form.



$r = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$

$\theta = \operatorname{Arctan}\left(\frac{2}{-2}\right) = \frac{3\pi}{4}$

$$\therefore\ -2+2i = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Complex Exponents

We shall show that $e^{i\theta} = \cos\theta + i\sin\theta$.

$$\therefore \boxed{z = re^{i\theta}}$$

Aside: Taylor's Theorem

A function $f(x)$ can be expanded as a polynomial near a point $x = a$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

where $f^{(n)}(a)$ is the nth derivative of $f(x)$ evaluated at $x = a$ and $n! = n(n-1)(n-2)\ldots 2.1$ $(0! \equiv 1)$.

1) Find the Taylor Expansion of $f(\theta) = \cos\theta$ near $\theta = 0$.

$$\begin{array}{ll}
f(\theta) = \cos\theta & f(0) = 1 \\
f'(\theta) = -\sin\theta & f'(0) = 0 \\
f^{(2)}(\theta) = -\cos\theta & f^{(2)}(0) = -1 \\
f^{(3)}(\theta) = \sin\theta & f^{(3)}(0) = 0 \\
f^{(4)}(\theta) = \cos\theta & f^{(4)}(0) = 1 \\
\vdots & \vdots
\end{array}$$

$$\therefore \cos\theta = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (\theta - 0)^n$$

$$= \frac{f^{(0)}(0)}{0!}\theta^0 + \frac{f^{(1)}(0)}{1!}\theta^1 + \frac{f^{(2)}(0)}{2!}\theta^2 + \frac{f^{(3)}(0)}{3!}\theta^3 + \cdots$$

$$= 1 + 0 - \frac{\theta^2}{2!} + 0 + \frac{\theta^4}{4!} - + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + - \cdots$$

2) Exercise: Show Taylor's expansion of $\sin\theta$ near $\theta = 0$ is:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + - \dots$$

3) Find Taylor expansion of e^z near $z=0$.

$f(z) = e^z \qquad f(0)=1$
$f'(z) = e^z \qquad f'(0)=1$
$f^{(2)}(z) = e^z \qquad f^{(2)}(0)=1$
$\vdots \qquad\qquad \vdots$

$$\therefore e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(z-0)^n$$
$$= \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

Let $z = i\theta \Rightarrow e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \dots$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - + \dots\right)$$
$$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \dots\right)$$

$$\therefore e^{i\theta} = \cos\theta + i\sin\theta$$

De Moivre's Theorem

The preceding result is very useful if we wish to raise a complex number to a power.

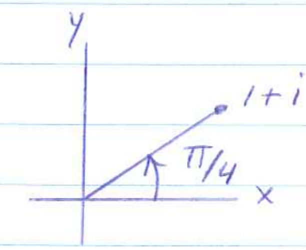
$$z^n = (r e^{i\theta})^n$$

$$= r^n e^{in\theta}$$

$$= r^n (\cos n\theta + i \sin n\theta)$$

Examples

1 Simplify $(1+i)^3$



$z = 1+i$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$\therefore 1+i = \sqrt{2}\, e^{i\pi/4}$

$$(1+i)^3 = \left(\sqrt{2}\, e^{i\pi/4}\right)^3$$

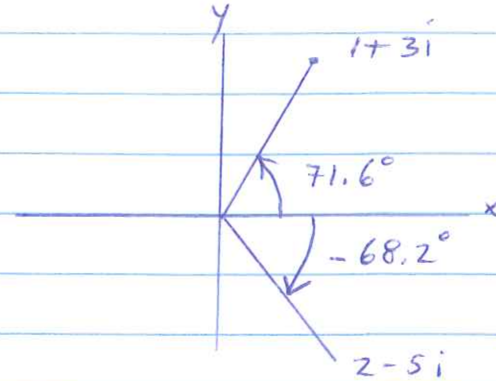
$$= 2^{3/2} e^{3i\pi/4}$$

$$= 2\sqrt{2}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$$

$$= 2\sqrt{2}\left(\frac{-1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right)$$

$$= -2 + 2i$$

2. Simplify $\dfrac{(1+3i)^2}{(2-5i)^3}$



$z_1 = 1+3i$

$r_1 = \sqrt{1+9} = \sqrt{10}$

$\theta_1 = \text{Arctan}\left(\frac{3}{1}\right) = 71.6°$

$z_2 = 2-5i$

$r_2 = \sqrt{4+25} = \sqrt{29}$

$\theta_2 = \text{Arctan}\left(\frac{-5}{2}\right) = -68.2°$

$$\therefore \frac{(1+3i)^2}{(2-5i)^3} = \frac{\left(\sqrt{10}\, e^{i71.6°}\right)^2}{\left(\sqrt{29}\, e^{-i68.2°}\right)^3}$$

$$= \frac{10}{29\sqrt{29}} \frac{e^{i\,143.2°}}{e^{-i204.6°}}$$

$$= \frac{10}{29\sqrt{29}}\, e^{i(143.2° + 204.6°)}$$

$$= \frac{10}{29\sqrt{29}}\, e^{i\,347.8°}$$

$$= \frac{10}{29\sqrt{29}} \left(\cos 347.8° + i \sin 347.8°\right)$$

$$= .0626 - i\,.0135$$

3. Find the cube roots of $z^3 = 1$

Now $1 = e^{i2k\pi}$ where $k \in \mathbb{Z}$ an integer

$$z^3 = e^{i2k\pi}$$

$$z = e^{i2k\pi/3}$$

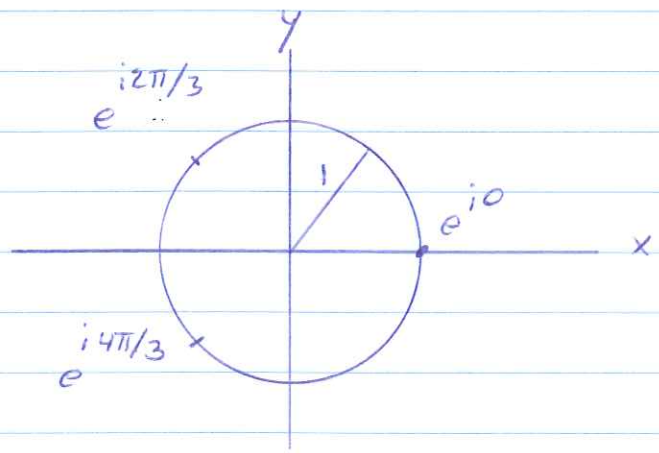
$k=0 \Rightarrow z = e^{i0} = 1$

$k=1 \Rightarrow z = e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k=2 \Rightarrow z = e^{i4\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$k=3 \Rightarrow z = e^{i2\pi} = 1$ same as for $k=0$.

For $k \geq 3$, the answers for $k=0,1,2$ repeat.



Note that the 3 roots all lie on a circle of radius 1 in the complex plane.

4. Find the fourth roots of $\sqrt{2}+\sqrt{2}\,i$.

i.e. solve $z^4 = \sqrt{2}+\sqrt{2}\,i$.

Exercise: Show $\sqrt{2}+\sqrt{2}\,i = 2e^{i(\pi/4+2k\pi)}$ $k \in \mathbb{Z}$

$$\therefore\ z^4 = 2e^{i(\pi/4+2k\pi)}$$

$$z = 2^{1/4}\, e^{i(\pi/16+k\pi/2)}$$

$k=0 \Rightarrow z = 2^{1/4} e^{\pi i/16}$

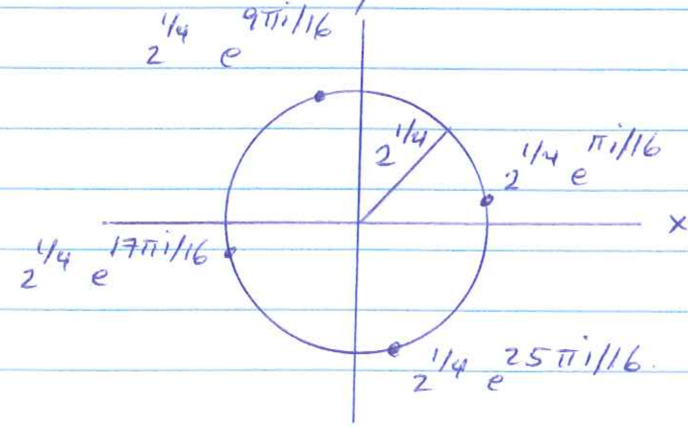
$k=1 \Rightarrow z = 2^{1/4} e^{9\pi i/16}$

$k=2 \Rightarrow z = 2^{1/4} e^{17\pi i/16}$

$k=3 \Rightarrow z = 2^{1/4} e^{25\pi i/16}$

$k=4 \Rightarrow z = 2^{1/4} e^{33\pi i/16} = 2^{1/4} e^{\pi i/16}$

i.e. solutions for $k \geq 4$ repeat.



Vectors

An n dimensional vector is an ordered group of n real numbers denoted by:

$$\vec{x} = (x_1, x_2, ---, x_n)$$

By ordered, we mean $(x_1, x_2, ---, x_n) \neq (x_2, x_1, ---, x_n)$

Examples

1) $\vec{x}$ = (# women in math, # men in math)

2) $\vec{x}$ = (E-W velocity component of plane, N-S velocity component of plane)

3) $\vec{x}$ = (# studs. in biology, # studs. in chemistry, # studs. in physics, # studs. in Comp. Sci, # studs in math)

4) $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$

x_1 = # people aged 0-20
x_2 = " " 21-40
x_3 = " " 41-60
x_4 = " " 61-80
x_5 = " " 81-100

Definition of Vector Addition

If $\vec{x} = (x_1, x_2, \ldots, x_n)$ & $\vec{y} = (y_1, y_2, \ldots, y_n)$ then

$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.

Definition of Scalar Multiplication

If $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $a \in R$ then

$a\vec{x} = (ax_1, ax_2, \ldots, ax_n)$

Properties

1) $\vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$ where $\vec{0} = (0, 0, \ldots, 0)$

2) $\vec{x} + (-\vec{x}) = \vec{0}$

3) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$

4) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

5) $0\vec{x} = \vec{0}$ and $1\vec{x} = \vec{x}$

6) $(a+b)\vec{x} = a\vec{x} + b\vec{x}$ $\quad a, b \in R$

7) $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$

Proof of 2: $\vec{x} + (-\vec{x}) = (x_1, x_2, \ldots, x_n) + (-x_1, -x_2, \ldots, -x_n)$

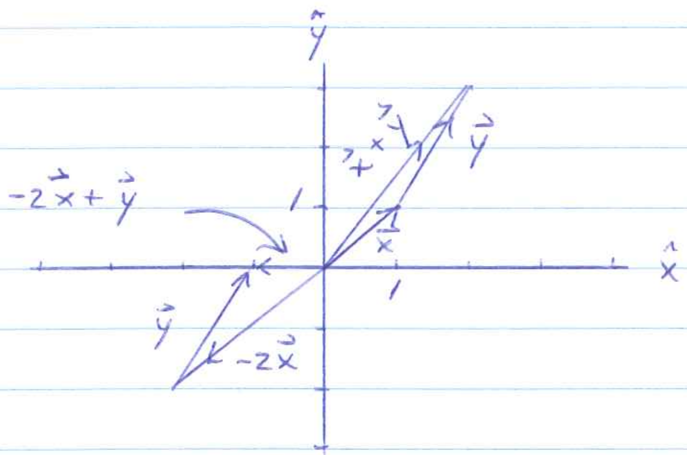
$$= (x_1 - x_1, x_2 - x_2, \ldots, x_n - x_n)$$
$$= (0, 0, \ldots, 0)$$
$$= \vec{0}$$

Examples

1) $\vec{x} = (1,1)$ Find $\vec{x} + \vec{y}$ and $-2\vec{x} + \vec{y}$.
$\vec{y} = (1,2)$

$$\vec{x} + \vec{y} = (1,1) + (1,2) = (2,3)$$

$$-2\vec{x} + \vec{y} = -2(1,1) + (1,2) = (-1,0)$$



2) Solve for a & b if $\vec{x} = 2\vec{y} - 3\vec{z}$ where $\vec{x} = (2,3)$
$\vec{y} = (4, a)$ & $\vec{z} = (b, 1)$.

$$\vec{x} = 2\vec{y} - 3\vec{z}$$

$$(2,3) = 2(4,a) - 3(b,1)$$

$$= (8, 2a) + (-3b, -3)$$
$$= (8-3b, 2a-3)$$

Equating x components $\Rightarrow$ $2 = 8 - 3b$
$6 = -3b$
$b = -2$

Equating y components $\Rightarrow$ $3 = 2a - 3$
$6 = 2a$
$a = 3$

Dot Product (Scalar Product)

If $\vec{x} = (x_1, x_2, \ldots, x_n)$ & $\vec{y} = (y_1, y_2, \ldots, y_n)$ then

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

Example

$\vec{N}$ = (# horses, # cows, # pigs)

$\vec{P}$ = (horse value, cow value, pig value)

$\therefore$ $\vec{N} \cdot \vec{P}$ = total value of all livestock

Properties of Dot Product

1) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$

2) $(a\vec{x}) \cdot (b\vec{y}) = (ab)(\vec{x} \cdot \vec{y})$

3) $\vec{x} \cdot (\vec{y} + \vec{w}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{w}$

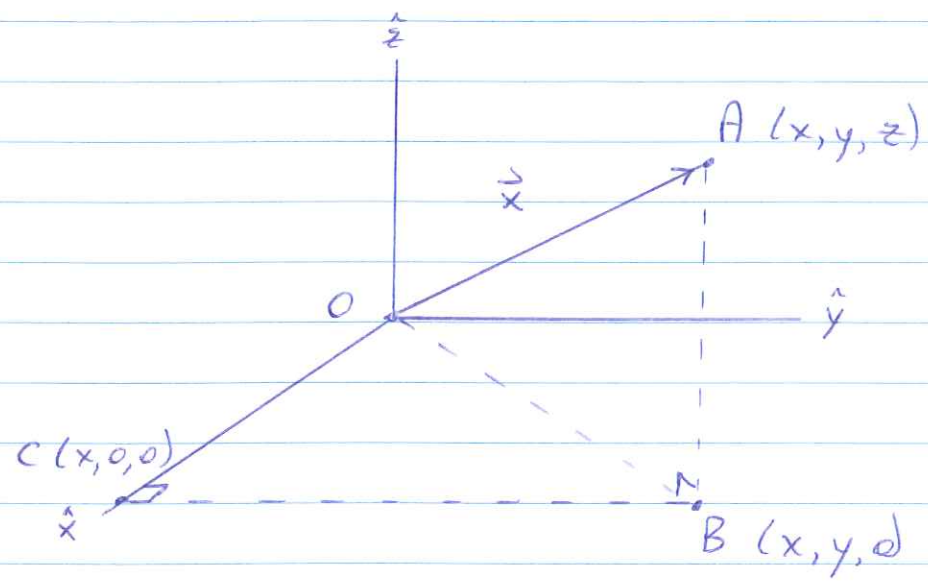
Length of $\vec{x}$

The length of a vector is denoted by $|\vec{x}|$ and is defined to be:

$$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}}$$

$$= \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$$

For a vector in 3 dimensions, $|\vec{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. This agrees with the result of the Pythagorean Theorem.



