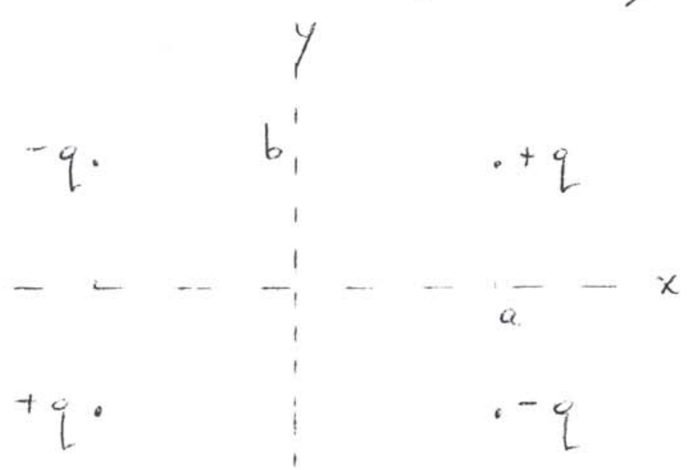
Assignment 4

1) Obviously $\Phi = 0$ is a solution. Because of uniqueness theorem, we know this is the only possible solution.

2) Consider the following problem.



Note: 1) $\nabla^2 \Phi = 4\pi \rho$ in the region $x, y > 0$ is the same as in problem.

2) $\Phi = 0$ on x & y planes.

$\therefore$ by uniqueness theorem, both problems have the same potential in the quadrant $x, y > 0$.

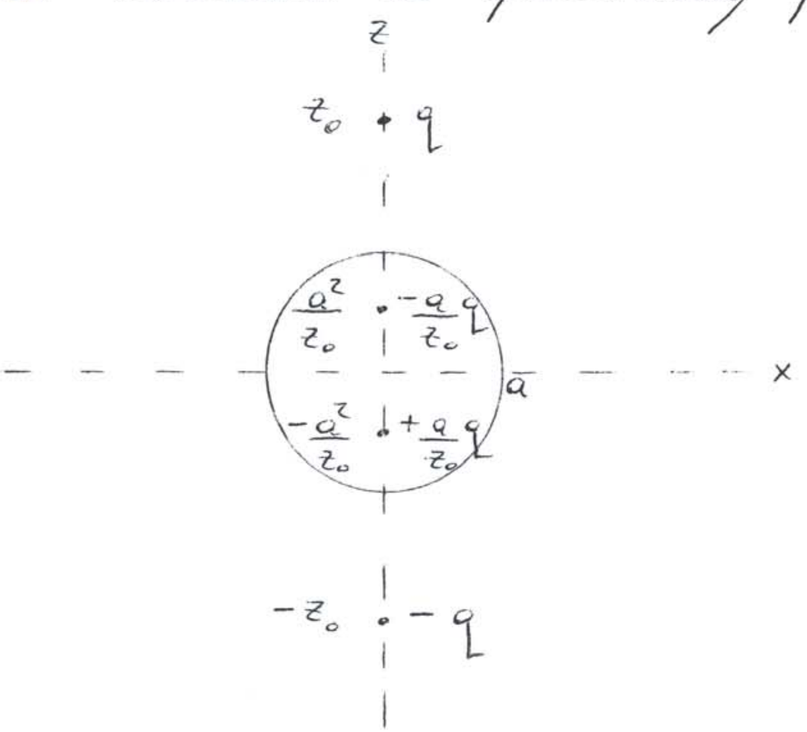
$$\therefore \Phi(x,y,z) = \frac{+q}{|(x-a, y-b, z)|} - \frac{q}{|(x-a, y+b, z)|}$$

$$- \frac{q}{|(x+a, y-b, z)|} + \frac{q}{|(x+a, y+b, z)|}$$

$$\Phi(x,y,z) = q\left\{ \left[(x-a)^2 + (y-b)^2 + z^2\right]^{-1/2} - \left[(x-a)^2 + (y+b)^2 + z^2\right]^{-1/2} \right.$$
$$\left. - \left[(x+a)^2 + (y-b)^2 + z^2\right]^{-1/2} + \left[(x+a)^2 + (y+b)^2 + z^2\right]^{-1/2} \right\}$$

This only holds in quadrant $x, y > 0$.
In conductor $\Phi = 0$.

3) Consider the following problem.



Note: 1) $\nabla^2 \Phi = 4\pi\rho$ in region above plane + hemispherical bubble is the same as in problem

2) $\Phi = 0$ on xy plane + on hemisphere.

