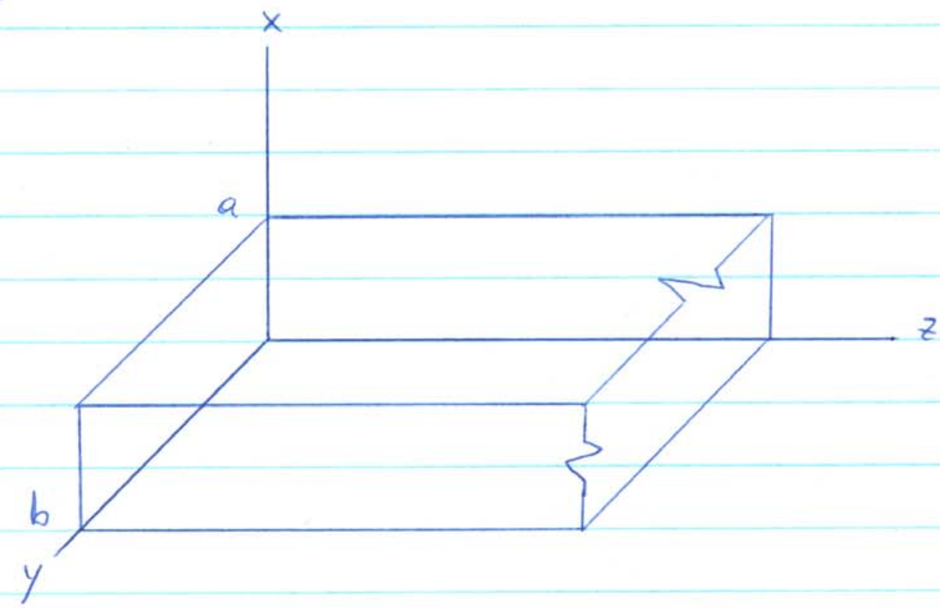
Assignment 4

10 1a. TM $\equiv$ transverse magnetic i.e. $B_{oz} = 0$.

$\therefore$ solve $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) E_{oz} + k_c^2 E_{oz} = 0$

where $k_c^2 \equiv \dfrac{\omega^2}{c^2} - k^2$ and $E_{oz} = 0$ on surfaces of waveguide.



Solution is $E_{oz} = E_o \sin \dfrac{m\pi x}{a} \sin \dfrac{n\pi y}{b}$ $\quad m, n = 1, 2, 3 \ldots$

where $k_c^2 = \dfrac{m^2\pi^2}{a^2} + \dfrac{n^2\pi^2}{b^2}$

$$k_c = \pi\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2}$$

$$E_{0x} = \frac{i}{k_c^2}\, k\, \frac{\partial E_{0z}}{\partial x} = \frac{ik}{k_c^2}\,\frac{m\pi}{a}\, E_0 \cos\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}$$

$$E_{0y} = \frac{i}{k_c^2}\, k\, \frac{\partial E_{0z}}{\partial y} = \frac{ik}{k_c^2}\,\frac{n\pi}{b}\, E_0 \sin\frac{m\pi x}{a}\,\cos\frac{n\pi y}{b}$$

$$B_{0x} = \frac{-i}{k_c^2}\,\frac{\omega}{c}\,\frac{\partial E_{0z}}{\partial y} = \frac{-i}{k_c^2}\,\frac{\omega}{c}\,\frac{n\pi}{b}E_0 \sin\frac{m\pi x}{a}\,\cos\frac{n\pi y}{b}$$

$$B_{0y} = \frac{i}{k_c^2}\,\frac{\omega}{c}\,\frac{\partial E_{0z}}{\partial x} = \frac{i}{k_c^2}\,\frac{\omega}{c}\,\frac{m\pi}{a}\, E_0 \cos\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}$$

5 b. $$k_c^2 \equiv \frac{\omega^2}{c^2} - k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$\therefore$ cutoff frequencies ($k=0$) occur at

$$\omega_{mn} = \pi c \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2}$$

$\therefore$ lowest cutoff frequency for TM modes is ω_{11}.

5 2. TM_{11} mode.

$$E_{0x} = \frac{ik}{k_c^2}\,\frac{\pi}{a}\, E_0 \cos\frac{\pi x}{a}\,\sin\frac{\pi y}{b}$$

$$E_{0y} = \frac{ik}{k_c^2}\,\frac{\pi}{b}\, E_0 \sin\frac{\pi x}{a}\,\cos\frac{\pi y}{b}$$

$$E_{0z} = E_0 \sin\frac{\pi x}{a}\,\sin\frac{\pi y}{b}.$$

$$B_{ox} = \frac{-i}{k_c^2} \frac{\omega}{c} \frac{\pi}{b} E_o \sin \frac{\pi x}{a} \cos \frac{\pi y}{b}$$

$$B_{oy} = \frac{i}{k_c^2} \frac{\omega}{c} \frac{\pi}{a} E_o \cos \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

$$B_{oz} = 0.$$

5 a) $$\langle \vec{S} \rangle = \frac{c}{8\pi} \mathrm{Re} \left(\vec{E} \times \vec{B}^* \right)$$

