Assignment 7

1. Lorentz Force $\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$

$\therefore$ magnetic field alone $\Rightarrow \vec{F}_M = q\,\frac{\vec{v}}{c} \times \vec{B}$

Work done when charge is displaced $\vec{v}\,dt$ is

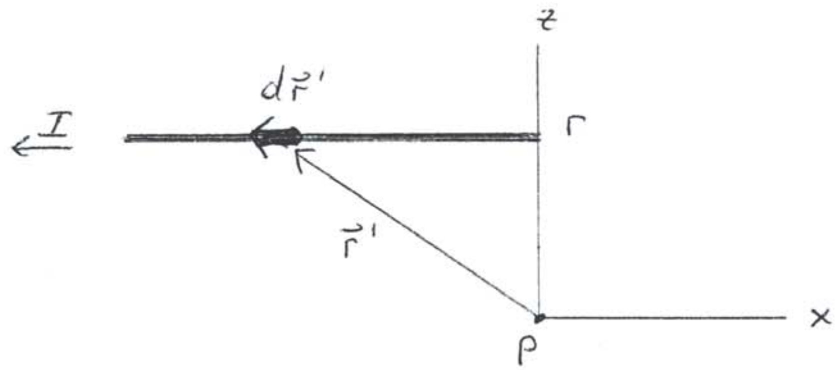
$$dW = \vec{F}_M \cdot \vec{v}\,dt$$

$$= 0$$

$\therefore$ magnetic fields do no work since $\vec{F}_M \perp \vec{v}$.

2) $\vec{B}(P)$ = Mag. field of Top part of wire + Mag. field of semicircle + Mag. field of Bottom part

<u>Magnetic Field of Top Part of Wire</u>



Consider infinitesimal current element at $\vec{r}' = (-x', 0, r)$

$$\therefore d\vec{r}' = -dx' \hat{x}$$

$$\vec{B}(\vec{r}) = \frac{I}{c} \int d\vec{r}' \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$\therefore \vec{B}(0) = \frac{I}{c} \int_0^{+\infty} -dx' \hat{x} \times \frac{(+x', 0, -r)}{(x'^2 + r^2)^{3/2}}$$

$$= \frac{I}{c} \int_0^{+\infty} \frac{(0, -r, 0)}{(x'^2 + r^2)^{3/2}} dx'$$

$$= -\frac{Ir}{c} \hat{y} \int_0^{+\infty} \frac{dx'}{(x'^2 + r^2)^{3/2}}$$

Let $x' = r \tan\theta$.

$dx' = r \sec^2\theta \, d\theta$.

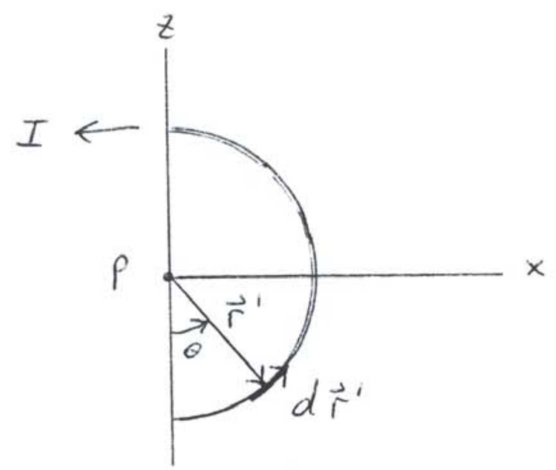
$$\therefore \vec{B}(0) = -\frac{I r}{c}\hat{y}\int_0^{+\pi/2} \frac{r\sec^2\theta\, d\theta}{r^3\sec^3\theta}$$

$$= \frac{-I}{rc}\hat{y}\int_0^{+\pi/2}\cos\theta\, d\theta$$

$$\vec{B}(0) = -\frac{I}{rc}\hat{y}$$

Similarly Bottom half of wire contributes $-\frac{I}{rc}\hat{y}$ to field at P.