$\therefore$ by uniqueness theorem, both problems have the same potential in region above plane + bubble.

$$\Phi(\vec{r}) = \frac{q}{|\vec{r}-(0,0,z_0)|} - \frac{\frac{a}{z_0}q}{|\vec{r}-(0,0,a^2/z_0)|} + \frac{\frac{a}{z_0}q}{|\vec{r}-(0,0,-a^2/z_0)|} - \frac{q}{|\vec{r}-(0,0,-z_0)|}$$

4). We must solve the two dimensional Laplace eqn. $\dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial y^2} = 0.$

Let $\Phi(x,y) = X(x)\, Y(y).$

Bound. Conds. are: $\Phi(0,y) = \Phi(a,y) = 0 \quad \forall y.$

$$\Rightarrow X(0) = X(a) = 0.$$

Hence we need periodic solution for $X(x)$. (Case 2)

$$X(x) = A \cos kx + B \sin kx.$$

$$X(0) = 0 \Rightarrow A = 0. \Rightarrow X(x) = B \sin kx.$$

$$X(a) = 0 \Rightarrow \quad B \sin ka = 0.$$

$$ka = n\pi.$$

$$k = \frac{n\pi}{a} \qquad n \in N.$$

Next $Y(y) = Ce^{ky} + De^{-ky}$

But $\Phi(x,\infty) = 0 \Rightarrow C = 0. \Rightarrow Y(y) = De^{-ky}$

$$\therefore \Phi(x,y) = \sum_{n=1}^{\infty} B_n \sin k_n x \; e^{-k_n y} \quad \text{where } k_n = \frac{n\pi}{a}.$$

$$= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \; e^{-\frac{n\pi y}{a}}$$

Remaining B.C. is $\Phi(x,0) = \Phi_0$.

$$\therefore \Phi_0 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}$$

$$\int_0^a \Phi_0 \sin\frac{m\pi x}{a}\,dx = \sum_{n=1}^{\infty} B_n \underbrace{\int_0^a \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}\,dx}_{=\frac{a}{2}\delta_{nm}}$$

$$\Phi_0\left[\frac{-\cos\frac{m\pi x}{a}}{\frac{m\pi}{a}}\right]_0^a = B_m\frac{a}{2}$$

$$B_m = \frac{2}{a}\frac{a}{m\pi}\left[-\cos m\pi + 1\right]\Phi_0$$

$$\therefore B_m = \frac{2}{m\pi}\left[1-(-1)^m\right]\Phi_0$$

$$\boxed{\therefore \Phi(x,y) = \sum_{n=1}^{\infty}\frac{2\Phi_0}{n\pi}\left(1-(-1)^n\right)\sin\frac{n\pi x}{a}\,e^{-n\pi y/a}}$$

5) Let $\Phi = R(\rho)\, Q(\phi)$.

$$0 = \nabla^2 \Phi = Q(\phi) \frac{1}{\rho} \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{R(\rho)}{\rho^2} \frac{d^2 Q(\phi)}{d\phi^2}$$

$$0 = \frac{\rho}{R} \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{Q} \frac{d^2 Q}{d\phi^2}$$

$$\therefore \quad \frac{1}{Q} \frac{d^2 Q}{d\phi^2} = -K.$$

$$\& \quad \frac{\rho}{R} \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = +K.$$

b) $K = 0 \implies \dfrac{d^2 Q}{d\phi^2} = 0 \implies Q = C\phi + D.$

$$\frac{\rho}{R} \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = 0.$$

$$\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = 0.$$

$$\rho \frac{dR}{d\rho} = A.$$

$$\frac{dR}{d\rho} = \frac{A}{\rho}$$

$$R(\rho) = A \ln\rho + B.$$

$K = +k^2 > 0 \qquad \dfrac{d^2 Q}{d\phi^2} = -k^2 Q.$

$$Q = C \cos k\phi + D \sin k\phi.$$

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = k^2.$$

Let $R = \rho^\lambda$. $\Rightarrow \rho \dfrac{d}{d\rho}\left(\rho \, \lambda \rho^{\lambda-1}\right) = k^2 \rho^\lambda.$

$$\rho \frac{d}{d\rho}\left(\lambda \rho^\lambda\right) = k^2 \rho^\lambda.$$

$$\lambda^2 = k^2.$$

$$\lambda = \pm k.$$

$$\Rightarrow R(\rho) = A\rho^k + B\rho^{-k}.$$

$K = -k^2 < 0$ Similar to above.

c) $Q(\phi) = Q(\phi + 2\pi)$.

If $K = -k^2 < 0$ one can quickly show $Q = 0$.

" $K = 0$ one can show $C = 0$.

" $K = +k^2 > 0$ " $C \cos k(2\pi + \phi) + D \sin k(2\pi + \phi)$

$= C \cos k\phi + D \sin k\phi.$

$\Rightarrow k = n$ an integer

$\therefore$ most general possible solution if
$\Phi(\rho,\phi) = \Phi(\rho,\phi+2\pi)$ is:

$$\Phi(\rho,\phi) = A\ln\rho + B + \sum_{n=1}^{\infty}\left(A_n\rho^{n} + B_n\rho^{-n}\right)$$

$$\cdot\left(C_n\cos n\phi + D_n\sin n\phi\right).$$

where constant D has been incorporated
into A & B.