$$= \frac{c}{8\pi} \hat{z} \left(E_{ox} B_{oy}^* - E_{oy} B_{ox}^* \right)$$

$$= \hat{z} \frac{c}{8\pi} \left\{ \frac{k}{k_c^4} \frac{\omega}{c} \frac{\pi^2}{a^2} E_o^2 \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} + \frac{k}{k_c^4} \frac{\omega}{c} \frac{\pi^2}{b^2} E_o^2 \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{b} \right\}$$

$$\langle \vec{S} \rangle = \hat{z} \frac{\omega k}{8\pi k_c^4} E_o^2 \left[\frac{\pi^2}{a^2} \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} + \frac{\pi^2}{b^2} \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{b} \right]$$

$\langle \vec{S} \rangle$ = energy passing per second through unit area down waveguide.

5 b) Power $P = \int_{\substack{\text{cross section} \\ \text{of waveguide}}} \langle \vec{S} \rangle \cdot \hat{z} \, da$

$$P = \int_0^a \int_0^b \frac{\omega k}{8\pi k_c^4} E_0^2 \left[\frac{\pi^2}{a^2} \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} + \frac{\pi^2}{b^2} \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{b} \right] dx\,dy$$

$$= \frac{\omega k}{8\pi k_c^4} E_0^2 \left[\frac{\pi^2}{a^2} \frac{a}{2} \frac{b}{2} + \frac{\pi^2}{b^2} \frac{a}{2} \frac{b}{2} \right]$$

$$= \frac{\omega k}{8\pi k_c^4} E_0^2 \frac{ab}{4} \left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} \right)$$

$$P = \frac{1}{32\pi} \frac{\omega k}{k_c^2} ab \, E_0^2 .$$

5 c) Average energy density

$$\langle u \rangle = \frac{1}{16\pi} \left(\vec{E} \cdot \vec{E}^* + \vec{B} \cdot \vec{B}^* \right)$$

$$= \frac{1}{16\pi} \left\{ \frac{k^2}{k_c^4} \frac{\pi^2}{a^2} E_0^2 \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} \right.$$

$$+ \frac{k^2}{k_c^4} \frac{\pi^2}{b^2} E_0^2 \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{b}$$

$$+ E_0^2 \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b}$$

$$+ \frac{1}{k_c^4} \frac{\omega^2}{c^2} \frac{\pi^2}{b^2} E_0^2 \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{b}$$

$$+ \frac{1}{k_c^4} \frac{\omega^2}{c^2} \frac{\pi^2}{a^2} E_0^2 \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} \Big\}$$

5 d) Average energy per unit length

$$\langle \mathcal{E} \rangle = \int_{\substack{\text{unit length volume} \\ \text{of waveguide}}} \langle u \rangle \, dV.$$

$$\langle \mathcal{E} \rangle = \int_0^a \int_0^b \langle u \rangle \, dy \, dx.$$

$$= \frac{1}{16\pi} \Big\{ \frac{k^2}{k_c^4} \frac{\pi^2}{a^2} E_0^2 \frac{a}{2} \frac{b}{2}$$

$$+ \frac{k^2}{k_c^4} \frac{\pi^2}{b^2} E_0^2 \frac{a}{2} \frac{b}{2}$$

$$+ E_0^2 \frac{a}{2} \frac{b}{2}$$

$$+ \frac{1}{k_c^4} \frac{\omega^2}{c^2} \frac{\pi^2}{b^2} E_0^2 \frac{a}{2} \frac{b}{2}$$

$$+ \frac{1}{k_c^4} \frac{\omega^2}{c^2} \frac{\pi^2}{a^2} E_0^2 \frac{a}{2} \frac{b}{2} \Big\}$$

$$= \frac{1}{16\pi} \left\{ \frac{k^2}{k_c^4} \left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} \right) + 1 + \frac{1}{k_c^4} \frac{\omega^2}{c^2} \left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} \right) \right\} E_0^2 \frac{ab}{4}.$$

$$\langle \mathcal{E} \rangle = \frac{1}{64\pi} \left\{ \frac{k^2}{k_c^2} + 1 + \frac{1}{k_c^2} \frac{\omega^2}{c^2} \right\} E_0^2 \, ab$$

$$= \frac{1}{64\pi} \frac{1}{k_c^2} \left[k^2 + \frac{\omega^2}{c^2} + k_c^2 \right] E_0^2 \, ab.$$

$$\therefore \langle \mathcal{E} \rangle = \frac{1}{32\pi} \frac{\omega^2}{k_c^2 c^2} \, ab \, E_0^2$$

5 e) <u>Speed of energy propagation</u>

$$v_g = \frac{P}{\langle \mathcal{E} \rangle}$$

$$= \frac{\omega k}{k_c^2} \frac{k_c^2 c^2}{\omega^2}$$

$$= \frac{c^2}{\omega / k}.$$

$$v_g = \frac{c^2}{v_p} \quad \text{where } v_p \equiv \frac{\omega}{k} = \text{phase velocity}$$

45