# Calculus

## Lecture Notes

W. A. van Wijngaarden

Physics Dept., York University

## Function

A function f is a rule that assigns a unique real number to a specified input number x.

$$\{\, x \,\} \xrightarrow{\qquad\qquad f \qquad\qquad} \{\, y = f(x) \,\}$$

Set of specified input numbers x called the domain.

Set of output numbers f(x) called the range.

Therefore to define a function, one must specify

1) the set of allowed input numbers - the domain

2) the rule f

## Examples

1) $f(x) = \sqrt{x+1} \qquad x \in R.$

$f(x)$ isn't a function since $f(x) = \sqrt{\text{neg. } \# } \quad \notin,$ when $x < -1,$

2) $f(x) = \sqrt{x+1} \qquad x \geq -1$

f is a function since 1) all outputs are real
2) each input x has only one output value.

Domain $= \{ x \mid , x \geq -1 \}$
Range $= \{ y \mid 0 \leq y < \infty \}$

3) $f(x) = \begin{cases} x+1 & x \ge 0 \\ +x & x \le 0 \end{cases}$

f isn't a function since $\begin{cases} f(0) = 1 & \text{from line 1} \\ f(0) = 0 & \text{" \quad " 2} \end{cases}$

4) $f(x) = \begin{cases} x+1 & x > 0 \\ +x & x \le 0 \end{cases}$

f is a function since 1) all outputs are real

2) each input x has only one output value.

Domain $= \{ x \mid x \in R \}$

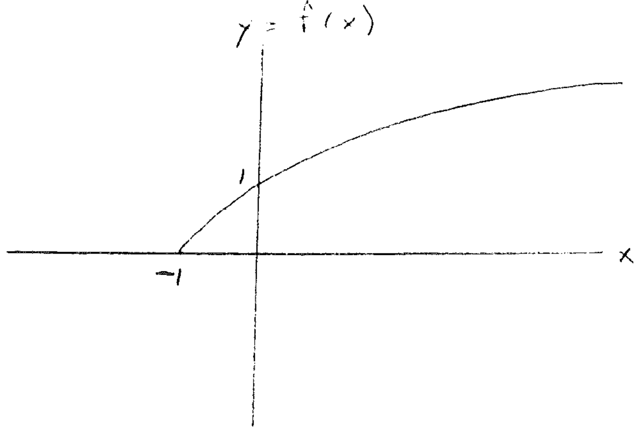
Range $= \{ y \mid \ y > 1 \text{ and } y \le 0 \}$.

## Graphing

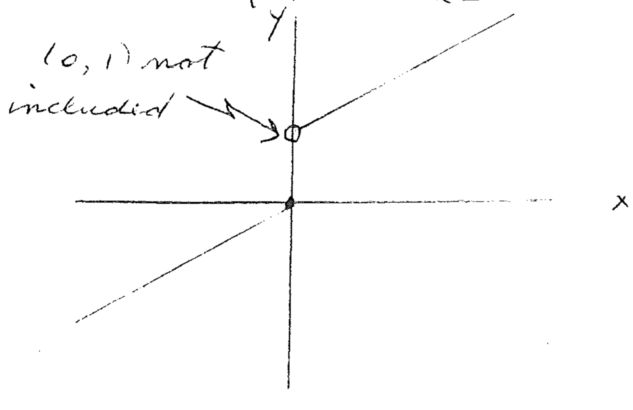
Pictures or graphs clarify definitions of functions.

Examples.

1) $f(x) = \sqrt{x+1} \qquad x \ge -1$



2) $f(x) = \begin{cases} x+1 & x > 0 \\ +x & x \le 0 \end{cases}$



# Inequalities

1. Find the largest possible domain of $f(x) = \sqrt{x+2}$.

$f(x)$ is a real number if $x + 2 \geq 0$

$x \geq -2$

$\therefore$ the domain is $\{x \mid x \geq -2\}$.

Alternatively one can write $x \in [-2, \infty)$.

[ means number is included, in this case $-2$.

( " " excluded, " " $\infty$.

Examples

$x \in [0, 54]$ is identical to $0 \leq x \leq 54$

$y \in (-\sqrt{2}, 1]$ " " $-\sqrt{2} < y \leq 1$

$z \in [\frac{\pi}{4}, \frac{\pi}{4}]$ " " $z = \frac{\pi}{4}$

Obviously the leftmost number < rightmost number for result to be meaningful.

$y \in [1, 0] \Rightarrow$ no such $y$!

2. Find the largest possible domain of $h(x) = \sqrt{1-x^2}$.

$h(x)$ is real when $1 - x^2 \geq 0$

$(1-x)(1+x) \geq 0$

This holds if 1) $1 - x \geq 0$ and $1 + x \geq 0$

or 2) $1 - x \leq 0$ and $1 + x \leq 0$.

Case 1: $1 - x \geq 0$ when $x \leq 1$
$1 + x \geq 0$ when $x \geq -1$
$\Big\}$ $\therefore$ both conditions hold when $-1 \leq x \leq 1$ or $x \in [-1, 1]$

Case 2: $1 - x \leq 0$ when $x \geq 1$
$1 + x \leq 0$ when $x \leq -1$
$\Big\}$ Both these conditions never hold at the same time

$\therefore$ largest possible domain is $x \in [-1, 1]$.

Laws of Inequalities

Let $a, b, c \in R$.

1) if $a < b$ then $a + c < b + c$

2) if $a < b$ then: $ac < bc$ if $c > 0$
$ac > bc$ if $c < 0$.

## Examples.

1) **Solve** $x^2 - 4x + 2 \leq -1$.

$x^2 - 4x + 3 \leq 0$.

$(x - 3)(x - 1) \leq 0$.

This holds if either 1) $x - 3 \leq 0$ and $x - 1 \geq 0$

$x \leq 3$ $\qquad$ $x \geq 1$

$\therefore$ $1 \leq x \leq 3$

or $x \in [1, 3]$

or 2) $x - 3 \geq 0$ and $x - 1 \leq 0$.

$x \geq 3$ $\qquad$ $x \leq 1$

$\therefore$ no such x exist.

$\therefore$ solution $x \in [1, 3]$.

2) **Solve** $\frac{1}{x} > \frac{1}{2}$.

First of all $x = 0$ isn't allowed.