$$OB = \sqrt{(OC)^2 + (CB)^2} = \sqrt{x^2 + y^2}$$

Hence, $|\vec{x}| = OA$

$$= \sqrt{(OB)^2 + (BA)^2}$$

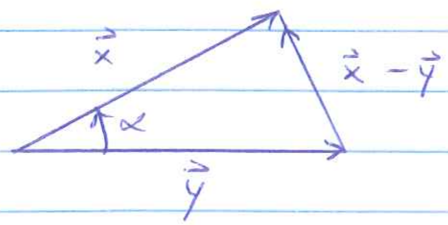
$$\therefore |\vec{x}| = \sqrt{x^2 + y^2 + z^2}$$

Example

$\vec{x} = (1, -3, 4)$ has length $|\vec{x}| = \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{26}$

A Useful Result (2 & 3 dim. vectors)

Consider vectors $\vec{x}$ & $\vec{y}$ intersecting at angle α.



$$|\vec{x}-\vec{y}|^2 = (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})$$

$$= \vec{x}\cdot\vec{x} - 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y}$$

$$= |\vec{x}|^2 + |\vec{y}|^2 - 2\vec{x}\cdot\vec{y} \quad (1)$$

Using the cosine law for the above triangle, we get:

$$|\vec{x}-\vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}||\vec{y}|\cos\alpha \quad (2)$$

$\therefore (1) + (2) \Longrightarrow \boxed{\vec{x}\cdot\vec{y} = |\vec{x}||\vec{y}|\cos\alpha}$

Orthogonal Vectors

Two n dimensional vectors $\vec{x}$ & $\vec{y}$ are said to be orthogonal if $\vec{x}\cdot\vec{y}=0$.

For 2 or 3 dim. vectors $\quad \vec{x}\cdot\vec{y}=0$

$$\Longrightarrow |\vec{x}||\vec{y}|\cos\alpha = 0$$

$$\cos\alpha = 0 \quad \text{if } |\vec{x}|, |\vec{y}| \neq 0$$

$$\alpha = \pi/2$$

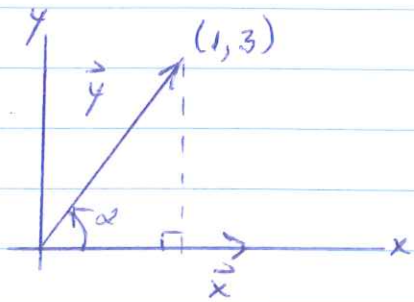
Examples

1) For $\vec{x} = (2,0)$ $\vec{y} = (1,3)$ show that $\vec{x}\cdot\vec{y} = |\vec{x}||\vec{y}|\cos\alpha$

$\vec{x}\cdot\vec{y} = (2,0)\cdot(1,3) = 2$

$|\vec{x}| = \sqrt{2^2+0^2} = 2$

$|\vec{y}| = \sqrt{1^2+3^2} = \sqrt{10}$



$$\cos\alpha = \frac{1}{\sqrt{1^2+3^2}} = \frac{1}{\sqrt{10}}$$

$\therefore$ $|\vec{x}||\vec{y}|\cos\alpha = 2\sqrt{10}\cdot\frac{1}{\sqrt{10}} = 2$

$\Longrightarrow$ $\vec{x}\cdot\vec{y} = |\vec{x}||\vec{y}|\cos\alpha$.

2) Find a vector orthogonal to $(3,4)$ having unit length.

Let vector be (x_1, x_2)

Two vectors are orthogonal $\Rightarrow$ $(x_1, x_2)\cdot(3,4) = 0$

$$3x_1 + 4x_2 = 0$$

$$x_1 = -\frac{4}{3}x_2 \quad (1)$$

(x_1, x_2) has unit length $\Longrightarrow$ $\sqrt{x_1^2+x_2^2} = 1$

$$x_1^2 + x_2^2 = 1 \quad (2)$$

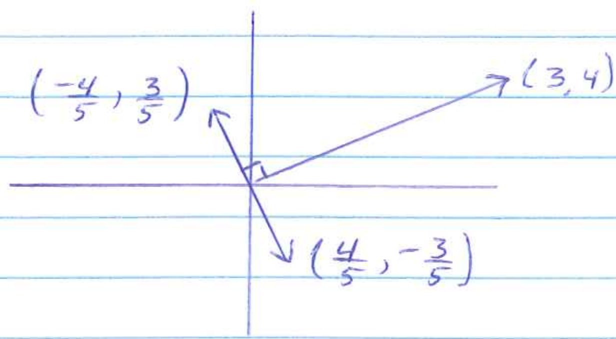
Substitute (1) into (2) $\Rightarrow$ $\frac{16}{9} x_2^2 + x_2^2 = 1$

$$\frac{25}{9} x_2^2 = 1$$

$$x_2 = \pm \frac{3}{5}$$

Subst. x_2 in (1) $\Rightarrow$ $x_1 = \mp \frac{4}{5}$.

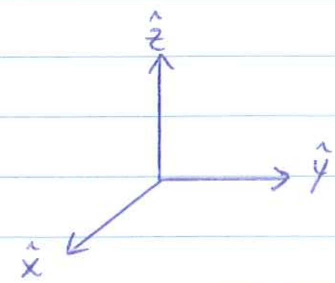
Hence, there are two possible answers $\left(\frac{-4}{5}, \frac{3}{5}\right)$ & $\left(\frac{4}{5}, \frac{-3}{5}\right)$.



Unit Vectors

Cartesian Coordinates

$\vec{r} = (x, y, z)$



$$\hat{x} = \frac{\partial \vec{r}/\partial x}{|\partial \vec{r}/\partial x|} = \frac{(1,0,0)}{|(1,0,0)|} = (1,0,0)$$

$$\hat{y} = \frac{\partial \vec{r}/\partial y}{|\partial \vec{r}/\partial y|} = (0,1,0)$$

$$\hat{z} = \frac{\partial \vec{r}/\partial z}{|\partial \vec{r}/\partial z|} = (0,0,1)$$

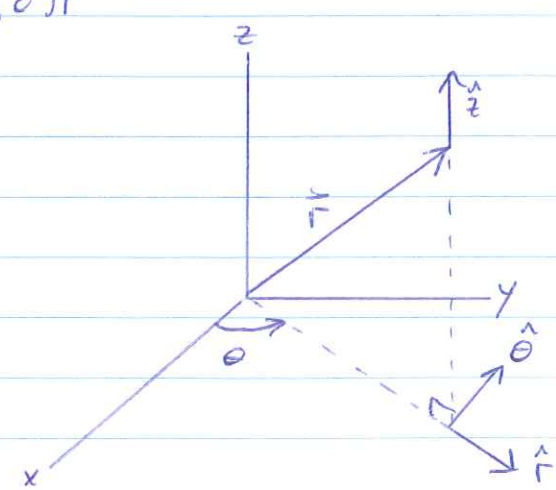
Cylindrical Coordinates

$\vec{r} = (r\cos\theta, r\sin\theta, z)$

$$\hat{r} = \frac{\partial \vec{r}/\partial r}{|\partial \vec{r}/\partial r|} = \frac{(\cos\theta, \sin\theta, 0)}{|(\cos\theta, \sin\theta, 0)|} = (\cos\theta, \sin\theta, 0)$$

$$\hat{\theta} = \frac{\partial \vec{r}/\partial \theta}{|\partial \vec{r}/\partial \theta|} = \frac{(-r\sin\theta, r\cos\theta, 0)}{|(-r\sin\theta, r\cos\theta, 0)|} = (-\sin\theta, \cos\theta, 0)$$

$$\hat{z} = \frac{\partial \vec{r}/\partial z}{|\partial \vec{r}/\partial z|} = (0,0,1)$$



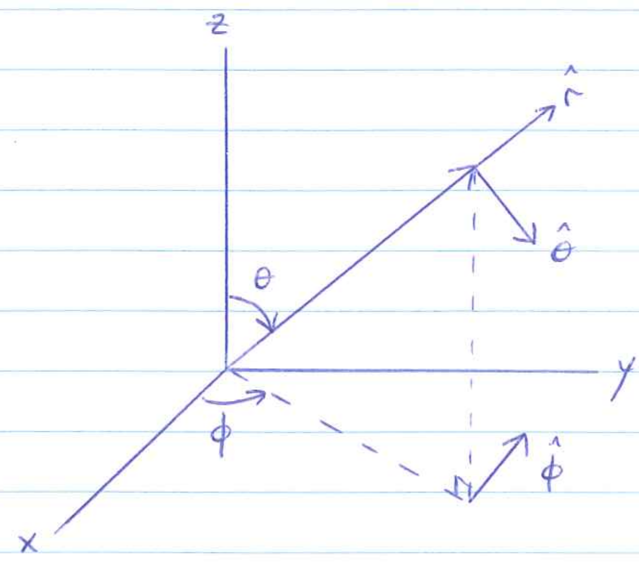
Spherical Coordinates

$$\vec{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\hat{r} = \frac{\partial\vec{r}/\partial r}{|\partial\vec{r}/\partial r|} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{|\partial\vec{r}/\partial\theta|} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$\hat{\phi} = \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|} = (-\sin\phi, \cos\phi, \cancel{\mathrm{si}})$$



Applications of Vectors

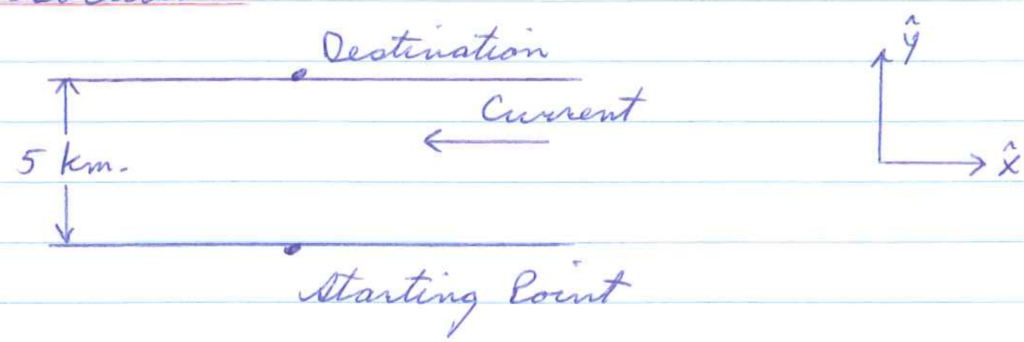
1) A boat crosses a 5 km. wide river having a 10 km/hr. current. In still water, the boat has a top speed of 20 km/hr. The destination is directly across from the starting point.

a) What direction should the captain steer so as to arrive in minimum time?

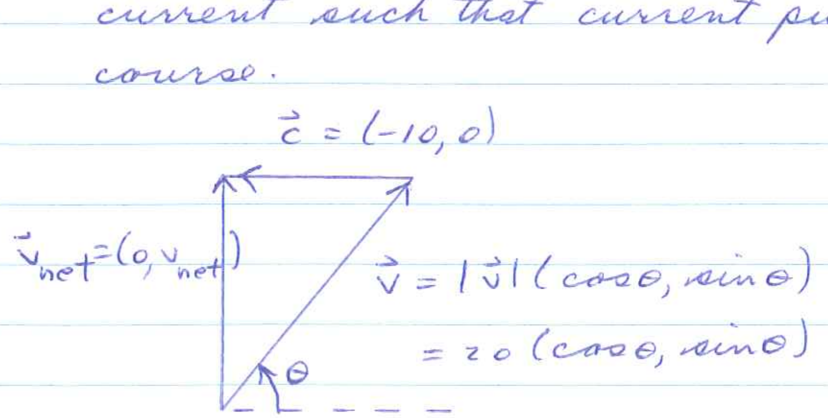
b) What is the resultant speed of the boat with respect to land?

c) What is the minimum time of the trip?

Solution



a) Fastest trip occurs when captain steers into current such that current pushes him back on course.



$$\vec{v}_{net} = \vec{v} + \vec{c}$$

$$(0, v_{net}) = (20\cos\theta, 20\sin\theta) + (-10, 0)$$

Equating $\hat{x}$ components $\Rightarrow$ $0 = 20\cos\theta - 10$

$$\cos\theta = \frac{1}{2}$$

$$\theta = 60°$$

$\therefore$ captain should steer a course $60°$ away from $\hat{x}$ direction towards $\hat{y}$.

b) Equating $\hat{y}$ components $\Rightarrow$ $v_{net} = 20\sin\theta + 0$

$$= 20\sin 60°$$

$$= 10\sqrt{3}\ \text{km/hr.}$$

$\therefore$ boat's net speed is $10\sqrt{3}$ km/hr.

c) Minimum time of trip $= \dfrac{5\ \text{km}}{10\sqrt{3}\ \text{km/hr.}}$

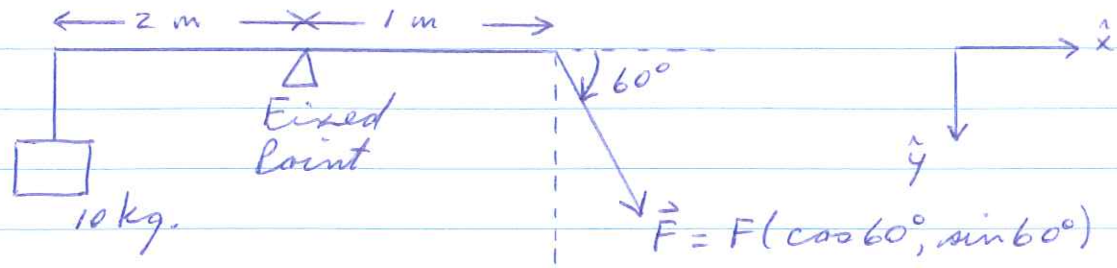
$$= .29\ \text{hr}$$

$$= 17\ \text{minutes}$$

2) Lever Principle

A lever balances when $\sum_i F_{i\perp} r_i = 0$ where $F_{i\perp}$ is component of force F_i exerted $\perp$ to lever. r_i is distance between pivot point to position where F_i is exerted.