Magnetic Field Due to Semicircle



Consider infinitesimal current element at
$\vec{r}' = r(\sin\theta, 0, -\cos\theta)$

$d\vec{r}' = r(\cos\theta, 0, \sin\theta)\, d\theta.$

$$\vec{B}(\vec{r}) = \frac{I}{c}\int d\vec{r}' \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{B}(0) = \frac{I}{c}\int_0^{\pi} r(\cos\theta, 0, \sin\theta)\, d\theta \times \frac{r(-\sin\theta, 0, \cos\theta)}{r^3}$$

$$\vec{B}(o) = \frac{I}{rc} \int_0^{\pi} (0, -1, 0) \, d\theta$$

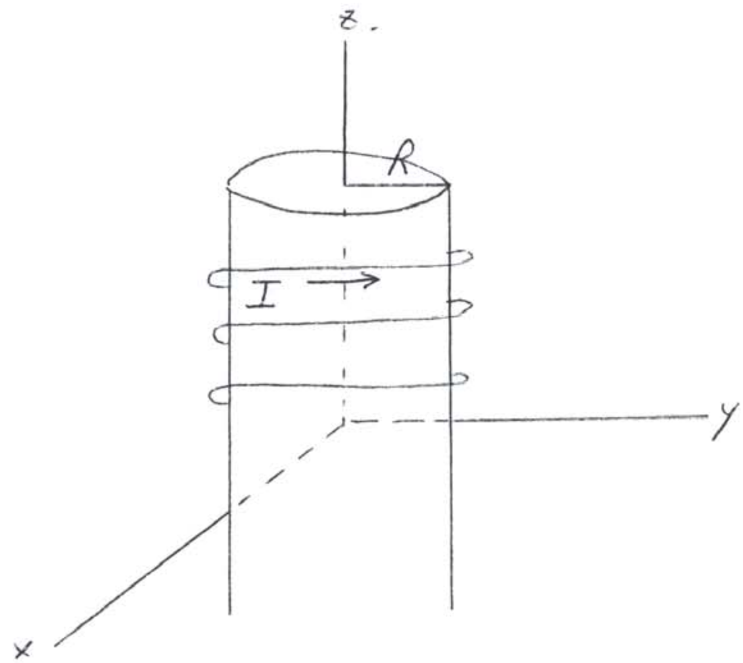
$$= -\frac{\pi I}{rc} \hat{y}$$

$$\therefore \vec{B}(P) = \left(-\frac{2I}{rc} - \frac{\pi I}{rc} \right) \hat{y}$$

$$= -(2+\pi) \frac{I}{rc} \hat{y}$$

where $\hat{y}$ is direction into page.

3)



a) $\vec{B} = \begin{cases} 0 & \text{outside solenoid} \\ \dfrac{4\pi NI}{c}\hat{z} & \text{inside solenoid} \end{cases}$

Due to cylindrical symmetry $\vec{A}(\vec{r}) = A(r)\,\hat{\phi}$.
(r = cylindrical radial coordinate)

$$\nabla \times \vec{A} = \vec{B}$$

$$\int_S \nabla \times \vec{A} \cdot d\vec{a} = \int \vec{B} \cdot d\vec{a}$$

S = circular surface of radius r
centered about z axis.

$$\oint_{\substack{\text{loop of} \\ \text{radius } r}} \vec{A} \cdot d\vec{l} = \int_S \vec{B} \cdot d\vec{a}$$

$$A(r)\, 2\pi r = \int_S \vec{B} \cdot d\vec{a}$$

$r < R \qquad A(r)\, 2\pi r = \frac{4\pi NI}{c}\, \pi r^2$

$$A(r) = \frac{2\pi NI}{c}\, r$$

$r > R \qquad \vec{A}(r) = \frac{2\pi NI}{c}\, r\, \hat{\phi}$

$r > R \qquad A(r)\, 2\pi r = \frac{4\pi NI}{c}\, \pi R^2$

$$A(r) = \frac{2\pi NI}{c}\, \frac{R^2}{r}$$

$$\vec{A}(r) = \frac{2\pi NI}{c}\, \frac{R^2}{r}\, \hat{\phi}.$$

Note that $\vec{A} = A_\phi(r)\hat{\phi}$.

b) $\nabla \times \vec{A} = \frac{1}{r}\, \frac{\partial}{\partial r}\left(r\, A_\phi \right) \hat{z}$

$r < R \qquad \nabla \times \vec{A} = \frac{1}{r}\, \frac{\partial}{\partial r}\left(\frac{2\pi NI}{c}\, r^2 \right) \hat{z}$

$$= \frac{4\pi NI}{c}\, \hat{z}$$

$r > R \qquad \nabla \times \vec{A} = \frac{1}{r}\, \frac{\partial}{\partial r}\left(\frac{2\pi NI}{c}\, R^2 \right) \hat{z}$

$$= 0.$$

$\Rightarrow \quad \vec{B} = \nabla \times \vec{A}$

c) $\nabla \cdot \vec{A} = \frac{1}{r}\, \frac{\partial A_\phi}{\partial \phi} = 0.$