**Case 1:** $x > 0$. $\qquad \frac{1}{x} > \frac{1}{2}$

$$x \cdot \frac{1}{x} > x \cdot \frac{1}{2}$$

$$1 > \frac{x}{2}$$

$$x < 2.$$

$\therefore$ $0 < x < 2$ or $x \in (0, 2)$.

Case 2: $x < 0$ $\qquad \frac{1}{x} > \frac{1}{2}$

$$x \cdot \frac{1}{x} < x \cdot \frac{1}{2}$$

$$1 < \frac{x}{2}$$

$$x > 2$$

But $x < 0 \Rightarrow$ no such $x$.

$\therefore$ solution is $x \in (0, 2)$.

3) Solve $\frac{x^2 - 4}{x - 3} < 0$. Obviously $x \neq 3$.

This holds if either

1) $x^2 - 4 < 0$ and $x - 3 > 0$

$x \in [-2, 2)$ $\qquad x > 3$.

$-2 < x < 2$

$\therefore$ this case never holds.

or 2) $x^2 - 4 > 0$ and $x - 3 < 0$

$x > 2$ or $x < -2$ $\qquad x < 3$.

$\therefore$ $x < -2$ or $2 < x < 3$.

$\therefore$ solution is $x < -2$ and $2 < x < 3$.

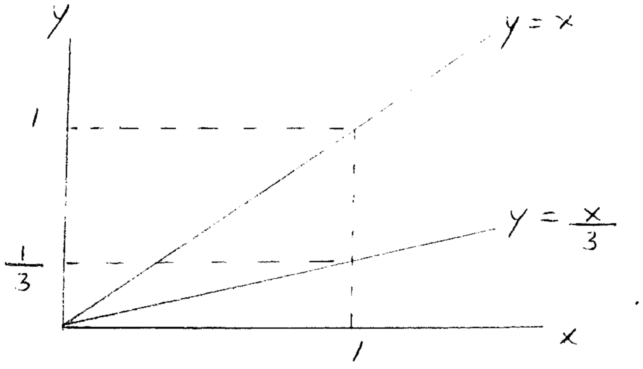
Check: $x = -3$ $\quad \frac{x^2 - 4}{x - 3} = + \frac{9 - 4}{-6} = \frac{-5}{6} < 0$. $\qquad x = 2.5$ $\quad \frac{x^2 - 4}{x - 3} < 0$

$x = 0$ $\quad \frac{x^2 - 4}{x - 3} = \frac{4}{3} > 0$ $\qquad x = 4$ $\quad \frac{x^2 - 4}{x - 3} > 0$

# Linear Functions

Section 3

Consider the functions $y = x$ and $y = \frac{x}{3}$.



The line $y = x$ rises or slopes upwards three times as fast as the line $y = \frac{x}{3}$.
We can represent these lines in the form

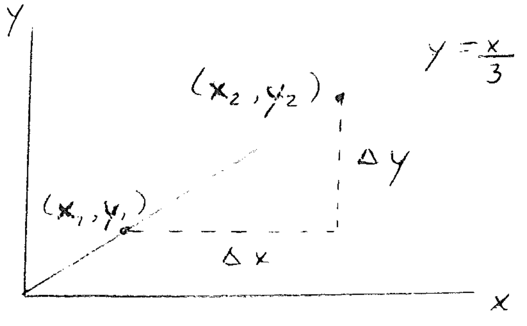
$$y = mx$$

where m is called the <u>slope</u>

$$\boxed{\text{slope } m \equiv \frac{\text{change of } y}{\text{corresponding change of } x}}$$

<u>Example</u>

Find slope of line $y = \frac{x}{3}$.



Consider any 2 points $(x_1, y_1)$ and $(x_2, y_2)$ on the line.

$$\text{slope } m = \frac{\text{change of } y \text{ when going from } (x_1,y_1) \to (x_2,y_2)}{\text{change of } x \text{ when going from } (x_1,y_1) \to (x_2,y_2)}$$

$$= \frac{\Delta y}{\Delta x}$$

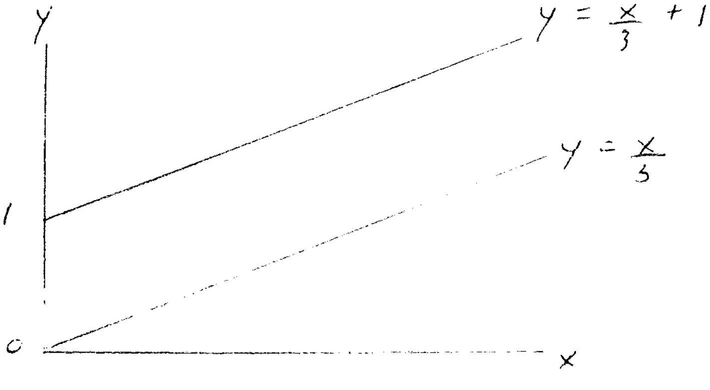
$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

For the line $y = \frac{x}{3}$, $y_1 = \frac{x_1}{3}$, $y_2 = \frac{x_2}{3}$.

$$\therefore \quad m = \frac{(x_2 - x_1)/3}{x_2 - x_1} = \frac{1}{3}.$$

i.e. line $y = \frac{x}{3}$ has slope $\frac{1}{3}$.

Next consider the functions $y = \frac{x}{3}$ and $y = \frac{x}{3} + 1$.



The line $y = \frac{x}{3} + 1$ intersects the $y$ axis at 1. We therefore call 1 the $y$ intercept.

A line has the algebraic form $y = mx + b$ where

1) m is the slope
2) b is the y intercept.

Examples

1. Find the equation of the line passing through points (2, 3) and (7, 10). Also find the slope and the x + y intercepts.

Solution

Equation of a line is $y = mx + b$.

$(2, 3) \Longrightarrow \quad 3 = 2m + b \qquad (1)$

$(7, 10) \Longrightarrow \quad 10 = 7m + b \qquad (2)$

Eqn. (2) - Eqn (1) $\Longrightarrow \quad 5m = 7$

$\therefore m = \frac{7}{5}$ is the slope.

Subst. m into eqn. (1) $\Longrightarrow \quad 3 = 2 \cdot \frac{7}{5} + b.$

$\therefore b = \frac{1}{5}$ is the y intercept.