Example



How large must F be to balance the 10 kg. mass?

Solution

Horizontal component of $\vec{F}$, $F \cos 60°$ does nothing to balance 10 kg.
Downward component of $\vec{F}$ is $F \sin 60°$.

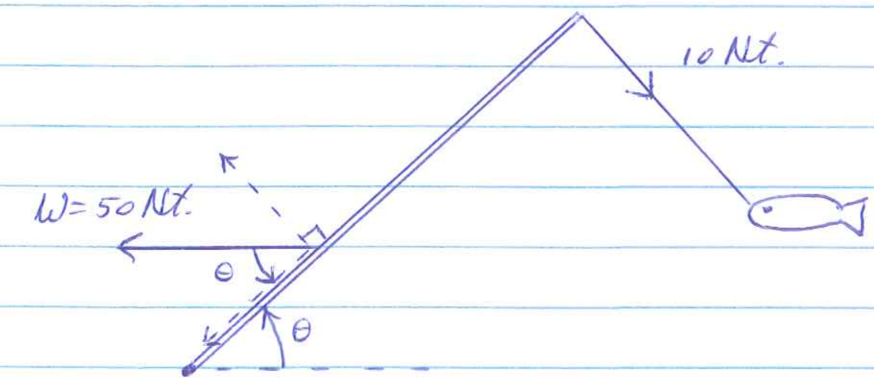
$\therefore$ for balance $\Rightarrow$ 10 kg. $\times\ g \times$ 2 meters $= F \sin 60° \times 1$ meter

accel. due to gravity 10 m/sec²

$$F = \frac{10 \times 10 \times 2}{1 \sin 60°}$$

$$= \frac{400\sqrt{3}}{3} \text{ Newtons}$$

3) A fish exerts a 10 Nt. force on a 2 meter long fishing rod stuck in mud. The fisherman has his hands 1/2 meter from the pivot end and pulls with a force of 50 Nt.



Find θ so that the rod remains stationary.

Solution

Component of $\vec{W}$ along rod, $W\cos\theta$ does nothing to balance the pull of the fish. This is done by the component of $\vec{W}$ perpendicular to the rod, $W\sin\theta$.

$$\therefore W\sin\theta \times \frac{1}{2} = 2 \times 10$$

$$\sin\theta = \frac{2\times2\times10}{50}$$

$$= .8.$$

$$\therefore \theta = 53^\circ.$$

Cross Product

For two 3 dimensional vectors $\vec{x} = (x_1, x_2, x_3)$ & $\vec{y} = (y_1, y_2, y_3)$ the cross product, denoted by $\vec{x} \times \vec{y}$ is defined by:

$$\vec{x} \times \vec{y} = (x_2 y_3 - x_3 y_2, -x_1 y_3 + x_3 y_1, x_1 y_2 - x_2 y_1)$$

Alternate Definition of $\vec{x} \times \vec{y}$

$$\vec{x} \times \vec{y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \qquad \left.\begin{array}{l} \hat{i} = (1,0,0) \\ \hat{j} = (0,1,0) \\ \hat{k} = (0,0,1) \end{array}\right\} \begin{array}{l} \text{Unit Vectors} \\ \text{in } x, y \,\&\, z \\ \text{directions.} \end{array}$$

$$= \hat{i} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \hat{j} \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \hat{k} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$= \hat{i}(x_2 y_3 - x_3 y_2) - \hat{j}(x_1 y_3 - x_3 y_1) + \hat{k}(x_1 y_2 - x_2 y_1)$$

$$= (x_2 y_3 - x_3 y_2, -x_1 y_3 + x_3 y_1, x_1 y_2 - x_2 y_1)$$

Properties of Cross Product

1) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

2) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

3) $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v}) \qquad k \in R$

4) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

Proof of 4

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2,\ -u_1 v_3 + u_3 v_1,\ u_1 v_2 - u_2 v_1)$$

$$\vec{v} \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$= (v_2 u_3 - v_3 u_2,\ -v_1 u_3 + v_3 u_1,\ v_1 u_2 - v_2 u_1)$$

$$\therefore \vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$$

Identities Involving Dot & Cross Products

1) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v})$

2) $\vec{u} \times (\vec{v} \times \vec{w}) = \vec{v}\,(\vec{u} \cdot \vec{w}) - \vec{w}\,(\vec{u} \cdot \vec{v})$

3) $(\vec{u} \times \vec{v})^2 = u^2 v^2 - (\vec{u} \cdot \vec{v})^2$ where $u \equiv |\vec{u}|$, $v \equiv |\vec{v}|$.

Direction of $\vec{x} \times \vec{y}$

Note that $\begin{cases} \vec{x} \cdot (\vec{x} \times \vec{y}) = 0 \\ \vec{y} \cdot (\vec{x} \times \vec{y}) = 0. \end{cases} \Rightarrow \vec{x} \times \vec{y}$ is $\perp$ to $\vec{x}$ & $\vec{y}$.

Proof that $\vec{x} \cdot (\vec{x} \times \vec{y}) = 0$.

$$\vec{x} \cdot (\vec{x} \times \vec{y}) = \vec{x} \cdot (x_2 y_3 - x_3 y_2, -x_1 y_3 + x_3 y_1, x_1 y_2 - x_2 y_1)$$

$$= x_1(x_2 y_3 - x_3 y_2) + x_2(-x_1 y_3 + x_3 y_1) + x_3(x_1 y_2 - x_2 y_1)$$

$$= x_1 x_2 y_3 - x_1 x_3 y_2 - x_1 x_2 y_3 + x_2 x_3 y_1 + x_1 x_3 y_2 - x_2 x_3 y_1$$

$$= 0$$

Exercise: Show that $\vec{y} \cdot (\vec{x} \times \vec{y}) = 0$

Magnitude of $\vec{x} \times \vec{y}$

Property 3 of previous page gives:

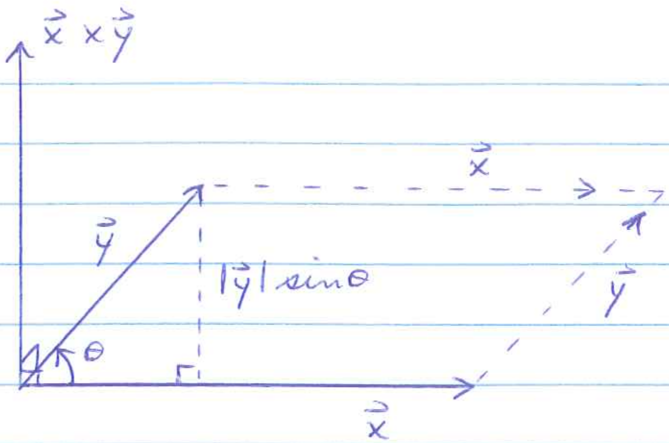
$|\vec{x} \times \vec{y}|^2 = x^2 y^2 - (\vec{x} \cdot \vec{y})^2$ where $x \equiv |\vec{x}|, y \equiv |\vec{y}|$

$= x^2 y^2 - x^2 y^2 \cos^2\theta$ where θ is angle between $\vec{x}$ & $\vec{y}$

$= x^2 y^2 (1 - \cos^2\theta)$

$= x^2 y^2 \sin^2\theta.$

$\therefore |\vec{x} \times \vec{y}| = xy \sin\theta$



Hence $\vec{x} \times \vec{y}$ is a vector perpendicular to both $\vec{x}$ + $\vec{y}$ having magnitude equal to the area of a parallelogram created by $\vec{x}$ + $\vec{y}$.

Right Hand Rule

1) Point fingers along $\vec{x}$ (of right hand!!!)

2) Move fingers toward $\vec{y}$

$\Rightarrow$ Thumb points along $\vec{x} \times \vec{y}$.

Examples

1) $\hat{x} \times \hat{y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$ $\quad (\hat{x} \equiv \hat{i}, \hat{y} \equiv \hat{j}, \hat{z} \equiv \hat{k})$

$= (0, 0, 1)$

$\therefore \hat{x} \times \hat{y} = \hat{z}$ in agreement with right hand rule.

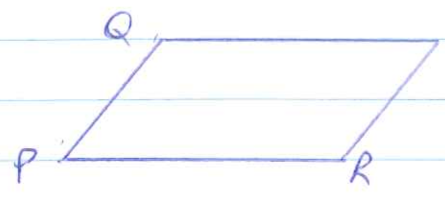
2) $(1, 2, 0) \times (1, 1, 0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix}$

$= (0, 0, 1.1 - 2.1)$

$= (0, 0, -1)$

Note that $(0,0,-1)$ is $\perp$ to $(1,2,0)$ + $(1,1,0)$ which each lie in xy plane.

3) Find the area of the parallelogram having vertices $P(1, 3, -2)$ $Q(2, 1, 4)$ + $R(-3, 1, 6)$.



$\vec{PQ} = (2, 1, 4) - (1, 3, -2) = (1, -2, 6)$

$\vec{PR} = (-3, 1, 6) - (1, 3, -2) = (-4, -2, 8)$

Area of Parallelogram is $|\vec{PQ} \times \vec{PR}|$

$= \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 6 \\ -4 & -2 & 8 \end{vmatrix} \right|$

$= |(-16 + 12, -8 + 24, -2 - 8)|$

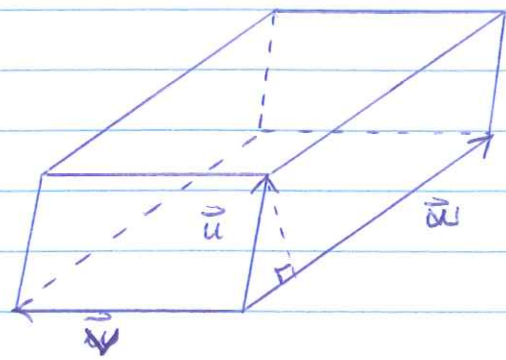
$= |(-4, 16, -10)|$

$= \sqrt{(-4)^2 + 16^2 + (-10)^2} = \sqrt{372}$ square units

Geometric Interpretation of $\vec{u}\cdot(\vec{v}\times\vec{w})$

$|\vec{v}\times\vec{w}|$ is the area of parallelogram created by $\vec{v}$ & $\vec{w}$

$|\vec{u}\cdot(\vec{v}\times\vec{w})|$ " " volume of parallelopiped created by $\vec{u}$, $\vec{v}$ & $\vec{w}$



Example

Find the parallelopiped volume created by
$\vec{u}=(1,1,0)$ $\vec{v}=(-2,0,1)$ & $\vec{w}=(1,2,-1)$.

$$\text{Volume} = \left|(1,1,0)\cdot\left[(-2,0,1)\times(1,2,-1)\right]\right|$$

$$= \left|(1,1,0)\cdot\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & 1\\ 1 & 2 & -1\end{vmatrix}\right|$$

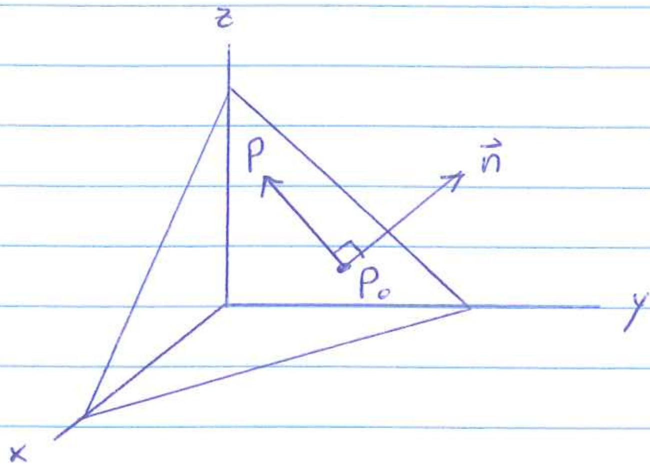
$$= \left|(1,1,0)\cdot(-2,\,-2+1,\,-4)\right|$$

$$= |-2-1+0|$$

$= 3$ cubic units

Equation of a Plane in 3 Dimensional Space

A plane contains points $P(x, y, z)$ and $P_0(x_0, y_0, z_0)$.
Vector $\vec{n} = (a, b, c)$ is perpendicular to the plane.



$\therefore \overrightarrow{P_0P}$ lies in plane.

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

$$(x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Eqn. of Plane. $ax + by + cz + d = 0$ where d is a constant

Example

Find equation of plane containing point $(3, -1, 7)$ and having perpendicular vector $\vec{n} = (4, 2, -5)$.

$\vec{n} = (a, b, c) \Rightarrow a = 4, b = 2, c = -5$

Eqn. of plane $4x + 2y - 5z + d = 0$.

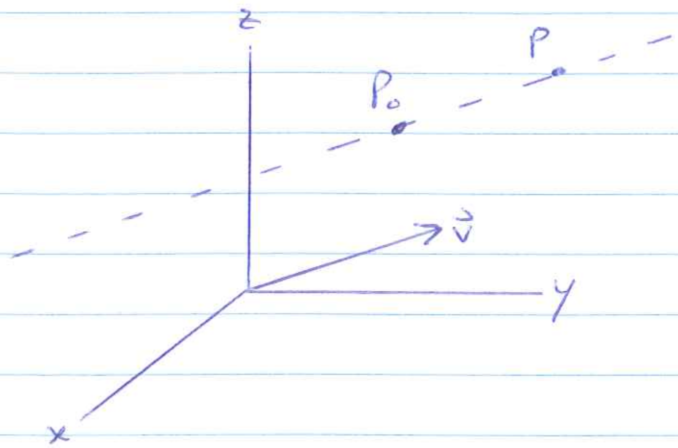
$(3, -1, 7)$ on plane $\Rightarrow 12 - 2 - 35 + d = 0$

$d = 25$

$\therefore$ plane is $4x + 2y - 5z + 25 = 0$.

Equation of a Line

Consider a line passing through point $P_0(x_0, y_0, z_0)$ and in direction parallel to $\vec{v} = (a, b, c)$.



Let $P(x, y, z)$ be an arbitrary point on the line.

i.e. $\overrightarrow{PP_0} = t\,\vec{v}$ where t is a scalar parameter

$$(x - x_0,\ y - y_0,\ z - z_0) = t(a, b, c)$$

OR: $x = x_0 + ta$ Parametric Equations of a Line

$y = y_0 + tb$

$z = z_0 + tc$

Example

Find the equation of a line passing through point $(1, 2, -3)$ and parallel to $\vec{v} = (4, 5, -7)$.