4 a) $\nabla \times \vec{A} = \nabla \times \left[-\frac{1}{2} \vec{r} \times \vec{B} \right]$

$$= -\frac{1}{2} \left\{ (\vec{B} \cdot \nabla) \vec{r} - \underbrace{(\vec{r} \cdot \nabla) \vec{B}}_{\substack{=0 \text{ since} \\ \vec{B} = \text{constant}}} + \vec{r} \underbrace{(\nabla \cdot \vec{B})}_{=0} - \vec{B} (\nabla \cdot \vec{r}) \right\}$$

$$= -\frac{1}{2} \left\{ (\vec{B} \cdot \nabla) \vec{r} - \vec{B} (\nabla \cdot \vec{r}) \right\}$$

$$= -\frac{1}{2} \left\{ \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} \right) (x, y, z) - 3 \vec{B} \right\}$$

$$= -\frac{1}{2} \left\{ B_x (1,0,0) + B_y (0,1,0) + B_z (0,0,1) - 3 \vec{B} \right\}$$

$$= -\frac{1}{2} \left\{ \vec{B} - 3 \vec{B} \right\}$$

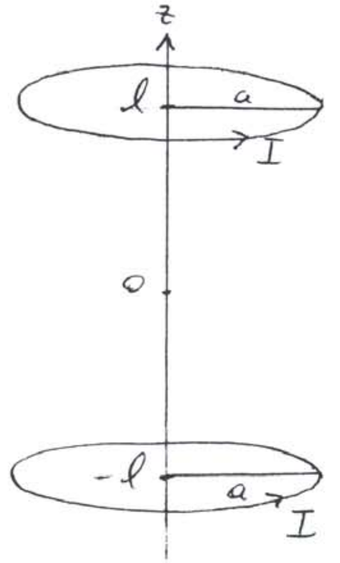
$\therefore \nabla \times \vec{A} = \vec{B}$

b) $\nabla \cdot \vec{A} = \nabla \cdot \left[-\frac{1}{2} \vec{r} \times \vec{B} \right]$

$$= -\frac{1}{2} \left\{ \vec{B} \cdot \underbrace{(\nabla \times \vec{r})}_{=0} - \vec{r} \cdot \underbrace{(\nabla \times \vec{B})}_{=0 \text{ for uniform } \vec{B}} \right\}$$

$\therefore \nabla \cdot \vec{A} = 0$

5)



a) Field along z axis due to coil at origin is

$$\vec{B}(0,0,z) = \frac{2\pi I a^2}{c(a^2+z^2)^{3/2}} \cdot \hat{z}.$$

Field due to coil 1 at $z = -l$ is:

$$\vec{B}_1(0,0,z) = \frac{2\pi I a^2}{c\left[a^2 + (z+l)^2\right]^{3/2}} \hat{z}.$$

Field due to coil 2 at $z = +l$ is;

$$\vec{B}_2(0,0,z) = \frac{2\pi I a^2}{c\left[a^2 + (z-l)^2\right]^{3/2}} \hat{z}$$

∴ field of both coils on z axis is

$$\vec{B}(0,0,z) = \vec{B}_1(0,0,z) + \vec{B}_2(0,0,z)$$

$$= \frac{2\pi I a^2}{c}\left\{\left[a^2+(z+l)^2\right]^{-3/2} + \left[a^2+(z-l)^2\right]^{-3/2}\right\}\hat{z}$$

b) $\vec{B}(0,0,z) = \hat{z}\; B(z)$

$$B(z) = B(0) + z \left.\frac{\partial B}{\partial z}\right|_0 + \frac{z^2}{2} \left.\frac{\partial^2 B}{\partial z^2}\right|_0 + \frac{z^3}{3} \left.\frac{\partial^3 B}{\partial z^3}\right|_0 + \cdots$$