Equ. of line is $y = \frac{7}{5}x + \frac{1}{5}$.

x- intercept occurs when line hits x axis i.e. $y = 0$.

$$\therefore \quad 0 = \frac{7}{5}x + \frac{1}{5}$$

$$7x = -1$$

$$\therefore \quad x = \frac{-1}{7} \text{ is x-intercept}$$

2. Find the linear relation between the Fahrenheit and Celsius temperature scales.

<u>Solution</u>

Let $F$ be temperature measured on Fahrenheit scale.
" $C$ " " Celsius " .

Freezing Point $F = 32, \; C = 0$
Boiling Point $F = 212 \quad C = 100$.

$$F = mC + b.$$

$(32, 0) \Longrightarrow 32 = b$ (1)
$(212, 100) \Longrightarrow 212 = 100m + b$ (2)

Subst. (1) into (2) $\Longrightarrow 212 = 100m + 32$

$$180 = 100m$$

$$m = \frac{9}{5}$$

$$\therefore \boxed{F = \frac{9}{5}C + 32}$$

## Polynomial and Rational Functions

Section 4

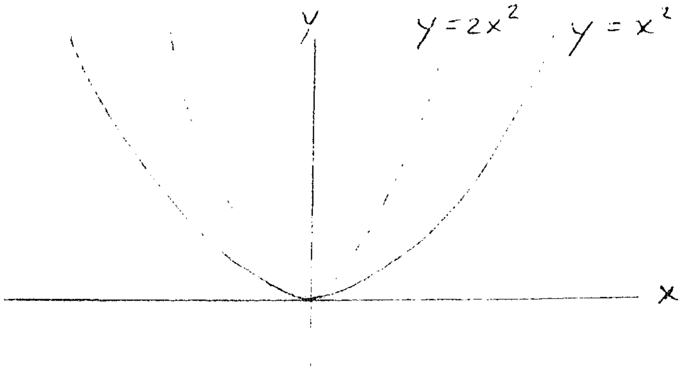
$y = a_2 X^2 + a_1 X + a_3$

Parabolas are functions having the form ~~$y = a(x-b)^2$~~ $+ c$ where $a, b, c \in R$.

How to plot simple parabolas
A simple parabola has the form $y = x^2$

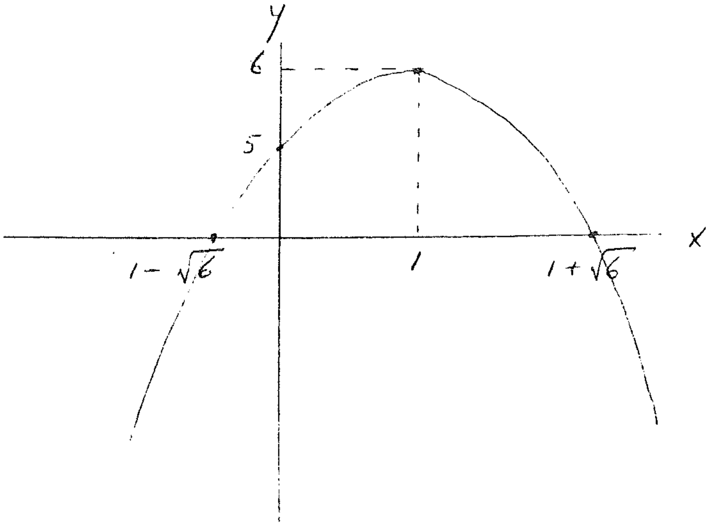
Examples

1) Plot $y = x^2$, $y = 2x^2$.



2) Plot

$$\begin{aligned} y &= -x^2 + 2x + 5 \\ &= -(x^2 - 2x) + 5 \\ &= -\left[ (x-1)^2 - 1 \right] + 5 \\ &= -(x-1)^2 + 6 \end{aligned}$$



y intercept ($x=0$) is $y=5$

x intercept ($y=0$) is

$$\begin{aligned} -(x-1)^2 + 6 &= 0 \\ (x-1)^2 &= 6 \\ x-1 &= \pm\sqrt{6} \\ x &= 1 \pm \sqrt{6}. \end{aligned}$$

Vertex is centre of curved end. Also by
~~The highest~~ or lowest point of the ~~parabola is called~~ the vertex. In the preceding case the vertex is the point V=(1,6
looking at simple parabola, $y=x^2$, the vertex is when $x^2$ term is zero,

3) Find the vertex of $y = a_2 x^2 + a_1 x + a_0$

$$y = a_2\left(x^2 + \frac{a_1 x}{a_2}\right) + a_0$$

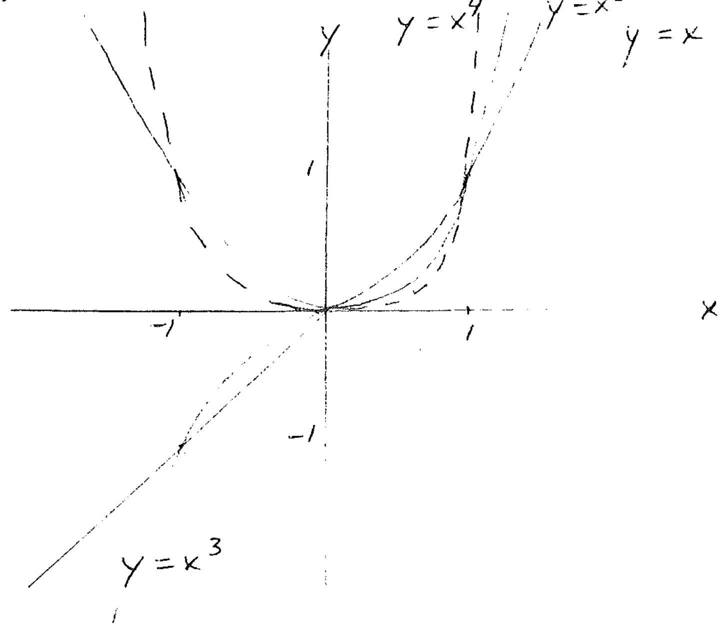
$$= a_2\left(x + \frac{a_1}{2a_2}\right)^2 - a_2\left(\frac{a_1}{2a_2}\right)^2 + a_0$$

$\therefore$ vertex occurs at point $x = \frac{-a_1}{2a_2}$.