$P_0(x_0, y_0, z_0) \Rightarrow x_0 = 1,\ y_0 = 2,\ z_0 = -3$ $\Longrightarrow$ $x = 1 + 4t$

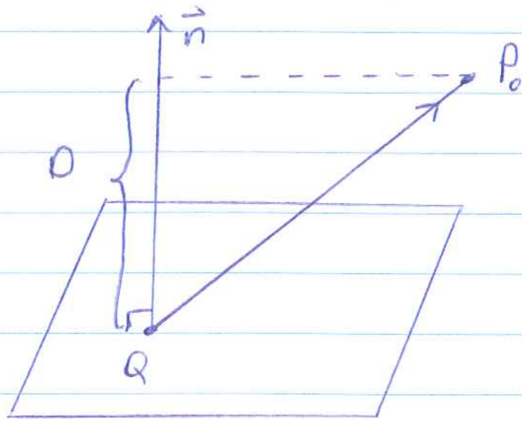
$\vec{v} = (a, b, c) \Rightarrow a = 4,\ b = 5,\ c = -7$ $y = 2 + 5t$

$z = -3 - 7t$

"Closest" Distance Between Point $P_0(x_0, y_0, z_0)$ & Plane $ax + by + cz + d = 0$

Consider point $Q(x_1, y_1, z_1)$ on plane.

i.e. $ax_1 + by_1 + cz_1 + d = 0$ (1)



$\overrightarrow{QP_0} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$

$\vec{n} = (a, b, c)$ normal to plane.

Closest Distance $D = |\overrightarrow{QP_0} \cdot \hat{n}|$ $\hat{n}$ = unit vector

$$= \frac{|\overrightarrow{QP_0} \cdot \vec{n}|}{|\vec{n}|}$$

$$= \frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (a, b, c)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_0 + by_0 + cz_0 - ax_1 - by_1 - cz_1|}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore \boxed{D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$ using (1)

Examples

Find distance between plane $2x-3y+6z=1$ and the points a) $(1,-4,-3)$
b) $(2,1,0)$

a) $\text{Distance} = \dfrac{|2\cdot 1 + (-3)(-4) + 6\cdot(-3) - 1|}{\sqrt{2^2+(-3)^2+6^2}} = \dfrac{3}{7}$

b) $\text{Distance} = \dfrac{|2\cdot 2 + (-3)\cdot 1 + 6\cdot 0 - 1|}{\sqrt{2^2+(-3)^2+6^2}} = 0$

Answer in b results because $(2,1,0)$ is on the plane.

Linear Combination

A vector $\vec{w}$ is said to be a linear combination of vectors $\vec{v}_1, \vec{v}_2, \dots \vec{v}_r$ if

$$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_r\vec{v}_r.$$

Example

Express $\vec{w} = (1,1)$ as a linear combination of $\vec{v}_1 = (1,-1)$ & $\vec{v}_2 = (3,0)$.

$$\begin{aligned} \text{Let } \vec{w} &= k_1\vec{v}_1 + k_2\vec{v}_2 \\ (1,1) &= k_1(1,-1) + k_2(3,0) \\ &= (k_1 + 3k_2, -k_1) \end{aligned}$$

Equating $\hat{y}$ components $\Rightarrow$ $1 = -k_1$

$k_1 = -1$

Equating $\hat{x}$ components $\Rightarrow$ $1 = k_1 + 3k_2$

$= -1 + 3k_2$

$k_2 = \frac{2}{3}$

$$\therefore \vec{w} = -\vec{v}_1 + \frac{2}{3}\vec{v}_2.$$

Linear Independence

A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is said to be linearly independent if the only solution of $k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_r\vec{v}_r = \vec{0}$ is $k_1 = k_2 = \ldots = k_r = 0$.

Examples

1) Are $\vec{v}_1 = (1,-1)$ & $\vec{v}_2 = (3,0)$ linearly independent?

Let $k_1\vec{v}_1 + k_2\vec{v}_2 = \vec{0}$

$$k_1(1,-1) + k_2(3,0) = (0,0)$$
$$(k_1 + 3k_2, -k_1) = (0,0)$$

Equating $\hat{y}$ components $\Rightarrow$ $-k_1 = 0$

$k_1 = 0$

Equating $\hat{x}$ components $\Rightarrow$ $k_1 + 3k_2 = 0$

$0 + 3k_2 = 0$

$k_2 = 0$

$\therefore k_1 = k_2 = 0 \Rightarrow \vec{v}_1$ & $\vec{v}_2$ are linearly independent.

2) Are $\vec{v}_1 = (1,-1)$ & $\vec{v}_2 = (-2,2)$ linearly independent?

Let $k_1\vec{v}_1 + k_2\vec{v}_2 = \vec{0}$

$$k_1(1,-1) + k_2(-2,2) = (0,0)$$
$$(k_1 - 2k_2, -k_1 + 2k_2) = (0,0)$$

Equating $\hat{x}$ components $\Rightarrow k_1 - 2k_2 = 0$
" " $\hat{y}$ " " $\Rightarrow -k_1 + 2k_2 = 0$

$$\left.\begin{array}{l} k_1 - 2k_2 = 0 \\ -k_1 + 2k_2 = 0 \end{array}\right\} \text{Identical Equations}$$

$\therefore$ $k_1\vec{v}_1 + k_2\vec{v}_2 = \vec{0}$ if $k_1 = 2k_2$ and $\vec{v}_1$ & $\vec{v}_2$ are not linearly independent.

3) Are $\vec{v}_1 = (1,-2,3)$, $\vec{v}_2 = (2,-2,0)$ & $\vec{v}_3 = (0,1,3)$ linearly independent?

Let $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0}$

$$k_1(1,-2,3) + k_2(2,-2,0) + k_3(0,1,3) = (0,0,0)$$
$$(k_1 + 2k_2, -2k_1 - 2k_2 + k_3, 3k_1 + 3k_3) = (0,0,0)$$

$\therefore$ $k_1 + 2k_2 = 0 \Longrightarrow k_2 = -\tfrac{1}{2}k_1$
$-2k_1 - 2k_2 + k_3 = 0$
$3k_1 + 3k_3 = 0 \Longrightarrow k_3 = -k_1$

Substituting k_2 & k_3 into the second equation gives:

$$-2k_1 - 2\left(-\tfrac{1}{2}k_1\right) - k_1 = 0.$$

$$-2k_1 = 0$$
$$k_1 = 0$$

$\therefore$ $k_1 = k_2 = k_3 = 0$ and $\vec{v}_1$, $\vec{v}_2$ & $\vec{v}_3$ are linearly independent.

Basis Set

An n dimensional vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ can be expressed as a linear combination of n linearly independent vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$.

i.e. $\vec{x} = x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_n\vec{u}_n$

The set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is said to form a basis for all n dimensional vectors and $\{x_1, x_2 \ldots, x_n\}$ are the coordinates of $\vec{x}$ relative to the basis vectors.

Examples

1) Consider $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. These three vectors are linearly independent and $\vec{x} = x_1\hat{\imath} + x_2\hat{\jmath} + x_3\hat{k}$.

eg. $(2,3,1) = 2\hat{\imath} + 3\hat{\jmath} + 1\hat{k}$

2) Consider $\{\vec{v}_1 = (1,-2,3), \vec{v}_2 = (2,-2,0), \vec{v}_3 = (0,1,3)\}$. These 3 vectors were previously shown to be linearly independent. Also any 3 dimensional vector can be expressed as a linear combination of $\vec{v}_1, \vec{v}_2$ & $\vec{v}_3$.

eg. $(2,3,1) = k_1(1,-2,3) + k_2(2,-2,0) + k_3(0,1,3)$

$$2 = k_1 + 2k_2 \implies k_2 = 1 - k_1$$
$$3 = -2k_1 - 2k_2 + k_3$$
$$1 = 3k_1 + 3k_3 \implies k_3 = -k_1 + \tfrac{1}{3}$$

Subst. k_2 + k_3 into the second equation gives:

$$3 = -2k_1 - 2(1-k_1) + \left(-k_1 + \frac{1}{3}\right)$$

$$= -2 - k_1 + \frac{1}{3}$$

$$k_1 = -\frac{14}{3}$$

Hence $k_2 = 1 - \left(-\frac{14}{3}\right) = \frac{17}{3}$, $k_3 = +\frac{14}{3} + \frac{1}{3} = 5$.

$$\therefore (2, 3, 1) = -\frac{14}{3}\vec{v}_1 + \frac{17}{3}\vec{v}_2 + 5\vec{v}_3.$$

Important Points

1. Any basis for n dimensional vectors has n elements.

2. Basis is not unique.

Orthonormal Basis

The basis vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ have unit length $|\vec{v}_i| = 1$ $i = 1, 2, \ldots n$ and are mutually perpendicular $\vec{v}_i \cdot \vec{v}_j = 0$ $\forall i \neq j$.

Examples

1) An orthonormal basis for 3 dimensional vectors is $\{\hat{i}, \hat{j}, \hat{k}\}$.

2) Another orthonormal basis " "
is: $\left\{ \vec{v}_1 = (0,1,0),\ \vec{v}_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),\ \vec{v}_3 = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right) \right\}$.

Theorem

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthonormal basis

then vector $\vec{x} = (\vec{x}\cdot\vec{v}_1)\vec{v}_1 + (\vec{x}\cdot\vec{v}_2)\vec{v}_2 + \cdots + (\vec{x}\cdot\vec{v}_n)\vec{v}_n$.

Examples

1) Obviously
$$\vec{x} = (x_1, x_2, x_3) = (\vec{x}\cdot\hat{i})\,\hat{i} + (\vec{x}\cdot\hat{j})\,\hat{j} + (\vec{x}\cdot\hat{k})\,\hat{k}$$

2) We shall check that $\vec{x} = (\vec{x}\cdot\vec{v}_1)\vec{v}_1 + (\vec{x}\cdot\vec{v}_2)\vec{v}_2 + (\vec{x}\cdot\vec{v}_3)\vec{v}_3$

$$\begin{aligned}
R.S. &= [(x_1, x_2, x_3)\cdot(0,1,0)]\,(0,1,0) \\
&\quad + \left[(x_1, x_2, x_3)\cdot\left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right)\right]\left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right) \\
&\quad + \left[(x_1, x_2, x_3)\cdot\left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{-1}{\sqrt{2}}\right)\right]\left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{-1}{\sqrt{2}}\right) \\
&= x_2(0,1,0) + \tfrac{1}{\sqrt{2}}(x_1 + x_3)\left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right) \\
&\quad + \tfrac{1}{\sqrt{2}}(x_1 - x_3)\left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{-1}{\sqrt{2}}\right) \\
&= (0, x_2, 0) + \tfrac{1}{2}(x_1 + x_3, 0, x_1 + x_3) + \tfrac{1}{2}(x_1 - x_3, 0, -x_1 + x_3) \\
&= (x_1, x_2, x_3) = \vec{x} = L.S.
\end{aligned}$$

Gram-Schmidt Process

This is a recipe for creating an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ from a basis $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$.

Recipe

1) $\vec{v}_1 = \vec{u}_1$

2) $\vec{v}_2 = \vec{u}_2 - \dfrac{\vec{u}_2 \cdot \vec{v}_1}{|\vec{v}_1|^2} \vec{v}_1$

3) $\vec{v}_3 = \vec{u}_3 - \dfrac{\vec{u}_3 \cdot \vec{v}_1}{|\vec{v}_1|^2} \vec{v}_1 - \dfrac{\vec{u}_3 \cdot \vec{v}_2}{|\vec{v}_2|^2} \vec{v}_2$

4) $\vec{v}_4 = \vec{u}_4 - \dfrac{\vec{u}_4 \cdot \vec{v}_1}{|\vec{v}_1|^2} \vec{v}_1 - \dfrac{\vec{u}_4 \cdot \vec{v}_2}{|\vec{v}_2|^2} \vec{v}_2 - \dfrac{\vec{u}_4 \cdot \vec{v}_3}{|\vec{v}_3|^2} \vec{v}_3$

$\vdots$

etc.

Example.

Consider the set of vectors that is a linear combination of $\vec{u}_1 = (1,1,1,1)$, $\vec{u}_2 = (0,1,1,1)$ + $\vec{u}_3 = (0,0,1,1)$. Find an orthogonal basis for this set of vectors.

1. $\vec{v}_1 = \vec{u}_1 = (1,1,1,1)$

2. $\vec{v}_2 = \vec{u}_2 - \dfrac{\vec{u}_2 \cdot \vec{v}_1}{|\vec{v}_1|^2} \vec{v}_1$

$$= (0,1,1,1) - \frac{(0,1,1,1)\cdot(1,1,1,1)}{(1^2+1^2+1^2+1^2)} (1,1,1,1)$$

$$= (0,1,1,1) - \frac{3}{4}(1,1,1,1)$$

$$= \left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

3. $$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{|\vec{v}_1|^2}\vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2}{|\vec{v}_2|^2}\vec{v}_2$$

$$= (0,0,1,1) - \frac{(0,0,1,1)\cdot(1,1,1,1)}{(1^2+1^2+1^2+1^2)}(1,1,1,1) - \frac{(0,0,1,1)\cdot(-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})}{(\frac{9}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16})}(-\tfrac{3}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4})$$

$$= (0,0,1,1) - \frac{2}{4}(1,1,1,1) - \frac{1/2}{12/16}(-\tfrac{3}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4})$$

$$= (0,0,1,1) - \frac{1}{2}(1,1,1,1) - \frac{2}{3}(-\tfrac{3}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4})$$

$$= (0,0,1,1) - \frac{1}{2}(1,1,1,1) - \frac{1}{6}(-3,1,1,1)$$

$$= (0,-\tfrac{2}{3},\tfrac{1}{3},\tfrac{1}{3})$$

To construct an orthonormal basis we simply divide by the length of each vector.

$$\frac{\vec{v}_1}{|\vec{v}_1|} = \frac{(1,1,1,1)}{\sqrt{1^2+1^2+1^2+1^2}} = \frac{1}{2}(1,1,1,1)$$

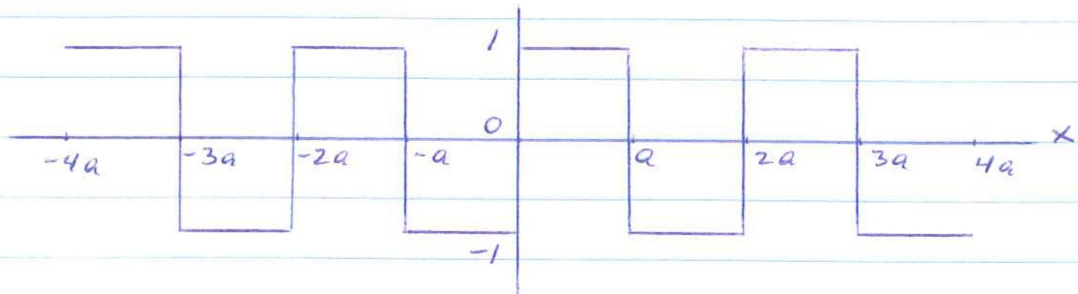
$$\frac{\vec{v}_2}{|\vec{v}_2|} = \frac{(-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})}{\sqrt{\frac{9}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}}} = \frac{2}{\sqrt{3}}(-\tfrac{3}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4}) = \frac{1}{2\sqrt{3}}(-3,1,1,1)$$

$$\frac{\vec{v}_3}{|\vec{v}_3|} = \frac{(0,-\frac{2}{3},\frac{1}{3},\frac{1}{3})}{\sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}} = \sqrt{\frac{3}{2}}(0,-\tfrac{2}{3},\tfrac{1}{3},\tfrac{1}{3}) = \frac{1}{\sqrt{6}}(0,-2,1,1)$$

Exercise: Check that $\vec{v}_1$, $\vec{v}_2$ & $\vec{v}_3$ are orthogonal.

