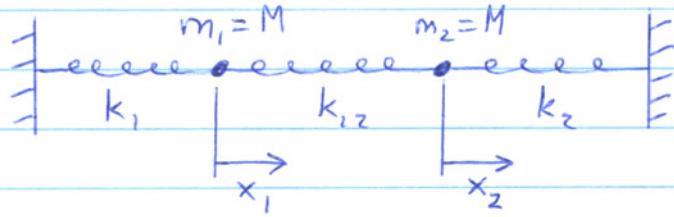


12.2 Let x_1 & x_2 be displacement of masses m_1 & m_2 from equilibrium.



$$\text{Lagrangian } L = \frac{M}{2} \dot{x}_1^2 + \frac{M}{2} \dot{x}_2^2 - \frac{k_1}{2} x_1^2 - \frac{k_{12}}{2} (x_2 - x_1)^2 - \frac{k_2}{2} x_2^2$$

$$x_1 \text{ eqn: } \frac{\partial L}{\partial x_1} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) = 0$$

$$-k_1 x_1 + k_{12} (x_2 - x_1) - M \ddot{x}_1 = 0$$

$$M \ddot{x}_1 + (k_1 + k_{12}) x_1 - k_{12} x_2 = 0 \quad (1)$$

$$x_2 \text{ eqn: } \frac{\partial L}{\partial x_2} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) = 0$$

$$-k_{12} (x_2 - x_1) - k_2 x_2 - M \ddot{x}_2 = 0$$

$$M \ddot{x}_2 + (k_{12} + k_2) x_2 - k_{12} x_1 = 0 \quad (2)$$

Let $x_1 = B_1 e^{i\omega t}$. (1) + (2) then give:

$$x_2 = B_2 e^{i\omega t}$$

$$(k_1 + k_{12} - M\omega^2) B_1 - k_{12} B_2 = 0$$

$$-k_{12} B_1 + (k_{12} + k_2 - M\omega^2) B_2 = 0$$

$$\text{OR} \quad \begin{pmatrix} k_1 + k_{12} - M\omega^2 & -k_{12} \\ -k_{12} & k_{12} + k_2 - M\omega^2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For a nontrivial solution, the determinant of first matrix is zero.

$$\therefore (k_1 + k_{12} - M\omega^2)(k_{12} + k_2 - M\omega^2) - k_{12}^2 = 0$$

$$M^2\omega^4 - M\omega^2(k_1 + k_{12} + k_2 + k_{12}) + (k_1 + k_{12})(k_{12} + k_2) - k_{12}^2 = 0$$

$$M^2\omega^4 - M\omega^2(k_1 + k_2 + 2k_{12}) + k_1k_2 + k_{12}(k_1 + k_2) = 0$$

$$\omega^2 = \frac{M(k_1 + k_2 + 2k_{12}) \pm \sqrt{M^2(k_1 + k_2 + 2k_{12})^2 - 4M^2(k_1k_2 + k_{12}(k_1 + k_2))}}{2M^2}$$

$$= \frac{1}{2M} \left\{ k_1 + k_2 + 2k_{12} \right. \\ \left. \pm \sqrt{k_1^2 + k_2^2 + 4k_{12}^2 + 2k_1k_2 + 4k_1k_{12} + 4k_2k_{12} - 4k_1k_2 - 4k_1k_{12} - 4k_2k_{12}} \right\}^{1/2}$$

$$\therefore \omega_{\pm}^2 = \frac{1}{2M} \left\{ k_1 + k_2 + 2k_{12} \pm \sqrt{(k_1 - k_2)^2 + 4k_{12}^2} \right\}^{1/2}$$

If the second mass were held fixed, then m_1 oscillates at frequency $\omega_1 = \sqrt{\frac{k_1 + k_{12}}{M}}$

Similarly if m_1 is held fixed, then m_2 oscillates at frequency

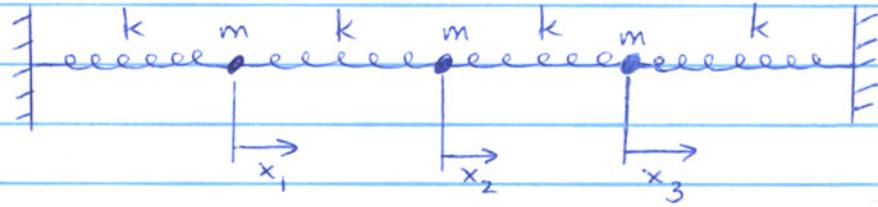
$$\omega_2 = \sqrt{\frac{k_2 + k_{12}}{M}}$$

$$\begin{aligned}
 \text{Now } \omega_+^2 &= \frac{1}{2M} \left\{ k_1 + k_2 + 2k_{12} + \left[(k_1 - k_2)^2 + 4k_{12}^2 \right]^{1/2} \right\} \\
 &> \frac{1}{2M} \left\{ k_1 + k_2 + 2k_{12} + (k_1 - k_2) \right\} \\
 &> \frac{k_1 + k_2}{M} \\
 &> \omega_1^2 \\
 \therefore \omega_+ &> \omega_1
 \end{aligned}$$