When $y = -x^2 + 2x + 5$, $a_2 = -1$, $a_1 = 2$ $\Rightarrow$ vertex at $\frac{-2}{-2} = 1$, as was found.

## Higher Order Polynomials

A polynomial function of degree $n$ $(n>0)$ has the form
$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$.



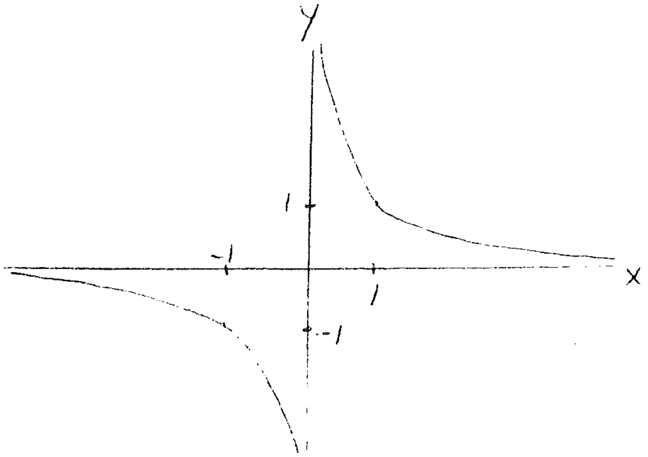
## Rational Functions

A rational function $f(x) = \frac{P_1(x)}{P_2(x)}$ where $P_1$ + $P_2$ are two polynomial functions.

### Examples

1) Plot $y = \frac{1}{x}$.

Domain $\{x \mid x \in R, x \neq 0\}$



As $|x|$ gets very large, y approaches 0.
The line $y = 0$ is called a horizontal asymptote.

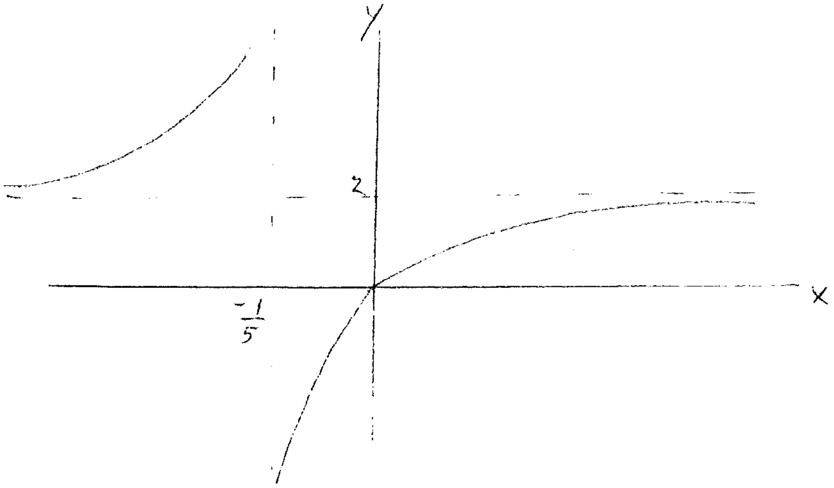
Similarly the line $x = 0$ is a vertical asymptote.

An asymptote is a line which the function ~~gets~~ ~~infinitely close~~ to but never touches.
approaches

2) Plot $y = \dfrac{10x}{1+5x}$ as $x \to \infty$, $y = \dfrac{10}{\frac{1}{x}+5}$

$= 2$

Domain $\{x \mid x \in R,\ x \neq -\frac{1}{5}\}$



When x gets very large + positive y gets close to 2.
When x gets close to $-\frac{1}{5}$ from the right, y gets large + negative

When x gets close to $-\frac{1}{5}$ from the left, y gets large + positive
When x gets large + negative, y gets close to 2.

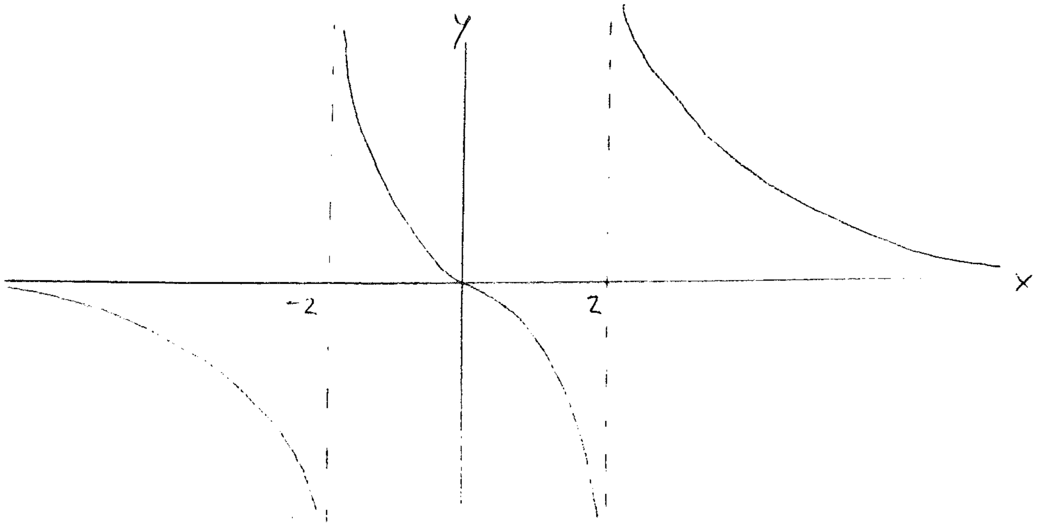
$\therefore$ $x = -\frac{1}{5}$ is a vertical asymptote.

+ $y = 2$ " "horizontal" " .

(Optional)

3) Plot $y = \frac{2x}{x^2-4}$.

Domain $\{x \mid x \in R, x \neq \pm 2\}$.



When x gets very large + positive, y gets small + positive.
When x gets close to 2 from the right, y gets big + positive

When x gets close to 2 from left, y gets big + negative.
When " " -2 " right " " positive.

When x " " -2 " left " " negative.
When x gets large + negative, y gets small + negative.