Application of a Basis:
(Fourier Analysis)

A periodic function can be expressed as a linear combination of cosine and sine functions which act as basis vectors. We shall consider a square wave function $f(x)$ shown below.



$$\text{Let } f(x) = \sum_{n=1}^{\infty} \left(c_n \cos \frac{\pi n x}{a} + d_n \sin \frac{\pi n x}{a} \right)$$

$$\text{Now } f(x) = -f(-x) \implies c_n = 0 \;\forall n.$$

$$\therefore f(x) = \sum_{n=1}^{\infty} d_n \sin \frac{\pi n x}{a}$$

$$\int_0^a f(x) \sin \frac{\pi m x}{a}\, dx = \sum_{n=1}^{\infty} d_n \int_0^a \sin \frac{\pi n x}{a} \sin \frac{\pi m x}{a}\, dx.$$

$$\int_0^a \sin \frac{\pi m x}{a}\, dx = \sum_{n=1}^{\infty} d_n \int_0^a \frac{1}{2}\left[\cos \frac{\pi(n-m)x}{a} - \cos \frac{\pi(n+m)x}{a} \right] dx$$

$$\left(\frac{-\cos \pi m x/a}{\pi m/a} \right)_0^a = \sum_{n=1}^{\infty} \frac{d_n}{2} \underbrace{\left[\frac{\sin \pi(n-m)x/a}{\pi(n-m)/a} - \frac{\sin \pi(n+m)x/a}{\pi(n+m)/a} \right]_0^a}_{=0 \text{ if } n \neq m}$$

$$\frac{a}{\pi m}\left[-\cos m\pi + \cos 0 \right] = \sum_{n=1}^{\infty} \frac{d_n}{2} \delta_{nm}\, a.$$

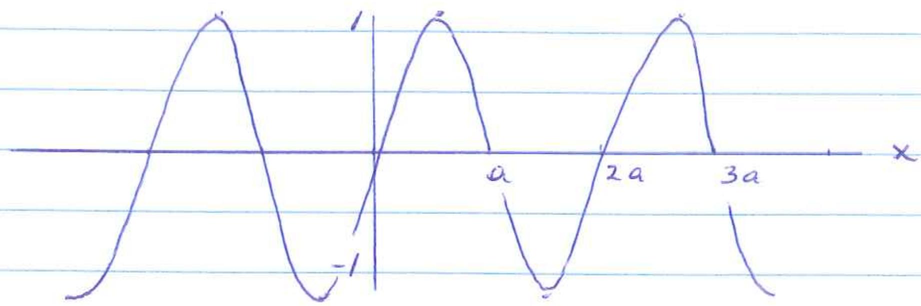
$$\frac{a}{\pi m}\left[-(-1)^m + 1 \right] = \frac{d_m}{2}\, a.$$

$$d_m = \frac{2}{\pi m}\left(1-(-1)^m\right)$$

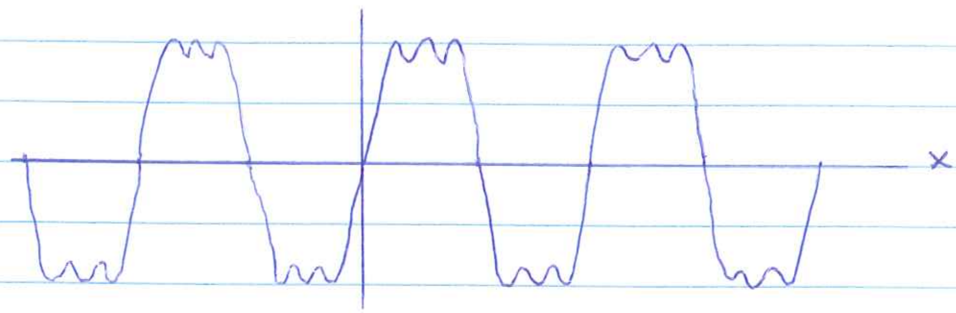
Note: $d_{even} = 0$, $d_{odd} = \frac{4}{\pi m}$.

$$\therefore f(x) = \sum_{m=1,3,5\ldots} \frac{4}{\pi m} \sin\frac{\pi m x}{a}.$$

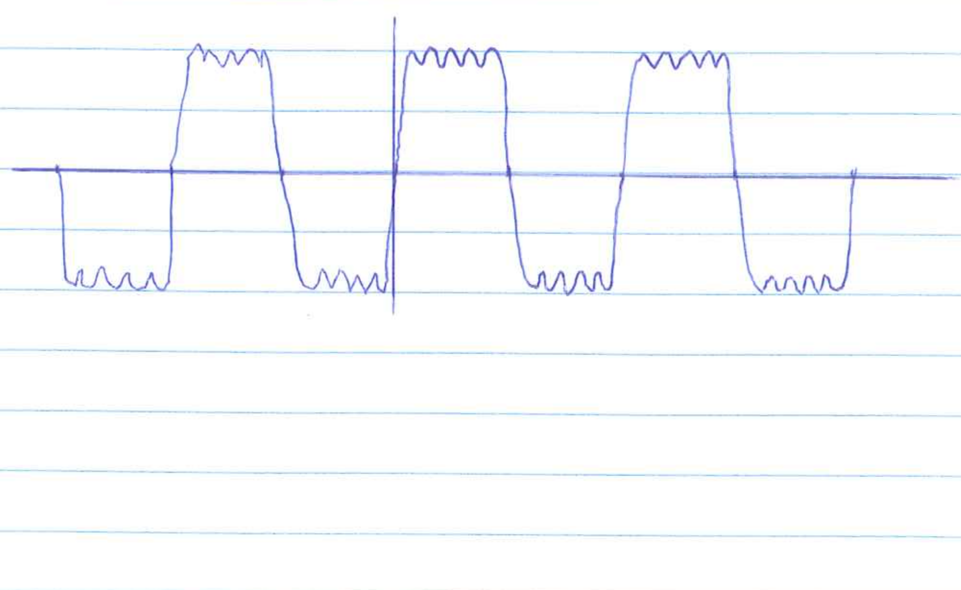
m=1 term



m=1 terms
+ m=3



m=1, 3 + 5
terms



Matrices

An $n \times m$ matrix is a rectangular array of numbers arranged in n rows and m columns. The element in the ith row and jth column of matrix A is denoted a_{ij}.

Examples

1) $A = \begin{pmatrix} 4 & -2 & 6 & 1 \\ 3 & 0 & 8 & 4 \end{pmatrix}$ is a 2×4 matrix.

   $a_{11} = 4 \quad a_{12} = -2 \quad a_{13} = 6 \ \ldots$

   $a_{21} = 3 \quad a_{22} = 0 \quad \ldots$

2) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a 3×1 matrix

Matrix Addition

Let A + B be $n \times m$ matrices.
Then $A + B$ is the $n \times m$ matrix whose element in the ith row and jth column is $a_{ij} + b_{ij}$.

Note: Matrix addition is defined only for matrices having the same number of rows + columns.

eg. $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \end{pmatrix}$ makes no sense!!

Scalar Multiplication of a Matrix

Let A be an $n \times m$ matrix.
Then cA, $c \in R$ is an $n \times m$ matrix whose element in the ith row & jth column is ca_{ij}.

Example

$$A = \begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -4 & 0 \end{pmatrix}$$

$$\begin{aligned} A - 2B &= \begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 & 3 \\ -1 & -4 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 3 \end{pmatrix} + \begin{pmatrix} -2 & -4 & -6 \\ 2 & 8 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -6 & -5 \\ 4 & 8 & 3 \end{pmatrix} \end{aligned}$$

Laws of Matrix Algebra

Let A, B & C be $n \times m$ matrices and $a, b \in R$.

1) $A + \underset{\sim}{0} = \underset{\sim}{0} + A = A$ where $\underset{\sim}{0}$ is $n \times m$ matrix with all elements equal to 0.

2) $A + (-A) = (-A) + A = \underset{\sim}{0}$

3) $A + (B + C) = (A + B) + C$

4) $A + B = B + A$

5) $(a+b)A = aA + bA$

6) $a(A+B) = aA + aB$

7) $0A = \underset{\sim}{0}$ and $1A = A$

Matrix Multiplication

Let A be an $n \times r$ matrix and B an $r \times m$ matrix. Then $C = AB$ is an $n \times m$ matrix with elements

$$c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}$$

$$= \text{row } i \text{ of } A \times \text{column } j \text{ of } B$$

Examples

1) $A = \begin{pmatrix} 4 & 0 & +2 \\ -3 & 1 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 6 & 3 \end{pmatrix}$

$$AB = \begin{pmatrix} 4 & 0 & 2 \\ -3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 4 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 6 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4.4 + 0.(-1) + 2.0 & 4.2 + 0.3 + 2.6 & 4.1 + 0.2 + 2.3 \\ -3.4 + 1.(-1) + (-2).0 & -3.2 + 1.3 + (-2).6 & -3.1 + 1.2 + (-2).3 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 20 & 10 \\ -13 & -15 & -7 \end{pmatrix}$$

Note $BA = \begin{pmatrix} 4 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 6 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 & 2 \\ -3 & 1 & -2 \end{pmatrix}$ isn't defined!

2) $C = \begin{pmatrix} 4 & 2 \\ -1 & 3 \\ 0 & 6 \end{pmatrix}$

$$AC = \begin{pmatrix} 4 & 0 & 2 \\ -3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ -1 & 3 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 16 & 20 \\ -13 & -15 \end{pmatrix}$$

$$CA = \begin{pmatrix} 4 & 2 \\ -1 & 3 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 4 & 0 & 2 \\ -3 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 10 & 2 & 4 \\ -13 & 3 & -8 \\ -18 & 6 & -12 \end{pmatrix}$$

$\therefore AC \neq CA$

More Laws of Matrix Algebra

Let A, B + C be matrices and $a, b \in \mathbb{R}$. Then provided A, B + C have dimensions for which multiplication is defined, we have:

1) $A(BC) = (AB)C$

2) $A(B+C) = AB + AC$

3) $(B+C)A = BA + CA$

4) $(aA)(bB) = (ab)(AB)$

Transpose of a Matrix

The transpose of a $m \times n$ matrix A, denoted by A^+ is an $n \times m$ matrix having elements $(A^+)_{ij} = A_{ji}$.

i.e. rows of A are columns of A^+ and vice versa.

Examples

1) $A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}$ $\quad A^+ = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

2) $B = \begin{pmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix}$ $\quad B^+ = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{pmatrix}$

Properties of Transpose Operation

1) $(A^+)^+ = A$

2) $(A+B)^+ = A^+ + B^+$ provided $A+B$ is defined.

3) $(kA)^+ = k\,A^+$ where $k \in R$

4) $(AB)^+ = B^+ A^+$ provided AB is defined.

Proof of 4

Let A be an $m \times r$ matrix.
" B " " $r \times n$ " ".

$$((AB)^{+})_{ij} = (AB)_{ji}$$

$$= \sum_{l=1}^{r} A_{jl} B_{li}$$

$$= \sum_{l=1}^{r} B_{li} A_{jl}$$

$$= \sum_{l=1}^{r} (B^{+})_{il} (A^{+})_{lj}$$

$$= (B^{+} A^{+})_{ij}$$

$$\therefore (AB)^{+} = B^{+} A^{+}$$

Square Matrices

A square matrix has an equal number of rows + columns.

Examples.

1) $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -3 & 6 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -2 & 7 \end{pmatrix}$$

$\therefore$ even for square matrices $AB \neq BA$

2) $I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$AI = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} = A$$

$\therefore I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the 2×2 identity matrix.

The general $k \times k$ identity matrix has 1's as the diagonal elements and all others 0.

