Assignment 9

1a, b $\nabla \cdot \vec{E} = 4\pi \rho$ Gauss Law

Flux of Electric field out of unit volume is $4\pi \times$ charge inside unit volume.

$\nabla \cdot \vec{B} = 0$

No magnetic monopoles exist.

$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$ Faraday Law

A rotating electric field is created by a changing magnetic field.

$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$ Ampere Law + Displacement Current

A rotating magnetic field is created by either a current or a changing electric field.

2. $\nabla \cdot \vec{B} = 0 \implies \vec{B} = \nabla \times \vec{A}$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$= -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \vec{A})$$

$$\nabla \times \left[\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right] = 0$$

$$\Rightarrow \vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\nabla \Phi \quad \text{or} \quad \vec{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}.$$

3a) Electric Displacement $r < a \quad \vec{D} = 0$

$r > a \quad \vec{D} = \frac{Q}{r^2}\,\hat{r}.$

Electric Field $r < a \quad \vec{E} = 0$

$b > r > a \quad \vec{E} = \frac{Q}{\epsilon r^2}\,\hat{r}$

$r > b \quad \vec{E} = \frac{Q}{r^2}\,\hat{r}.$

Energy Density $r < a \quad \mathcal{U}_E = \frac{\vec{E}\cdot\vec{D}}{8\pi} = 0$

$b > r > a \quad \mathcal{U}_E = \frac{Q^2}{8\pi\epsilon r^4}$

$r > b \quad \mathcal{U}_E = \frac{Q^2}{8\pi r^4}.$

b) Total Energy

$$U_{TOT} = \int_{\text{all space}} \mathcal{U}_E \, dV$$

$$= \int_a^b \frac{Q^2}{8\pi\epsilon r^4}\, 4\pi r^2 dr + \int_b^\infty \frac{Q^2}{8\pi r^4}\, 4\pi r^2 dr$$

$$= \frac{Q^2}{2}\left\{ \frac{1}{\epsilon}\int_a^b \frac{dr}{r^2} + \int_b^\infty \frac{dr}{r^2} \right\}$$

$$= \frac{Q^2}{2}\left\{ \frac{1}{\epsilon}\left[\frac{-1}{r}\right]_a^b + \left[\frac{-1}{r}\right]_b^\infty \right\}$$

$$\therefore U_{TOT} = \frac{Q^2}{2}\left\{ \frac{1}{\epsilon}\left(\frac{-1}{b} + \frac{1}{a}\right) + \frac{1}{b} \right\}$$

4) $$W = \frac{1}{8\pi} \int_{\text{all space}} \vec{H} \cdot \vec{B} \, d^3x .$$

$$= \frac{1}{8\pi} \int_{\text{all space}} \vec{H} \cdot (\nabla \times \vec{A}) \, d^3x$$

$$= \frac{1}{8\pi} \int \left[\nabla \cdot (\vec{H} \times \vec{A}) + \vec{A} \cdot (\nabla \times \vec{H}) \right] d^3x$$

$$= \frac{1}{8\pi} \int_{\substack{\text{surface at} \\ \text{infinity}}} \vec{H} \times \vec{A} \cdot d\vec{a} \quad + \frac{1}{8\pi} \int_{\text{all space}} \vec{A} \cdot \frac{4\pi}{c} \vec{J} \, d^3x$$

$$= 0 \quad + \frac{1}{2c} \int \vec{A} \cdot \vec{J} \, d^3x$$

since $\vec{H} \times \vec{A}$ falls off faster than $1/r^2$

$$\therefore W = \frac{1}{2c} \int_{\text{all space}} \vec{A} \cdot \vec{J} \, d^3x .$$