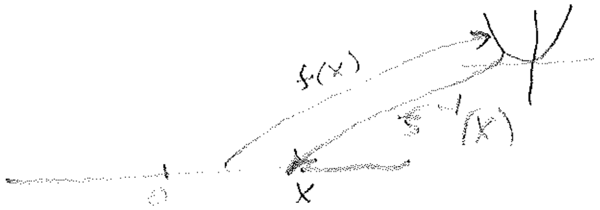
$x = \pm 2$ are vertical asymptotes.
$y = 0$ is horizontal " ".

## Inverse Functions

Section 5

We wish to find a function that undoes what $f(x)$ has done. This so called inverse function is denoted by $f^{-1}(x)$.

( Note $f^{-1}(x) \neq \frac{1}{f(x)}$ !)



Examples

1. $f(x) = 2x$

Obviously $f^{-1}(x) = \frac{x}{2}$

$x = 3 \Longrightarrow f(3) = 6.$

$f^{-1}(f(3)) = f^{-1}(6) = 3.$

In general $f^{-1}(f(x)) = x$. In this case we have:

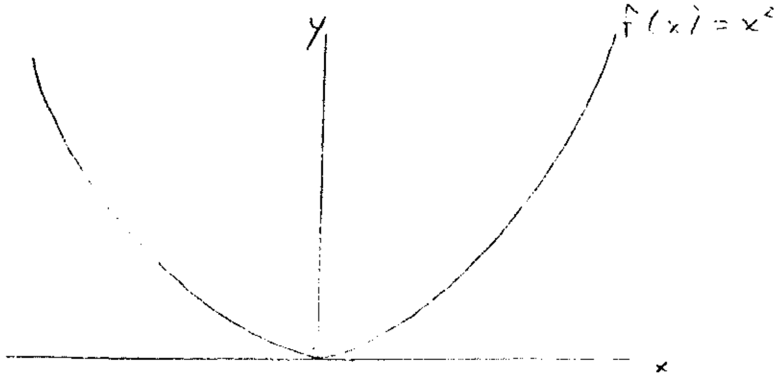
$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(2x) \\ &= \frac{2x}{2} \\ &= x. \end{aligned}$$

2. $f(x) = x^2$

inverse Function $f^{-1}(x) = \sqrt{x}$ ??

$x = -3$ $f(-3) = 9$ But $f^{-1}(9) = \sqrt{9} = 3 \overset{\text{and}}{\cancel{\neq}} -3$.

Trouble The problem is that 2 x-values have the same $f(x)$ value.



$f(x) = f(-x)$ Hence we cannot define an inverse functio
But could if take 1/2 of parabola
eg $f(x) = x^2$ $f^{-1}(x) = \sqrt{x}$
for $x \geq 0$

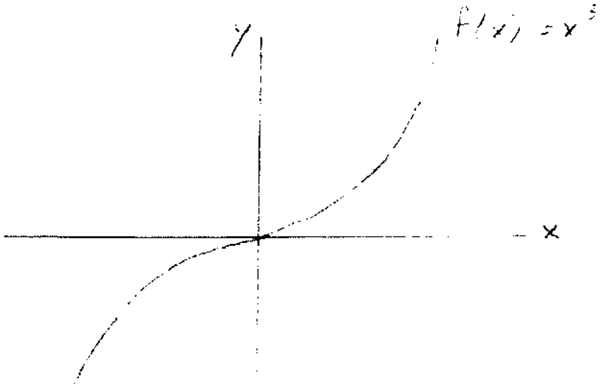
## One-to-One Functions

A function $f(x)$ is said to be one-to-one if no two inputs of the domain have the same output.

i.e. $\forall\ x_1, x_2 \in$ domain $f(x_1) \neq f(x_2)$.

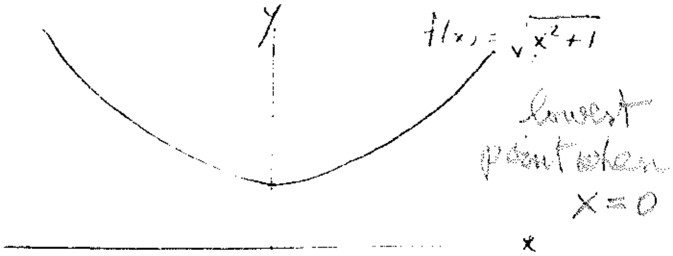
Examples

1) $f(x) = x^3$



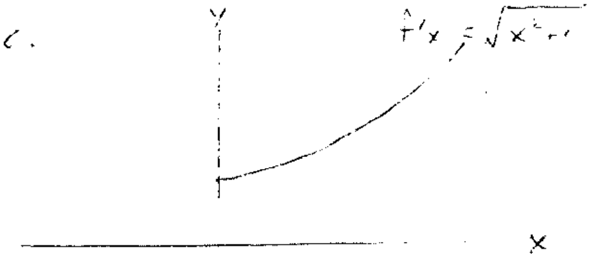
Since no 2 x values have the same $f(x)$, $x^3$ is a 1-1 function.

2) $f(x) = \sqrt{x^2+1}$



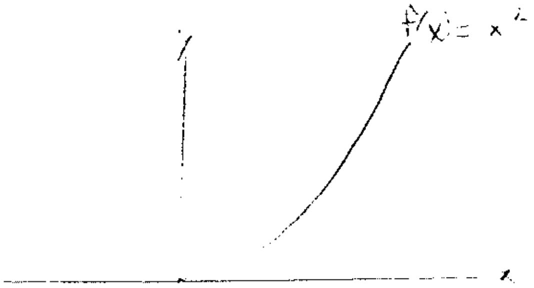
since $f(x) = f(-x)$,
$f(x)$ isn't 1-1.