Symmetric Matrix

A square matrix A is said to be symmetric if $A^t = A$.
i.e. $A_{ij} = A_{ji}$

eg. $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 4 & 5 & 6 \end{pmatrix}$

Skew-Symmetric Matrix

A square matrix A is said to be skew-symmetric if $A^t = -A$. i.e. $A_{ij} = -A_{ji}$.

eg. $A = \begin{pmatrix} 0 & 2 & 4 \\ -2 & 0 & 5 \\ -4 & -5 & 0 \end{pmatrix}$

Note that all the diagonal elements must be zero since

$$A_{ii} = -A_{ii}$$
$$2A_{ii} = 0$$
$$\therefore \ A_{ii} = 0$$

Trace of a Matrix

The trace of a square $n \times n$ matrix A, denoted by $\mathrm{Tr}\,A$ is the sum of the diagonal elements.

$$\text{i.e. } \mathrm{Tr}\,A = \sum_{i=1}^{n} A_{ii}$$

Inverse of a Matrix

Consider matrices $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ $\quad C = \frac{1}{6}\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$

$$AC = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1/2 & 1/6 \\ 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

C is called the inverse of A and is denoted by A^{-1}.

$$\therefore AA^{-1} = A^{-1}A = I$$

Inverse of a 2×2 Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

One can show $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Matrices with Complex Entries

Conjugate Transpose

The conjugate transpose of a matrix A, denoted by A^* is defined by:

$$A^* = (\bar{A})^T$$

i.e. First take the complex conjugate of every entry and then take the transpose of the matrix

eg. $$A = \begin{pmatrix} 1+i & -i & 0 \\ 2 & 3-2i & i \end{pmatrix}$$

$$A^* = \begin{pmatrix} 1-i & 2 \\ i & 3+2i \\ 0 & -i \end{pmatrix}$$

Properties

1) $(A^*)^* = A$

2) $(A+B)^* = A^* + B^*$ assuming $A+B$ is defined

3) $(kA)^* = \bar{k}\, A^*$

4) $(AB)^* = B^* A^*$ assuming AB is defined

Unitary Matrix

A square matrix is unitary if $A^{-1} = A^*$.

eg. $A = \begin{pmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{1-i}{2} & \frac{-1+i}{2} \end{pmatrix}$ $\quad A^* = \begin{pmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1-i}{2} & \frac{-1-i}{2} \end{pmatrix}$

$$AA^* = \frac{1}{4}\begin{pmatrix} 1+i & 1+i \\ 1-i & -1+i \end{pmatrix}\begin{pmatrix} 1-i & 1+i \\ 1-i & -1-i \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} (1+i)(1-i)2 & (1+i)^2-(1+i)^2 \\ (1-i)^2-(1-i)^2 & (1+i)(1-i)2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore A^* = A^{-1}$.

Hermitian Matrix

A square matrix is hermitian if $A = A^*$.

eg. $A = \begin{pmatrix} 1 & i & 1+i \\ -i & -5 & 2-i \\ 1-i & 2+i & 3 \end{pmatrix}$

$$\bar{A} = \begin{pmatrix} 1 & -i & 1-i \\ i & -5 & 2+i \\ 1+i & 2-i & 3 \end{pmatrix}$$

$$A^{*} = (\bar{A})^{+} = \begin{pmatrix} 1 & i & 1+i \\ -i & -5 & 2-i \\ 1-i & 2+i & 3 \end{pmatrix}$$

$$\therefore A^{*} = A.$$

Note that the diagonal elements of a Hermitian matrix are real since:

$$A_{ii} = A^{*}_{ii}$$

$$= ((\bar{A})^{+})_{ii}$$

$$\therefore A_{ii} = \bar{A}_{ii}$$

Determinants

Recall if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (1)

The denominator of (1) is called the determinant of A and is denoted as:

$$\det A \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc$$

If $\det A = 0$, A^{-1} doesn't exist. Hence the determinant determines whether A^{-1} exists.

Determinant of a 3×3 Matrix

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \equiv a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Obviously, one can define determinant of a $n \times n$ matrix in terms of determinants of $(n-1) \times (n-1)$ matrices:

Properties of Determinants

1) $\det A = 0$ if all the entries of a row or column of A are zero.

2) $\det A = \det A^t$

eg. $A = \begin{pmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{pmatrix}$ $\quad A^t = \begin{pmatrix} 1 & 2 & 1 \\ 7 & 0 & 4 \\ 5 & 3 & 7 \end{pmatrix}$

$$\det A = 1\begin{vmatrix} 0 & 3 \\ 4 & 7 \end{vmatrix} - 7\begin{vmatrix} 2 & 3 \\ 1 & 7 \end{vmatrix} + 5\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}$$
$$= 1(0-12) - 7(14-3) + 5(8-0)$$
$$= -12 - 77 + 40$$
$$= -49$$

$$\det A^t = 1\begin{vmatrix} 0 & 4 \\ 3 & 7 \end{vmatrix} - 2\begin{vmatrix} 7 & 4 \\ 5 & 7 \end{vmatrix} + 1\begin{vmatrix} 7 & 0 \\ 5 & 3 \end{vmatrix}$$
$$= 1(0-12) - 2(49-20) + 1(21-0)$$
$$= -12 - 58 + 21$$
$$= -49$$

3) If a row (col.) of matrix B is k times a row (col.) of matrix A and all other entries are the same, then $\det B = k \det A$

eg. $B = \begin{pmatrix} 1/7 & 1 & 5/7 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{pmatrix}$ & $A = \begin{pmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{pmatrix}$

$$\det B = \frac{1}{7}\begin{vmatrix} 0 & 3 \\ 4 & 7 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ 1 & 7 \end{vmatrix} + \frac{5}{7}\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}$$

$$= \frac{1}{7}(0-12) - 1(14-3) + \frac{5}{7}(8-0)$$

$$= \frac{-12}{7} - 11 + \frac{40}{7}$$

$$= -7$$

$$\det B = \frac{1}{7} \det A$$

4) If two rows (cols.) of matrix A are interchanged to give matrix B then $\det B = -\det A$.

eg. $B = \begin{pmatrix} 1 & 5 & 7 \\ 2 & 3 & 0 \\ 1 & 7 & 4 \end{pmatrix}$

$$\det B = 1\begin{vmatrix} 3 & 0 \\ 7 & 4 \end{vmatrix} - 5\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} + 7\begin{vmatrix} 2 & 3 \\ 1 & 7 \end{vmatrix}$$

$$= 1(12-0) - 5(8-0) + 7(14-3)$$

$$= 12 - 40 + 77$$

$$= 49$$

$$\det B = -\det A$$

5) If a multiple of one row (col.) of matrix A is added to another row (col.) to give matrix B then $\det B = \det A$.

eg. $B = \begin{pmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+2 & 4+0 & 7+3 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 3 & 4 & 10 \end{pmatrix}$

$$\det B = 1\begin{vmatrix} 0 & 3 \\ 4 & 10 \end{vmatrix} - 7\begin{vmatrix} 2 & 3 \\ 3 & 10 \end{vmatrix} + 5\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix}$$

$$= 1(0-12) - 7(20-9) + 5(8-0)$$

$$= -12 - 77 + 40$$

$$= -49$$

$\det B = \det A$

6) If two rows (cols) of matrix A are proportional to each other then $\det A = 0$.

eg. $A = \begin{pmatrix} 4 & 0 & 6 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{pmatrix}$

$$\det A = 4\begin{vmatrix} 0 & 3 \\ 4 & 7 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 1 & 7 \end{vmatrix} + 6\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}$$

$$= 4(0-12) - 0(14-3) + 6(8-0)$$

$$= -48 + 0 + 48$$

$$= 0$$

7) $\det(A+B) \neq \det A + \det B$

eg. $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ $A+B = \begin{pmatrix} 4 & 3 \\ 3 & 8 \end{pmatrix}$

$\det A = 5 - 4 = 1$

$\det B = 9 - 1 = 8$

$\det(A+B) = 4.8 - 3.3 = 23$

$\neq \det A + \det B$

8) $\det(AB) = (\det A)(\det B)$

eg. $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ $\quad B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ $\quad AB = \begin{pmatrix} 5 & 7 \\ 11 & 17 \end{pmatrix}$

$$\begin{aligned} \det(AB) &= 5 \cdot 17 - 11 \cdot 7 \\ &= 8 \\ &= 1 \cdot 8 \\ &= (\det A)(\det B) \end{aligned}$$

9) $\det(A^{-1}) = \dfrac{1}{\det A}$

eg. $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ $\quad A^{-1} = \frac{1}{8}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$

$$AA^{-1} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \det(A^{-1}) &= (3/8)(3/8) - (-1/8)(-1/8) \\ &= \frac{9}{64} - \frac{1}{64} \\ &= \frac{1}{8} \end{aligned}$$

$\therefore \det(A^{-1}) = \dfrac{1}{\det A}$

10) If A, B & C differ only in the rth row (col.) such that the rth row (col.) is the sum of the rth rows (cols.) of A & B then $\det C = \det A + \det B$.

eg. $A = \begin{pmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 5 & 6 \end{pmatrix}$

$$\det A = 1\begin{vmatrix} 0 & 3 \\ 4 & 7 \end{vmatrix} - 7\begin{vmatrix} 2 & 3 \\ 1 & 7 \end{vmatrix} + 5\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}$$
$$= 1(0-12) - 7(14-3) + 5(8-0)$$
$$= -12 - 77 + 40$$
$$= -49$$

$$\det B = 1\begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} - 7\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} + 5\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}$$
$$= 1(0-3) - 7(-2-0) + 5(2-0)$$
$$= -3 + 14 + 10$$
$$= 21$$

$$\det C = 1\begin{vmatrix} 0 & 3 \\ 5 & 6 \end{vmatrix} - 7\begin{vmatrix} 2 & 3 \\ 1 & 6 \end{vmatrix} + 5\begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix}$$
$$= 1(0-15) - 7(12-3) + 5(10-0)$$
$$= -15 - 63 + 50$$
$$= -28$$
$$= -49 + 21$$

$\therefore \det C = \det A + \det B$

Finding the Inverse of a Matrix

The method to find the inverse of a matrix will be demonstrated with some examples.

Example 1

Find A^{-1} if $A = \begin{pmatrix} 4 & -6 \\ 2 & -8 \end{pmatrix}$

Solution

$\det A = 4(-8) - 2.(-6) = -20 \neq 0 \quad \therefore A^{-1}$ exists.

$$\left(\underbrace{\begin{matrix} 4 & -6 \\ 2 & -8 \end{matrix}}_{A} \middle| \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I}\right) \quad \text{row } 1 \div 4 \rightarrow \text{row } 1$$

We perform operations to change A to I by adding rows & multiplying rows by appropriate numbers as follows.

$$\left(\begin{matrix} 1 & -3/2 \\ 2 & -8 \end{matrix} \middle| \begin{matrix} 1/4 & 0 \\ 0 & 1 \end{matrix}\right) \quad 2 \times \text{row } 1 - \text{row } 2 \rightarrow \text{row } 2$$

$$\left(\begin{matrix} 1 & -3/2 \\ 0 & 5 \end{matrix} \middle| \begin{matrix} 1/4 & 0 \\ 1/2 & -1 \end{matrix}\right) \quad \text{row } 2 \div 5 \rightarrow \text{row } 2$$

$$\left(\begin{matrix} 1 & -3/2 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 1/4 & 0 \\ 1/10 & -1/5 \end{matrix}\right) \quad \text{row } 1 + \frac{3}{2} \text{row } 2 \rightarrow \text{row } 1$$

$$\left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 2/5 & -3/10 \\ 1/10 & -1/5 \end{matrix}\right)$$

$\therefore \; A^{-1} = \begin{pmatrix} 2/5 & -3/10 \\ 1/10 & -1/5 \end{pmatrix}$

Check: $AA^{-1} = \begin{pmatrix} 4 & -6 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} 2/5 & -3/10 \\ 1/10 & -1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Example 2

Find inverse of $A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 0 & 3 \\ -1 & 0 & 0 \end{pmatrix}$

Solution

$\det \begin{pmatrix} 4 & -1 & 6 \\ 2 & 0 & 3 \\ -1 & 0 & 0 \end{pmatrix} = 4(0-0) + 1(0+3) + 6(0-0) = 3 \neq 0$

$\therefore A^{-1}$ exists.

$\left(\begin{array}{ccc|ccc} 4 & -1 & 6 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$ row 1 $\div$ 4 $\rightarrow$ row 1

$\left(\begin{array}{ccc|ccc} 1 & -1/4 & 3/2 & 1/4 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$ row 2 $-$ 2 row 1 $\rightarrow$ row 2
row 1 + row 3 $\rightarrow$ row 3

$\left(\begin{array}{ccc|ccc} 1 & -1/4 & 3/2 & 1/4 & 0 & 0 \\ 0 & 1/2 & 0 & -1/2 & 1 & 0 \\ 0 & -1/4 & 3/2 & 1/4 & 0 & 1 \end{array}\right)$ 2 x row 2 $\rightarrow$ row 2

$$\left(\begin{array}{ccc|ccc} 1 & -1/4 & 3/2 & 1/4 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & -1/4 & 3/2 & 1/4 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \text{row } 1 + \frac{1}{4}\text{ row } 2 \rightarrow \text{row } 1 \\ \\ \text{row } 3 + \frac{1}{4}\text{ row } 2 \rightarrow \text{row } 3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 3/2 & 0 & 1/2 & 1 \end{array}\right) \quad \frac{2}{3}\text{ row } 3 \rightarrow \text{row } 3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1/3 & 2/3 \end{array}\right) \quad \text{row } 1 - \frac{3}{2}\text{ row } 3 \rightarrow \text{row } 1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1/3 & 2/3 \end{array}\right)$$

$$\therefore\ A^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix}$$

Check: $A A^{-1} = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 0 & 3 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solving Equations

Frequently one has to solve a set of n equations having n unknowns $x_1, x_2, x_3 \cdots x_n$.

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = y_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = y_2$$
$$\vdots$$
$$a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = y_n$$

Using matrices, we can rewrite this as follows.

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{Y}$$

$$A X = Y$$

If we know A^{-1} then $A^{-1} A X = A^{-1} Y$

$$X = A^{-1} Y$$

Examples

1) Solve $4x_1 - 6x_2 = 1$
$2x_1 - 8x_2 = 0$

New Way: $\begin{pmatrix} 4 & -6 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Previously we found inverse of $A = \begin{pmatrix} 4 & -6 \\ 2 & -8 \end{pmatrix}$ to be $A^{-1} = \begin{pmatrix} 2/5 & -3/10 \\ 1/10 & -1/5 \end{pmatrix}$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/5 & -3/10 \\ 1/10 & -1/5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2/5 \\ 1/10 \end{pmatrix}$$

Check: $4x_1 - 6x_2 = 4 \cdot \frac{2}{5} - 6 \cdot \frac{1}{10} = 1$

$2x_1 - 8x_2 = 2 \cdot \frac{2}{5} - 8 \cdot \frac{1}{10} = 0$

Old Way: $4x_1 - 6x_2 = 1$ $\quad \frac{1}{4}$ eqn. 1 $\rightarrow$ eqn. 1
$2x_1 - 8x_2 = 0$

$x_1 - \frac{3}{2}x_2 = \frac{1}{4}$ $\quad$ eqn 2 $-$ 2 eqn 1 $\rightarrow$ eqn 2
$2x_1 - 8x_2 = 0$

$x_1 - \frac{3}{2}x_2 = \frac{1}{4}$ $\quad -\frac{1}{5}$ eqn. 2 $\rightarrow$ eqn. 2
$0x_1 - 5x_2 = -\frac{1}{2}$

$$x_1 - \frac{3}{2}x_2 = \frac{1}{4} \qquad \text{eqn.1} + \frac{3}{2}\,\text{eqn.2} \rightarrow \text{eqn.1}$$
$$0x_1 + x_2 = \frac{1}{10}$$

$$x_1 + 0x_2 = \frac{2}{5}$$
$$0x_1 + x_2 = \frac{1}{10}$$

Hence, steps taken when finding matrix inverse are exactly the same as those taken when solving equations except x's are not written!