But due to reflection symmetry about the xy plane (containing $z=0$) we require

$$B(z) = B(-z) \Rightarrow \left.\frac{\partial B}{\partial z}\right|_0 = \left.\frac{\partial^3 B}{\partial z^3}\right|_0 = \cdots = 0.$$

c) $$\frac{\partial B}{\partial z} = \frac{2\pi I a^2}{c} \left\{ -\frac{3}{2}\, 2(z+\ell)\left[a^2 + (z+\ell)^2\right]^{-5/2} - \frac{3}{2}\, 2\cdot(z-\ell)\left[a^2 + (z-\ell)^2\right]^{-5/2} \right\}$$

$$= -\frac{6\pi I a^2}{c} \left\{ (z+\ell)\left[a^2 + (z+\ell)^2\right]^{-5/2} + (z-\ell)\left[a^2+(z-\ell)^2\right]^{-5/2} \right.$$

$$\frac{\partial^2 B}{\partial z^2} = -\frac{6\pi I a^2}{c} \left\{ \left[a^2 + (z+\ell)^2\right]^{-5/2} - \frac{5}{2}\, 2\cdot(z+\ell)^2\left[a^2 + (z+\ell)^2\right]^{-7/2} \right.$$

$$+ \left[a^2 + (z-\ell)^2\right]^{-5/2} - \frac{5}{2}\, 2\,(z-\ell)^2\left[a^2 + (z-\ell)^2\right]^{-7/2}$$

$$\left.\frac{\partial^2 B}{\partial z^2}\right|_{z=0} = -\frac{6\pi I a^2}{c}\left\{ 2\cdot\left[a^2+\ell^2\right]^{-5/2} - 10\,\ell^2\left[a^2+\ell^2\right]^{-7/2} \right\}$$

$$0 = \left.\frac{\partial^2 B}{\partial z^2}\right|_{z=0} \Longrightarrow 0 = 2\left[a^2+\ell^2\right]^{-5/2} - 10\ell^2\left[a^2+\ell^2\right]^{-7/2}$$

$$= a^2 + \ell^2 - 5\ell^2$$

$0 = a^2 - 4l^2.$

$\therefore \; a = 2l.$

Hence if $a = 2l$, $B(z) = B(0) + O(z^4)$.

d. charge of electron $= 4.8 \times 10^{-10}$ esu
$= 1.6 \times 10^{-19}$ Coul.

$\therefore 1.6 \times 10^{-19}$ Coul $= 4.8 \times 10^{-10}$ esu.

1 Coul $= 3 \times 10^{9}$ esu.

1 amp $= 3 \times 10^{9}$ esu/sec.

$\therefore \; I \text{(amps)} \cdot 3 \times 10^{9} = I \text{(esu/sec)}$

$$I \text{(amp)} \cdot \frac{3 \times 10^{9}}{3 \times 10^{10}} = \frac{I \text{(esu/sec)}}{3 \times 10^{10}}.$$

$$\frac{I \text{(amp)}}{10} \qquad \frac{I \text{(esu/sec)}}{c}$$

e) From part a we find:

$$\vec{B}(0,0,0) = \hat{z}\ \frac{2\pi I a^2}{c} \left\{ 2\left[a^2 + \ell^2\right]^{-3/2} \right\}$$

Helmholtz Criterion is $a = 2\ell$.

$$\vec{B}(0,0,0) = \hat{z}\ \frac{2\pi I a^2}{c}\ 2\ \left(a^2 + \frac{a^2}{4}\right)^{-3/2}$$

$$= \hat{z}\ \frac{4\pi I a^2}{c}\ \frac{1}{a^3}\ \left(\frac{5}{4}\right)^{-3/2}$$

$$= \hat{z}\ \frac{4\pi I}{a c}\ \left(\frac{4}{5}\right)^{3/2}$$

$$= \hat{z}\ \frac{32\pi}{5^{3/2}}\ \frac{I}{a c}$$

For Helmholtz coil with N turns

$$\vec{B}(0,0,0) = \hat{z}\ \frac{32\pi}{5^{3/2}}\ \frac{NI}{a c}$$

$B_{Earth} \simeq 0.5$ gauss.

$$\frac{I}{c} = \frac{5^{3/2}}{32\pi}\ B(0)\ \frac{a}{N} N$$

$$\frac{I(\text{amps})}{10} = \frac{5^{3/2}}{32\pi}\ B(0)\,(\text{gauss})\ \frac{a\,(\text{cm})}{N\,(\text{turns})}$$

$$I(\text{amps}) = 10\ \frac{5^{3/2}}{32\pi}\ \frac{1}{2}\ \text{gauss}\ \frac{30\ \text{cm}}{50\ \text{turns}}$$

$$= \frac{1}{3}\ \text{amp}.$$

One would need to be careful that direction of field opposes $\vec{B}_{Earth}$ rather than adding to it!