3) $f(x) = \sqrt{x^2+1} \qquad x \geq 0.$



All values of the domain $x \geq 0$, have different outputs $f(x)$. $\therefore$ $f(x)$ (on the specified domain) is 1-1

4) $f(x) = x^2 \qquad x \geq 0$



As in the previous case
$f(x)$ is 1-1.

but define one-to-one
P17

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## Recipe For Finding Inverse Function
## of a 1-1 Function $y = f(x)$

Recall $f^{-1}$ means the argument in $f(x)$, ~~so~~ $f^{-1}(f(x)) = x$

1. Interchange $x$ and $y$ in the formula $y = f(x)$ to obtain $x = f(y)$.

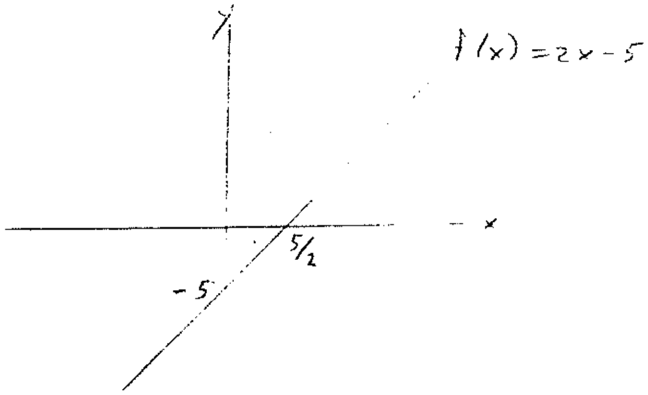
   if write $x = f(y)$ & take $f^{-1}$ each side
   $f^{-1}(x) = f^{-1}(f(y)) = y$

2. Solve $x = f(y)$ for $y$ in terms of $x$. The resulting expression is $y = f^{-1}(x)$.

Examples

1. $y = f(x) = 2x - 5$

   $f(x)$ is 1-1.



Interchange $x$ & $y$ gives:

$$x = 2y - 5$$
$$x + 5 = 2y$$
$$y = \frac{x+5}{2}$$

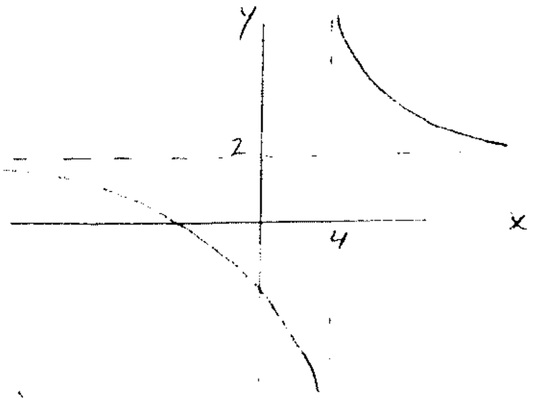
$$\therefore \quad f^{-1}(x) = \frac{x+5}{2}$$

Check: $f^{-1}(f(x)) = f^{-1}(2x-5)$

$$= \frac{(2x-5)+5}{2}$$

$$= x.$$

2. $y = f(x) = \frac{2x+3}{x-4}$



$f(x)$ is 1-1

Interchanging x & y gives:

$$x = \frac{2y+3}{y-4}$$

$$x(y-4) = 2y+3$$

$$xy - 4x = 2y+3$$

$$xy - 2y = 4x+3$$

$$y(x-2) = 4x+3$$

$$y = \frac{4x+3}{x-2}$$

$$\therefore \; f^{-1}(x) = \frac{4x+3}{x-2} \,.$$

$$\frac{4\,\frac{2x+3}{x-4} + 3}{\frac{2x+3}{x-4} - 2}$$

$$= 4(2x+3) + 3(x-4)$$

$$2x+3 - 2x + 8$$

$$= \frac{11x}{11}$$

$$= x$$

## Laws of Exponents

Sections 5, 15
(2 lectures)

$x, y, k_1, k_2 \in R$

1. $x^{k_1} x^{k_2} = x^{k_1 + k_2}$

2. $(x^{k_1})^{k_2} = x^{k_1 k_2}$

3. $(xy)^k = x^k y^k$

4. $\left(\frac{x}{y}\right)^k = \frac{x^k}{y^k}$

### Examples

$\sqrt[3]{-8} = (-8)^{1/3} = -2$

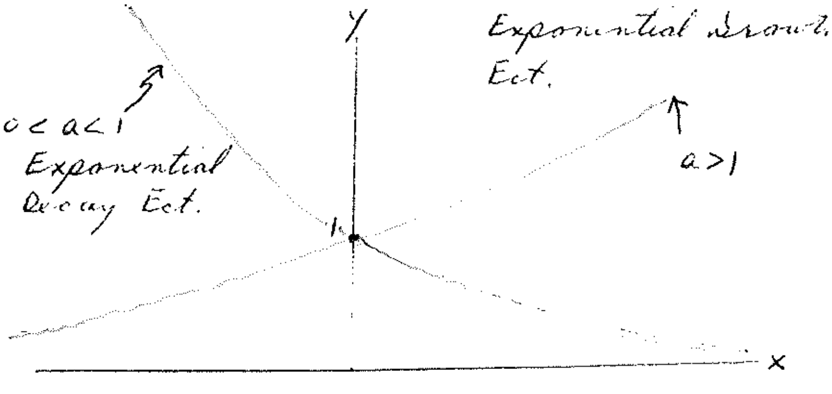
$16^{-3/4} = (16^{1/4})^{-3} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

$(-32)^{2/5} = \left((-32)^{1/5}\right)^2 = (-2)^2 = 4.$

### Exponential Function

$y = a^x \quad a > 0.$

$y(0) = a^0 = 1. \quad \forall a.$



Case 1: $a > 1$.

As $x$ gets very large + positive so does $y$.

As $x$ " " negative, $y$ nears zero.

Case 2: $a < 1$

As $x$ gets very large + positive, $y$ nears zero.

As $x$ gets " " negative, $y$ gets very large + pos

## Logarithms

The inverse function of $y = a^x$ is defined to be the logarithm.