2) Solve
$$4x_1 - x_2 + 6x_3 = 1$$
$$2x_1 + 3x_3 = 4$$
$$-x_1 = 2$$

In matrix form
$$\begin{pmatrix} 4 & -1 & 6 \\ 2 & 0 & 3 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

Previously we found inverse of A to be $A^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix}$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 7 \\ 8/3 \end{pmatrix}$$

3) Solve $x_1 + x_2 = 4$

$2x_1 + 2x_2 = 8$

In matrix form $\underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

$\det A = 2 - 2 = 0 \Rightarrow A^{-1}$ doesn't exist.

The problem is that the second equation is the first equation multiplied by 2. Hence, $\det A = 0$ means not enough information is given for there to be a single answer. All we can say is that the solution is $\{(x_1, x_2) \mid x_1 + x_2 = 4\}$.

4) Solve $x_1 + x_2 = 4$

$2x_1 + 2x_2 = 7$

Note that these 2 equations are inconsistent and therefore no solution exists.

Gauss Jordan Elimination

This is a somewhat faster way to solve a system of equations and will be illustrated by an example.

$$-x_1 + 3x_2 + 2x_3 = 1$$
$$x_1 + 2x_2 - 3x_3 = -9$$
$$2x_1 + x_2 - 2x_3 = -3$$

This is rewritten as: $\left(\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & -2 & -3 \end{array}\right)$ $-\text{Row }1 \to \text{Row }1$

$$\left(\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & -2 & -3 \end{array}\right) \begin{array}{l} \text{Row }2 - \text{Row }1 \to \text{Row }2 \\ \text{Row }3 - 2\text{Row }1 \to \text{Row }3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & 5 & -1 & -8 \\ 0 & 7 & 2 & -1 \end{array}\right) \frac{1}{5}\text{Row }2 \to \text{Row }2$$

$$\left(\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & 1 & -1/5 & -8/5 \\ 0 & 7 & 2 & -1 \end{array}\right) \begin{array}{l} \text{Row }1 + 3\text{Row }2 \to \text{Row }1 \\ \text{Row }3 - 7\text{Row }2 \to \text{Row }3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -13/5 & -29/5 \\ 0 & 1 & -1/5 & -8/5 \\ 0 & 0 & 17/5 & 51/5 \end{array}\right) \frac{5}{17}\text{Row }3 \to \text{Row }3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -13/5 & -29/5 \\ 0 & 1 & -1/5 & -8/5 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} \text{Row }1 + \frac{13}{5}\text{Row }3 \to \text{Row }1 \\ \text{Row }2 + \frac{1}{5}\text{Row }3 \to \text{Row }2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$\therefore\ x_1 = 2,\ x_2 = -1\ \&\ x_3 = 3.$

Cramer's Rule

A system of linear equations $AX = B$ has solution $x_i = \dfrac{\det A_i}{\det A}$ if $\det A \neq 0$ where A_i is matrix obtained by replacing the ith column of A by matrix B.

Example 1

Solve

$$3x - 2y + z = -9$$
$$x + 2y - z = 5$$
$$2x - y + 3z = -10$$

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 3 \end{pmatrix} \qquad \det A = 3\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= 3(6-1) + 2(3+2) + 1(-1-4)$$

$$= 15 + 10 - 5$$

$$= 20$$

$$A_1 = \begin{pmatrix} -9 & -2 & 1 \\ 5 & 2 & -1 \\ -10 & -1 & 3 \end{pmatrix} \qquad \det A_1 = -9\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 5 & -1 \\ -10 & 3 \end{vmatrix} + 1\begin{vmatrix} 5 & 2 \\ -10 & -1 \end{vmatrix}$$

$$= -9(6-1) + 2(15-10) + 1(-5+20)$$

$$= -45 + 10 + 15$$

$$= -20$$

$$A_2 = \begin{pmatrix} 3 & -9 & 1 \\ 1 & 5 & -1 \\ 2 & -10 & 3 \end{pmatrix} \qquad \det A_2 = 3\begin{vmatrix} 5 & -1 \\ -10 & 3 \end{vmatrix} + 9\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ 2 & -10 \end{vmatrix}$$

$$= 3(15-10) + 9(3+2) + 1(-10-10)$$

$$= 15 + 45 - 20$$

$$= 40$$

$$A_3 = \begin{pmatrix} 3 & -2 & -9 \\ 1 & 2 & 5 \\ 2 & -1 & -10 \end{pmatrix} \qquad \det A_3 = 3\begin{vmatrix} 2 & 5 \\ -1 & -10 \end{vmatrix} + 2\begin{vmatrix} 1 & 5 \\ 2 & -10 \end{vmatrix} - 9\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= 3(-20+5) + 2(-10-10) - 9(-1-4)$$

$$= -45 - 40 + 45$$

$$= -40$$

$$\therefore\ x = \frac{\det A_1}{\det A} = \frac{-20}{20} = -1$$

$$y = \frac{\det A_2}{\det A} = \frac{40}{20} = 2$$

$$z = \frac{\det A_3}{\det A} = \frac{-40}{20} = -2$$

Example 2

Solve $2x_1 + 4x_2 + 6x_3 = 18$

$4x_1 + 5x_2 + 6x_3 = 24$

$3x_1 + x_2 - 2x_3 = 4$

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{pmatrix} \qquad \det A = 2\begin{vmatrix} 5 & 6 \\ 1 & -2 \end{vmatrix} - 4\begin{vmatrix} 4 & 6 \\ 3 & -2 \end{vmatrix} + 6\begin{vmatrix} 4 & 5 \\ 3 & 1 \end{vmatrix}$$

$$= 2(-10-6) - 4(-8-18) + 6(4-15)$$

$$= -32 + 104 - 66$$

$$= 6$$

$$A_1 = \begin{pmatrix} 18 & 4 & 6 \\ 24 & 5 & 6 \\ 4 & 1 & -2 \end{pmatrix}$$

$$\det A_1 = 18\begin{vmatrix} 5 & 6 \\ 1 & -2 \end{vmatrix} - 4\begin{vmatrix} 24 & 6 \\ 4 & -2 \end{vmatrix} + 6\begin{vmatrix} 24 & 5 \\ 4 & 1 \end{vmatrix}$$

$$= 18(-10-6) - 4(-48-24) + 6(24-20)$$

$$= -288 + 288 + 24$$

$$= 24$$

$$A_2 = \begin{pmatrix} 2 & 18 & 6 \\ 4 & 24 & 6 \\ 3 & 4 & -2 \end{pmatrix}$$

$$\det A_2 = 2\begin{vmatrix} 24 & 6 \\ 4 & -2 \end{vmatrix} - 18\begin{vmatrix} 4 & 6 \\ 3 & -2 \end{vmatrix} + 6\begin{vmatrix} 4 & 24 \\ 3 & 4 \end{vmatrix}$$

$$= 2(-48-24) - 18(-8-18) + 6(16-72)$$

$$= -144 + 468 - 336$$

$$= -12$$

$$A_3 = \begin{pmatrix} 2 & 4 & 18 \\ 4 & 5 & 24 \\ 3 & 1 & 4 \end{pmatrix}$$

$$\det A_3 = 2\begin{vmatrix} 5 & 24 \\ 1 & 4 \end{vmatrix} - 4\begin{vmatrix} 4 & 24 \\ 3 & 4 \end{vmatrix} + 18\begin{vmatrix} 4 & 5 \\ 3 & 1 \end{vmatrix}$$

$$= 2(20-24) - 4(16-72) + 18(4-15)$$

$$= -8 + 224 - 198$$

$$= 18$$

$$\therefore x_1 = \frac{\det A_1}{\det A} = \frac{24}{6} = 4$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{-12}{6} = -2$$

$$x_3 = \frac{\det A_3}{\det A} = \frac{18}{6} = 3$$

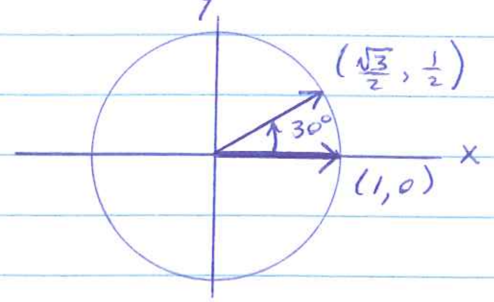
Matrix Rotation of a Vector

Rotation of 2 Dim. Vectors

Consider $\vec{x}' = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} \cos 30^\circ \\ \sin 30^\circ \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

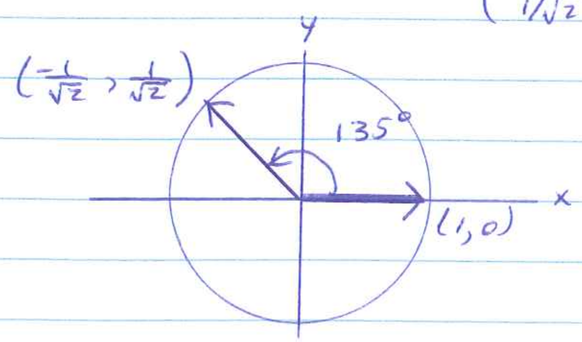


Hence, the matrix has rotated $\hat{x} = (1,0)$ by 30°.

Next consider $\vec{x}' = \begin{pmatrix} \cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^\circ \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} \cos 135^\circ \\ \sin 135^\circ \end{pmatrix}$$

$$= \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$



Hence, the matrix has rotated $\hat{x} = (1,0)$ by 135°.

Hence, $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotates a vector counterclockwise through an angle θ.

Successive Rotations

A vector $\vec{x}$ is first rotated through angle θ_1 and then through angle θ_2. Find the matrix describing the two rotations.

Let $\vec{x}'$ be vector produced by first rotation.
" $\vec{x}''$ " " " second " "

$$\therefore \vec{x}'' = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \vec{x}'$$

$$= \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \vec{x}$$

$$= \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1 \\ \sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{pmatrix} \vec{x}$$

$$= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} \vec{x}$$

This of course agrees with our intuition that two successive rotations through angles θ_1 & θ_2 is equivalent to one rotation through angle $\theta_1+\theta_2$.

Rotation of 3 Dim. Vectors

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ rotates vector about $\hat{z}$ axis by angle ϕ.

Proof

We shall examine effect of this matrix on $\hat{x}$, $\hat{y}$ & $\hat{z}$.

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$ i.e. no effect on $\hat{z}$.

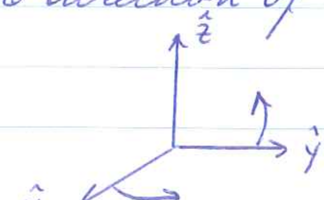
To see effect on $\hat{x}$ + $\hat{y}$, we consider $\phi = 90°$ for simplicity.

$$\begin{pmatrix} \cos 90° & -\sin 90° & 0 \\ \sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{x} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hat{y}$$

$$\begin{pmatrix} \cos 90° & -\sin 90° & 0 \\ \sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{y} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -\hat{x}$$

Right Hand Rule

For the above matrix, the z axis is called the rotation axis. If one places one's right thumb along $\hat{z}$, then one's fingers point in the direction of the rotation.



Exercise: Show that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ rotates a vector about the x axis by angle θ.

Successive Rotations

Suppose a vector $\vec{x}$ is first rotated about the z axis by angle ϕ and next rotated about the new x axis by angle θ. Find the matrix that describes the two rotations

Let $\vec{x}'$ be vector produced by first rotation.
" $\vec{x}''$ " " 2nd " "

$$\therefore \vec{x}'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \vec{x}'$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \cos\theta\sin\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \end{pmatrix} \vec{x}$$

Eigenvalues and Eigenvectors

We consider the problem of finding X and λ where

$$AX = \lambda X \qquad \lambda \in R \ , \ A \text{ a matrix}$$

X is called the eigenvector and λ the eigenvalue. This problem is encountered in Chemistry, Economics, Engineering, Physics etc.

$$(A - \lambda I)\, X = 0 \qquad \text{where } I \text{ is identity matrix}$$

$$\text{or} \qquad BX = 0 \qquad \text{where } B = A - \lambda I$$

If B^{-1} exists then $B^{-1} B X = 0$

$$X = 0$$

This is called the trivial solution and isn't very interesting.

If $\det B = 0$, B^{-1} doesn't exist and there are nontrivial solutions for X. Hence, we wish to solve

$$\boxed{\det (A - \lambda I) = 0}$$

Example 1

Find eigenvectors and eigenvalues for $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$.

$$0 = \det(A - \lambda I)$$

$$= \det\left[\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \det\begin{pmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{pmatrix}$$

$$= (1-\lambda)(2-\lambda) - 12$$
$$= \lambda^2 - 3\lambda + 2 - 12$$
$$= \lambda^2 - 3\lambda - 10$$
$$0 = (\lambda - 5)(\lambda + 2)$$
$$\therefore \lambda = 5, -2$$

We now find the eigenvector associated with $\lambda = 5$.

$$(A - \lambda I)\, X = 0$$

$$\begin{pmatrix} 1-5 & 3 \\ 4 & 2-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-4x_1 + 3x_2 = 0 \implies x_1 = \frac{3}{4}x_2$$
$$4x_1 - 3x_2 = 0$$

$\therefore X = \begin{pmatrix} 3/4\, x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3/4 \\ 1 \end{pmatrix}$

Hence, eigenvector corresponding to $\lambda = 5$ is $\begin{pmatrix} 3/4 \\ 1 \end{pmatrix}$.

Next, we find eigenvector associated with $\lambda = -2$.

$$(A - (-2)I)X = 0$$

$$\begin{pmatrix} 1-(-2) & 3 \\ 4 & 2-(-2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3x_1 + 3x_2 = 0 \implies x_1 = -x_2$$
$$4x_1 + 4x_2 = 0$$

$\therefore X = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Hence, eigenvector corresponding to $\lambda = -2$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Check: $A \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} = \begin{pmatrix} 15/4 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 3/4 \\ 1 \end{pmatrix}$

$A \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Note that a general 2 dimensional vector can be expressed as a linear combination of the two eigenvectors

i.e. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

OR $x_1 = \frac{3}{4} a - b \quad (1)$
$x_2 = a + b \quad (2)$

$(1) + (2) \Longrightarrow \quad x_1 + x_2 = \frac{7}{4} a$

$$a = \frac{4}{7}(x_1 + x_2)$$

Subst. a into (2) $\Longrightarrow \quad x_2 = \frac{4}{7}(x_1 + x_2) + b$

$$b = \frac{1}{7}\left(-4x_1 + 3x_2\right)$$

The above information is useful to evaluate AX.