$$\begin{aligned}
 \text{Also } \omega_-^2 &= \frac{1}{2M} \left\{ k_1 + k_2 + 2k_{12} - \left[(k_1 - k_2)^2 + 4k_{12}^2 \right]^{1/2} \right\} \\
 &< \frac{1}{2M} \left\{ k_1 + k_2 + 2k_{12} - (k_1 - k_2) \right\} \\
 &< \frac{k_2 + 2k_{12}}{M} \\
 &< \omega_2^2 \\
 \therefore \omega_- &< \omega_2
 \end{aligned}$$

Hence, if $k_1 > k_2$ then $\omega_+ > \omega_1 > \omega_2 > \omega_-$

12.17 Let $x_1, x_2 + x_3$ be displacement of masses m_1, m_2 & m_3 from equilibrium.



$$\text{Kinetic Energy } T = \frac{m \dot{x}_1^2}{2} + \frac{m \dot{x}_2^2}{2} + \frac{m \dot{x}_3^2}{2}$$

$$\text{Potential Energy } U = \frac{k x_1^2}{2} + \frac{k (x_2 - x_1)^2}{2} + \frac{k (x_3 - x_2)^2}{2} + \frac{k x_3^2}{2}$$

$$\text{Lagrangian } L = \frac{m \dot{x}_1^2}{2} + \frac{m \dot{x}_2^2}{2} + \frac{m \dot{x}_3^2}{2}$$

$$- \frac{k x_1^2}{2} - \frac{k (x_2 - x_1)^2}{2} - \frac{k (x_3 - x_2)^2}{2} - \frac{k x_3^2}{2}$$

Lagrange's equations for $x_1, x_2 + x_3$ yield:

$$2kx_1 - kx_2 - m\ddot{x}_1 = 0$$

$$2kx_2 - kx_1 - kx_3 - m\ddot{x}_2 = 0$$

$$2kx_3 - kx_2 - m\ddot{x}_3 = 0$$

Substituting $x_i = a_i e^{i\omega t}$ gives:

$$\begin{pmatrix} 2k - mw^2 & -k & 0 \\ -k & 2k - mw^2 & -k \\ 0 & -k & 2k - mw^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (1)$$

For nontrivial solution, determinant of first matrix is zero.

$$(2k - mw^2) [(2k - mw^2)^2 - k^2] + k(-k)(2k - mw^2) = 0$$

$$(2k - \omega^2 m) \left[(2k - \omega^2 m)^2 - 2k^2 \right] = 0$$

First term is zero for $\omega = \omega_1 = \sqrt{\frac{2k}{m}}$

Next term is zero $\Rightarrow 2k - \omega^2 m = \pm \sqrt{2} k$

$$\omega^2 = (2 \mp \sqrt{2}) \frac{k}{m}$$

Hence, other two roots are $\omega_2 = \sqrt{2 + \sqrt{2}} \sqrt{\frac{k}{m}}$ & $\omega_3 = \sqrt{2 - \sqrt{2}} \sqrt{\frac{k}{m}}$.

$\omega = \omega_1$: Eqn. (1) then yields:

$$\begin{pmatrix} 0 & -k & 0 \\ -k & 0 & -k \\ 0 & -k & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{OR } -k a_2 = 0 \Rightarrow a_2 = 0$$

$$-k a_1 - k a_3 = 0 \Rightarrow a_1 = -a_3$$

$\omega = \omega_2$: Eqn. (1) then yields:

$$\begin{pmatrix} -\sqrt{2}k & -k & 0 \\ -k & -\sqrt{2}k & -k \\ 0 & -k & -\sqrt{2}k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{OR } -\sqrt{2}k a_1 - k a_2 = 0 \Rightarrow a_2 = -\sqrt{2} a_1$$

$$-k a_1 - \sqrt{2}k a_2 - k a_3 = 0$$

$$-k a_2 - \sqrt{2}k a_3 = 0 \Rightarrow a_2 = -\sqrt{2} a_3$$

$a_1 = a_3$

$w = w_3$: Eqn. (i) then yields:

$$\begin{pmatrix} \sqrt{2}k & -k & 0 \\ -k & \sqrt{2}k & -k \\ 0 & -k & \sqrt{2}k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

OR $\sqrt{2}ka_1 - ka_2 = 0 \Rightarrow a_2 = \sqrt{2}a_1$,
 $-ka_1 + \sqrt{2}ka_2 - ka_3 = 0 \Rightarrow a_1 = a_3$
 $-ka_2 + \sqrt{2}ka_3 = 0 \Rightarrow a_2 = \sqrt{2}a_3$