If $\boxed{y = a^x, \text{ then } \log_a y \equiv x. \quad a > 0}$

i.e. if $f(x) = a^x$ then $f^{-1}(x) = \log_a x$.

$$
\begin{aligned}
f^{-1}(f(x)) &= f^{-1}(a^x) \\
&= \log_a a^x \\
&= x.
\end{aligned}
$$

## Examples

$\log_{10} 100 = \log_{10} 10^2 = 2$

$\log_3 \sqrt{3} = \log_3 3^{1/2} = \frac{1}{2}$

$\log_{\sqrt{2}} 8 = \log_{\sqrt{2}} (\sqrt{2})^6 = 6$

← Feb 13/90

$\log_{10} .0001 = \log_{10} 10^{-4} = -4.$

If the base is omitted, assume it to be 10. i.e. $\log x \equiv \log_{10} x$

## Properties

1) $\log_a (xy) = \log_a x + \log_a y$

2) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

3) $\log_a x^k = k \log_a x$

4) $\log_a a = 1$ and $\log_a 1 = 0.$

5) $\log_a x = \dfrac{\log_b x}{\log_b a}.$

## Proof: of above properties

1) By definition of logarithm $\log_a y = x \Rightarrow y = a^x$

$X = a^{\log_a X}$

~~Let $w = \log_a x. \Rightarrow \log_a a^{\log_a x} = \log_a x.$~~

~~$\therefore x = a^{\log_a x}$~~

Similarly $X = a^{\log_a X}$

Know $x = a^{\log_a x}$, $y = a^{\log_a y}$ so

$$xy = (a^{\log_a x}).(a^{\log_a y})$$

$$= a^{(\log_a x + \log_a y)}$$ using law of exponents

since $\log_a a^z = z$

$\therefore \log_a xy = \log_a x + \log_a y$ by definition of logarithm

2) ~~Homework~~

3) $x = a^{\log_a x}$

$x^k = a^{k \log_a x}$ from exponent law $(X^{k_1})^{k_2} = X^{k_1 k_2}$

$\therefore \log_a x^k = k \log_a x$

4) $\log_a a = \log_a a^1 = 1.$

$\log_a 1 = \log_a a^0 = 0.$

5) $x = a^{\log_a x}$

Similarly $a = b^{\log_b a}$

$$\therefore x = a^{\log_a x}$$

$$= (b^{\log_b a})^{\log_a x}$$

$$= b^{(\log_b a) \cdot (\log_a x)}$$

$\therefore \log_b x = \log_b a \cdot \log_a x$

## Notation

$e \equiv 2.718 \ldots$

$\log_e x \equiv \ln x$. but in many texts $\log_e x = \log x$

Next month we shall see the importance of this funny numb

## Examples.

1) $4\,(3^{2x}) = 8.$

$3^{2x} = 2.$

$\log_3 2 = 2x$

$x = \frac{1}{2} \log_3 2.$

But $\log_3 2 = \dfrac{\log_{10} 2}{\log_{10} 3}$

$= \dfrac{.3011}{.4771}$

$= .6309.$

$\therefore x = .3155$

2) $5 - 2^{3x} = 3$

$-2^{3x} = -2.$

$2^{3x} = 2$

$2^{3x-1} = 1$

$\log_2 1 = 3x - 1.$

$\log_2 \mathbf{2}^0 = 3x - 1$

$0 = 3x - 1$

$x = \frac{1}{3}$

3) $3^{x^2 - x} = 9$

$\log_3 9 = x^2 - x.$

$2 = x^2 - x$

$x^2 - x - 2 = 0.$

$(x-2)(x+1) = 0$

$x = -1, 2.$

4) $\log_8 (x+4) = 2.$

$8^2 = x+4.$

$64 = x+4$

$x = 60.$

5) $\log_a (xy^{-2}) = \log_a x + \log_a (y^{-2})$

$= \log_a x - 2\log_a y.$

6) $\log_a (x^5 y^2) = \log_a x^5 + \log_a y^2$

$= 5\log_a x + 2\log_a y.$

## Richter Scale

The strength of earthquakes is measured by a seismograph that measures the amplitude A of the ground's movement. Since A varies enormously from quake to quake, it is useful to employ logarithms.

Richter Scale $R = \log_{10}\left(\frac{A}{A_0}\right)$ or $\frac{A}{A_0} = 10^R$

$A_0 = \overset{\text{standard}}{\text{amplitude}}$ of a quake that rattles dishes

Suppose quake 1 has $R_1 = 7$
" 2 " $R_2 = 6$

Then $\frac{A_1}{A_2} = \frac{10^{R_1}}{10^{R_2}} = 10^{R_1 - R_2} = 10^{7-6} = 10.$

$\therefore$ earthquake 1 is 10 times stronger than earthquake 2.

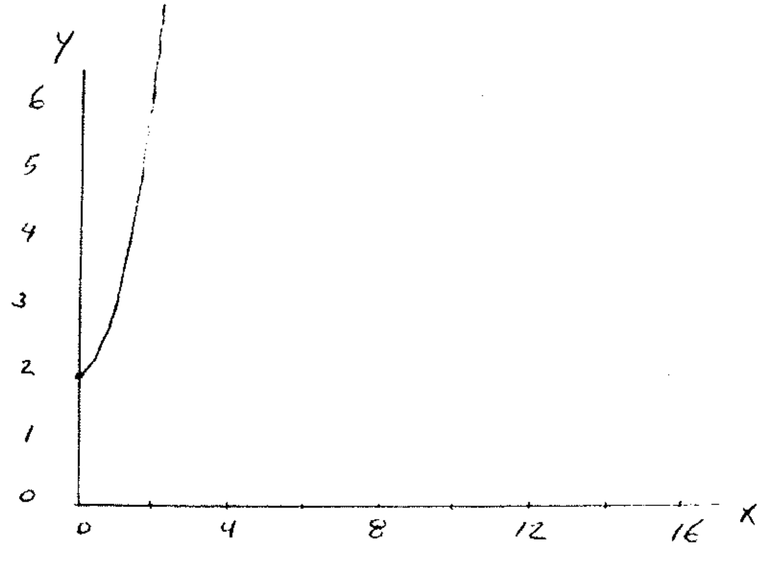
Largest recorded quake 1935 Japan $R = 8.9$
1971 Los Angeles $R = 6.7$

# Plotting on Semilog Paper

begin
lecture

Plot the function $y = 2 \cdot 10^{x/4}$.

| $x$ | $y$ | $\log_{10} y$ |
|---|---|---|
| 0 | 2 | .30 |
| 2 | 6.3 | .80 |
| 4 | 20 | 1.30 |
| 6 | 63.3 | 1.80 |
| 8 | 200 | 2.30 |
| 10 | 632.5 | 2.80 |
| 12 | 2000 | 3.30 |
| 14 | 6325 | 3.80 |



Not very interesting!