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \left\{ a \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

$$= a A \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} + b A \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= a\, 5 \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} - b\, 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \frac{20}{7}(x_1 + x_2) \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} - \frac{2}{7}(-4x_1 + 3x_2) \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Example 2

Find eigenvalues and eigenvectors for $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{pmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 1 \\ -17 & 8-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 4 & 8-\lambda \end{vmatrix} + 0 = 0$$

$$-\lambda\left[-\lambda(8-\lambda) + 17\right] - 1\left[0 - 4\right] = 0$$

$$-\lambda(-8\lambda + \lambda^2 + 17) + 4 = 0$$

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

Note that $\lambda = 4$ is one solution. Hence, $(\lambda - 4)$ is factor of left side.

$$\begin{array}{r|l} & \lambda^2 - 4\lambda + 1 \\ \hline \lambda - 4 & \lambda^3 - 8\lambda^2 + 17\lambda - 4 \\ & \underline{\lambda^3 - 4\lambda^2} \\ & \quad -4\lambda^2 + 17\lambda \\ & \quad \underline{-4\lambda^2 + 16\lambda} \\ & \qquad \lambda - 4 \\ & \qquad \underline{\lambda - 4} \\ & \qquad 0 \end{array}$$

$$\therefore (\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

Roots of $\lambda^2 - 4\lambda + 1 = 0$ are $\lambda = \dfrac{4 \pm \sqrt{16-4}}{2}$

$$= 2 \pm \sqrt{3}$$

Eigenvector for $\lambda = 4$
$$\begin{pmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 4 & -17 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-4x_1 + x_2 = 0 \implies x_1 = x_2/4$$
$$-4x_2 + x_3 = 0 \implies x_3 = 4x_2$$
$$4x_1 - 17x_2 + 4x_3 = 0 \implies \text{No add. info.}$$

$$\therefore X_{\lambda=4} = \begin{pmatrix} x_2/4 \\ x_2 \\ 4x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1/4 \\ 1 \\ 4 \end{pmatrix}$$

Eigenvector for $\lambda = 2 + \sqrt{3}$
$$\begin{pmatrix} -2-\sqrt{3} & 1 & 0 \\ 0 & -2-\sqrt{3} & 1 \\ 4 & -17 & 6-\sqrt{3} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-(2+\sqrt{3})x_1 + x_2 = 0 \implies x_1 = x_2/(2+\sqrt{3})$$
$$-(2+\sqrt{3})x_2 + x_3 = 0 \implies x_3 = (2+\sqrt{3})x_2$$
$$4x_1 - 17x_2 + (6-\sqrt{3})x_3 = 0 \implies \text{No new info.}$$

$$\therefore X_{\lambda=2+\sqrt{3}} = \begin{pmatrix} x_2/(2+\sqrt{3}) \\ x_2 \\ (2+\sqrt{3})x_2 \end{pmatrix} = x_2 \begin{pmatrix} (2+\sqrt{3})^{-1} \\ 1 \\ 2+\sqrt{3} \end{pmatrix} = x_2 \begin{pmatrix} 2-\sqrt{3} \\ 1 \\ 2+\sqrt{3} \end{pmatrix}$$

Exercise: Show eigenvector for $\lambda = 2 - \sqrt{3}$ is $\begin{pmatrix} 2+\sqrt{3} \\ 1 \\ -2+\sqrt{3} \end{pmatrix}$.

Note that the three eigenvectors $\begin{pmatrix} 1/4 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2-\sqrt{3} \\ 1 \\ 2+\sqrt{3} \end{pmatrix}, \begin{pmatrix} 2+\sqrt{3} \\ 1 \\ -2+\sqrt{3} \end{pmatrix}$ are linearly independent and any 3 dimensional vector can be expressed as a linear combination of these three eigenvectors.

Example 3

Find eigenvalues + eigenvectors for $A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$.

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix} = 0$$

$$(-3-\lambda)(1-\lambda) + 4 = 0.$$

$$\lambda^2 + 2\lambda + 1 = 0.$$

$$(\lambda+1)^2 = 0$$

$$\lambda = -1$$

The eigenvalue $\lambda = -1$ is said to have a multiplicity of 2 and not be distinct.

Eigenvector for $\lambda = -1$ $\quad (A - (-1)I)X = 0$

$$\begin{pmatrix} -3+1 & 2 \\ -2 & 1+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + 2x_2 = 0 \implies x_1 = x_2$$
$$-2x_1 + 2x_2 = 0$$

$$\therefore X = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hence, eigenvector corresponding to $\lambda = -1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Theorem

If $X_1, X_2 \cdots X_n$ are eigenvectors of an $n \times n$ matrix A corresponding to distinct eigenvalues $\lambda_1, \lambda_2 \cdots \lambda_n$ then any n dimensional vector can be expressed as a linear combination of the n eigenvectors.

Matrix Diagonalization

Consider an $n \times n$ matrix A having n distinct eigenvalues. Matrix P has the eigenvectors of A as its columns. Then $P^{-1}AP$ is a diagonal matrix having the eigenvalues of A as its diagonal elements.

Example 1

Previously we found $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ has eigenvectors $\begin{pmatrix} 3/4 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ corresponding to eigenvalues 5 & -2 respectively.

$$\therefore P = \begin{pmatrix} 3/4 & -1 \\ 1 & 1 \end{pmatrix}$$

Exercise: Show $P^{-1} = \frac{1}{7}\begin{pmatrix} 4 & 4 \\ -4 & 3 \end{pmatrix}$

$$\therefore P^{-1}AP = \frac{1}{7}\begin{pmatrix} 4 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3/4 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 4 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 15/4 & 2 \\ 5 & -2 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 35 & 0 \\ 0 & -14 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$$

Example 2

Exercise: Show eigenvalues & eigenvectors of

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \text{ are } X_{\lambda=1} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \; X_{\lambda=-2} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \; X_{\lambda=3} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

$$\therefore \; P = \begin{pmatrix} -1 & -1 & 1 \\ 4 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Exercise: Show $P^{-1} = \frac{1}{6}\begin{pmatrix} -1 & 2 & -3 \\ -2 & -2 & 6 \\ 3 & 0 & 3 \end{pmatrix}$

$$\therefore \; P^{-1}AP = \frac{1}{6}\begin{pmatrix} -1 & 2 & -3 \\ -2 & -2 & 6 \\ 3 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 & 1 \\ 4 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -1 & 2 & -3 \\ -2 & -2 & 6 \\ 3 & 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 & 3 \\ 4 & -2 & 6 \\ 1 & -2 & 3 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 6 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Computing Powers of a Matrix

Consider a matrix A that is diagonalizable.

i.e. $P^{-1}AP = D$

Multiplying the above equation on the left by P and on the right by P^{-1} gives:

$$PP^{-1}APP^{-1} = PDP^{-1}$$

$$IAI = PDP^{-1}$$

$$A = PDP^{-1}$$

Hence, $A^k = (PDP^{-1})^k$

$$= PD\underbrace{P^{-1}P}_{=I}D\underbrace{P^{-1}P}_{=I\ldots}DP^{-1} \ldots PDP^{-1}$$

$$= PDIDI \ldots IDP^{-1}$$

$$= PD^kP^{-1}$$

Example

Example

For $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ find A^4.

Previously we found $P = \begin{pmatrix} 3/4 & -1 \\ 1 & 1 \end{pmatrix}$, $P^{-1} = \frac{1}{7}\begin{pmatrix} 4 & 4 \\ -4 & 3 \end{pmatrix}$ & $D = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$

$$\therefore\ A^4 = P\,D^4\,P^{-1}$$

$$= \begin{pmatrix} 3/4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5^4 & 0 \\ 0 & (-2)^4 \end{pmatrix}\frac{1}{7}\begin{pmatrix} 4 & 4 \\ -4 & 3 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 3/4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 625 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} 4 & 4 \\ -4 & 3 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 3/4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2500 & 2500 \\ -64 & 48 \end{pmatrix}$$

$$= \begin{pmatrix} 277 & 261 \\ 348 & 364 \end{pmatrix}$$

Vector Operators

"Del" Operator $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

$$= \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Gradient Operation

$$\nabla \Phi = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z} \right)$$

eg. $\Phi = x y^2 z$

$$\frac{\partial \Phi}{\partial x} = y^2 z \qquad \frac{\partial \Phi}{\partial y} = 2xyz \qquad \frac{\partial \Phi}{\partial z} = xy^2$$

$$\therefore \nabla \Phi = (y^2 z, 2xyz, xy^2)$$

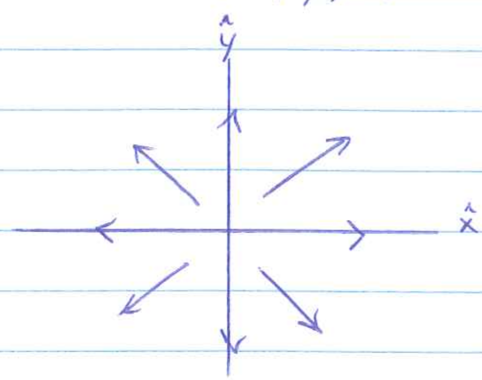
Significance: One can think of the gradient as a three dimensional slope. It explains how fast Φ changes in the x, y & z directions.

Divergence Operation

$$\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

eg 1. $\vec{V} = \vec{r}$

$= (x, y, z)$



$$\nabla \cdot \vec{V} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$= 1 + 1 + 1$

$= 3$

$\nabla \cdot \vec{V} > 0$

Note if $\vec{V}$ represented water flow, we would have a tap gushing water at the origin. Hence $\nabla \cdot \vec{V} > 0$ means there is a source of $\vec{V}$.

eg. 2 $\vec{V} = -\vec{r}$

Exercise: Draw vector field and show if $\vec{V}$ represents water flow that there is a sink at the origin. Also show $\nabla \cdot \vec{V} = -3$.

Curl Operation

$$\nabla \times \vec{V} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

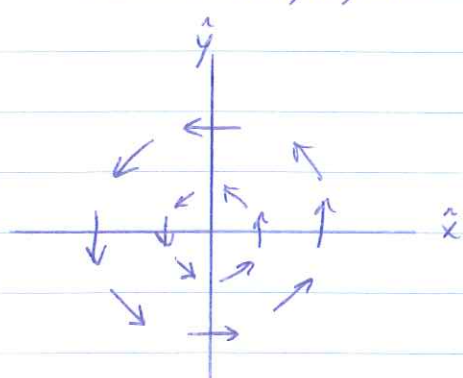
$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, -\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

eg. $\vec{V} = -y\,\hat{x} + x\,\hat{y}$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$= \left(-\frac{\partial x}{\partial z}, -\frac{\partial y}{\partial z}, \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right)$$

$$= (0, 0, 2)$$



$\vec{V} = -y\,\hat{x} + x\,\hat{y}$

Note the vector field curls around the origin:
i.e. $\nabla \times \vec{V} \neq 0$ means there is a whirlpool at the origin. Placing fingers of right hand in direction of $\vec{V}$ we find thumb points in direction of $\nabla \times \vec{V}$.

Laplacian Operator

$$\nabla^2 \equiv \nabla \cdot \nabla$$
$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\therefore \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

eg. $\Phi = x^2 + y^2 + z^2$

$$\frac{\partial \Phi}{\partial x} = 2x \qquad \frac{\partial^2 \Phi}{\partial x^2} = 2$$
$$\frac{\partial \Phi}{\partial y} = 2y \qquad \frac{\partial^2 \Phi}{\partial y^2} = 2$$
$$\frac{\partial \Phi}{\partial z} = 2z \qquad \frac{\partial^2 \Phi}{\partial z^2} = 2$$
$$\therefore \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 6$$

∇ Vector Identities

1. $\nabla(fg) = f\nabla g + g\nabla f$

Proof: $\nabla(fg) = \frac{\partial}{\partial x}(fg)\,\hat{x} + \frac{\partial}{\partial y}(fg)\,\hat{y} + \frac{\partial}{\partial z}(fg)\,\hat{z}$

$$= f\frac{\partial g}{\partial x}\hat{x} + g\frac{\partial f}{\partial x}\hat{x} + f\frac{\partial g}{\partial y}\hat{y} + g\frac{\partial f}{\partial y}\hat{y}$$

$$+ f\frac{\partial g}{\partial z}\hat{z} + g\frac{\partial f}{\partial z}\hat{z}$$

$$= f\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) + g\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\therefore \nabla(fg) = f\nabla g + g\nabla f$$

2. $\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot\nabla f$

Proof: $\nabla\cdot(f\vec{A}) = \nabla\cdot(fA_x, fA_y, fA_z)$

$$= \frac{\partial(fA_x)}{\partial x} + \frac{\partial(fA_y)}{\partial y} + \frac{\partial(fA_z)}{\partial z}$$

$$= f\frac{\partial A_x}{\partial x} + A_x\frac{\partial f}{\partial x} + f\frac{\partial A_y}{\partial y} + A_y\frac{\partial f}{\partial y}$$

$$+ f\frac{\partial A_z}{\partial z} + A_z\frac{\partial f}{\partial z}$$

$$= f\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$+ A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} + A_z\frac{\partial f}{\partial z}$$

$$= f(\nabla\cdot\vec{A}) + \vec{A}\cdot\nabla f$$

3. $\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}$

4. $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

5. $\nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) - \vec{A}\times(\nabla f)$

6. $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$

7. $\nabla\cdot(\nabla\times\vec{A}) = 0$

Proof:
$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$
$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, -\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla\cdot(\nabla\times\vec{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x\,\partial y} - \frac{\partial^2 A_y}{\partial x\,\partial z} - \frac{\partial^2 A_z}{\partial y\,\partial x} + \frac{\partial^2 A_x}{\partial y\,\partial z} + \frac{\partial^2 A_y}{\partial z\,\partial x} - \frac{\partial^2 A_x}{\partial z\,\partial y}$$

$\therefore\ \nabla\cdot(\nabla\times\vec{A}) = 0$

8. $\nabla \times (\nabla f) = \vec{0}$

Proof: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, -\frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 f}{\partial z \partial x}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

$\therefore \nabla \times (\nabla f) = \vec{0}$

9. $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$