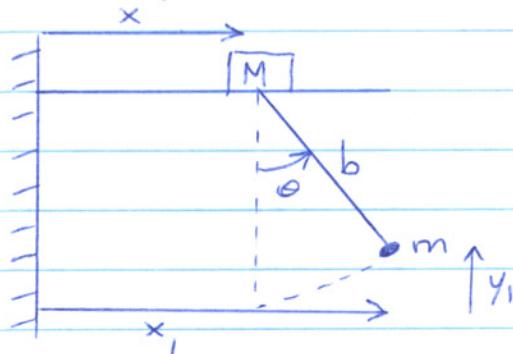
Summary of Mode Motions

$$w_1 \rightarrow \circ \leftarrow a_2 = 0, a_1 = -a_3$$

$$w_2 \leftarrow \circ \rightarrow \leftarrow a_2 = -\sqrt{2}a_1 = -\sqrt{2}a_3$$

$$w_3 \rightarrow \circ \rightarrow \rightarrow a_2 = \sqrt{2}a_1 = \sqrt{2}a_3$$

12.18 Consider pendulum attached to sliding mass.
Generalized coordinates are x & θ .



Position of mass m is:
 $x_1 = x + b \sin \theta$
 $y_1 = b - b \cos \theta$

Kinetic Energy of 2 masses is:

$$\begin{aligned} T &= \frac{M}{2} \dot{x}^2 + \frac{m}{2} (\dot{x}_1^2 + \dot{y}_1^2) \\ &= \frac{M}{2} \dot{x}^2 + \frac{m}{2} [(\dot{x} + b \cos \theta \dot{\theta})^2 + b^2 \sin^2 \theta \dot{\theta}^2] \\ &= \frac{M}{2} \dot{x}^2 + \frac{m}{2} [\dot{x}^2 + b^2 \dot{\theta}^2 + 2b \dot{x} \dot{\theta} \cos \theta] \end{aligned}$$

Potential Energy $U = mg y_1$
 $= mg b (1 - \cos \theta)$

Lagrangian $L = \frac{M}{2} \dot{x}^2 + \frac{m}{2} [\dot{x}^2 + b^2 \dot{\theta}^2 + 2b \dot{x} \dot{\theta} \cos \theta]$
 $-mg b (1 - \cos \theta)$

$$\approx \frac{M}{2} \dot{x}^2 + \frac{m}{2} [\dot{x}^2 + b^2 \dot{\theta}^2 + 2b \dot{x} \dot{\theta}] - \frac{mgb}{2} \dot{\theta}^2$$

where we considered small θ such that
 $\cos \theta = 1 - \frac{\theta^2}{2}$

$$\begin{aligned} \text{x egn: } & \frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0 \\ & 0 - \frac{d}{dt} \left(M\dot{x} + m\dot{x} + mb\ddot{\theta} \right) = 0 \\ & (M+m)\ddot{x} + mb\ddot{\theta} = 0 \quad (1) \end{aligned}$$

$$\begin{aligned} \theta \text{ egn: } & \frac{\partial L}{\partial \theta} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0 \\ & -mg b \theta - \frac{d}{dt} \left(mb^2 \dot{\theta} + mb\dot{x} \right) = 0 \\ & g\theta + \frac{d}{dt} (b\dot{\theta} + \dot{x}) = 0 \\ & \ddot{x} + b\ddot{\theta} + g\theta = 0 \quad (2) \end{aligned}$$

Let $x = A e^{i\omega t}$. (1) + (2) then yield:

$$\begin{aligned} -(M+m)\omega^2 A - mb\omega^2 B &= 0 \\ -\omega^2 A - \omega^2 b B + gB &= 0 \end{aligned}$$

$$\text{OR } \begin{pmatrix} -(M+m)\omega^2 & -mb\omega^2 \\ -\omega^2 & -\omega^2 b + g \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3)$$

For nontrivial solution, determinant of first matrix is zero.

$$-(M+m)\omega^2 (-\omega^2 b + g) - mb\omega^4 = 0$$

$$\omega^2 [(M+m)(-\omega^2 b + g) + mb\omega^2] = 0$$

$$\omega^2 [-\omega^2 Mb + (M+m)g] = 0$$

$$\therefore \text{solutions are: } w_1 = 0 \quad + w_2 = \sqrt{\frac{(M+m)}{Mb} g}$$

$w = w_1$: Eqn. (3) yields:

$$\begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore B = 0 \Rightarrow \theta = 0.$$

$w = w_2$: Eqn. (3) yields:

$$\begin{pmatrix} -\frac{(M+m)^2}{Mb} g & -m \frac{(M+m)}{M} g \\ -\frac{(M+m)}{Mb} g & -\frac{(M+m)}{M} g + g \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{OR: } \frac{(M+m)}{b} A + m B = 0 \implies B = -\frac{(M+m)}{b m} A$$

$$-\frac{(M+m)}{Mb} A - \frac{m}{M} B = 0 \quad (\text{yields same as above})$$