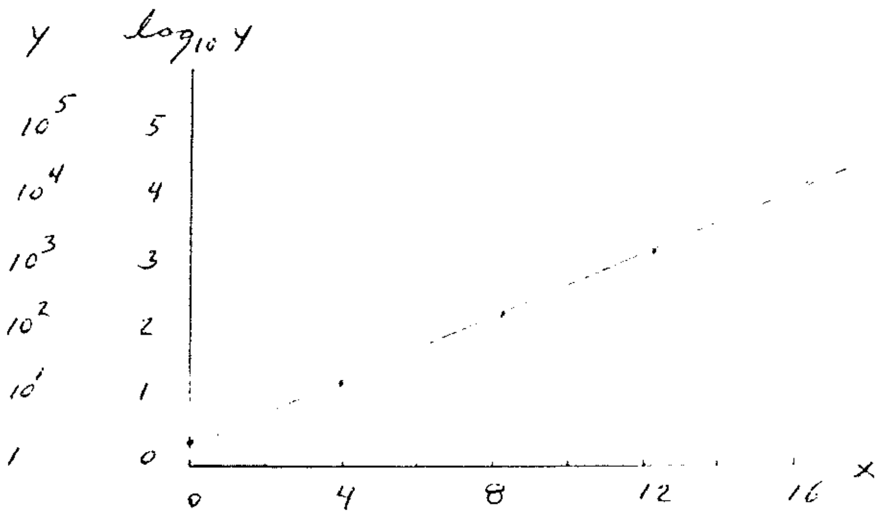
Suppose we plot $\log_{10} y$ vs. x.

$$\log_{10} y = \log_{10}(2 \cdot 10^{x/4})$$

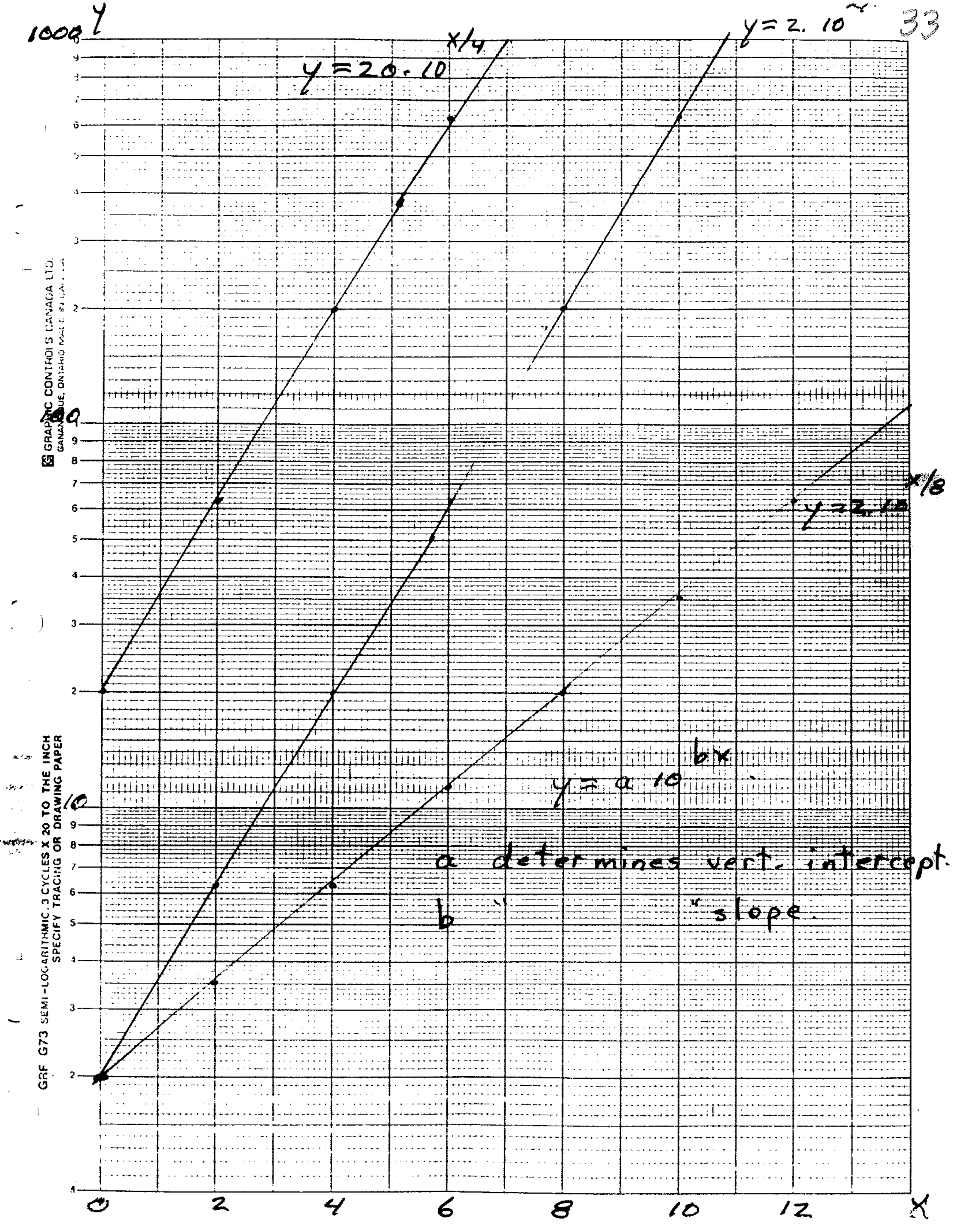
$$= \log_{10} 2 + \log_{10} 10^{x/4}$$

$$= \log_{10} 2 + \frac{x}{4}$$

Result is a line!



| $x$ | $y = 2 \cdot 10^{x/4}$ | $y = 2 \cdot 10^{x/8}$ | $y = 20 \cdot 10^{x/4}$ |
|---|---|---|---|
| 0 | 2 | 2 | 20 |
| 1 | | 2.67 | |
| 2 | 6.3 | 3.56 | 63 |
| 3 | | 4.74 | |
| 4 | 20 | 6.32 | 200 |
| 5 | | 8.43 | |
| 6 | 63.3 | 11.25 | 633 |
| 7 | | 15.00 | |
| 8 | 200 | 20.0 | 2000 |
| 9 | | 26.7 | |
| 10 | 633 | 35.6 | |
| 11 | | 47.4 | |
| 12 | 2000 | 63.2 | |
| 13 | | 84.3 | |
| 14 | 6325 | 112.55 | |

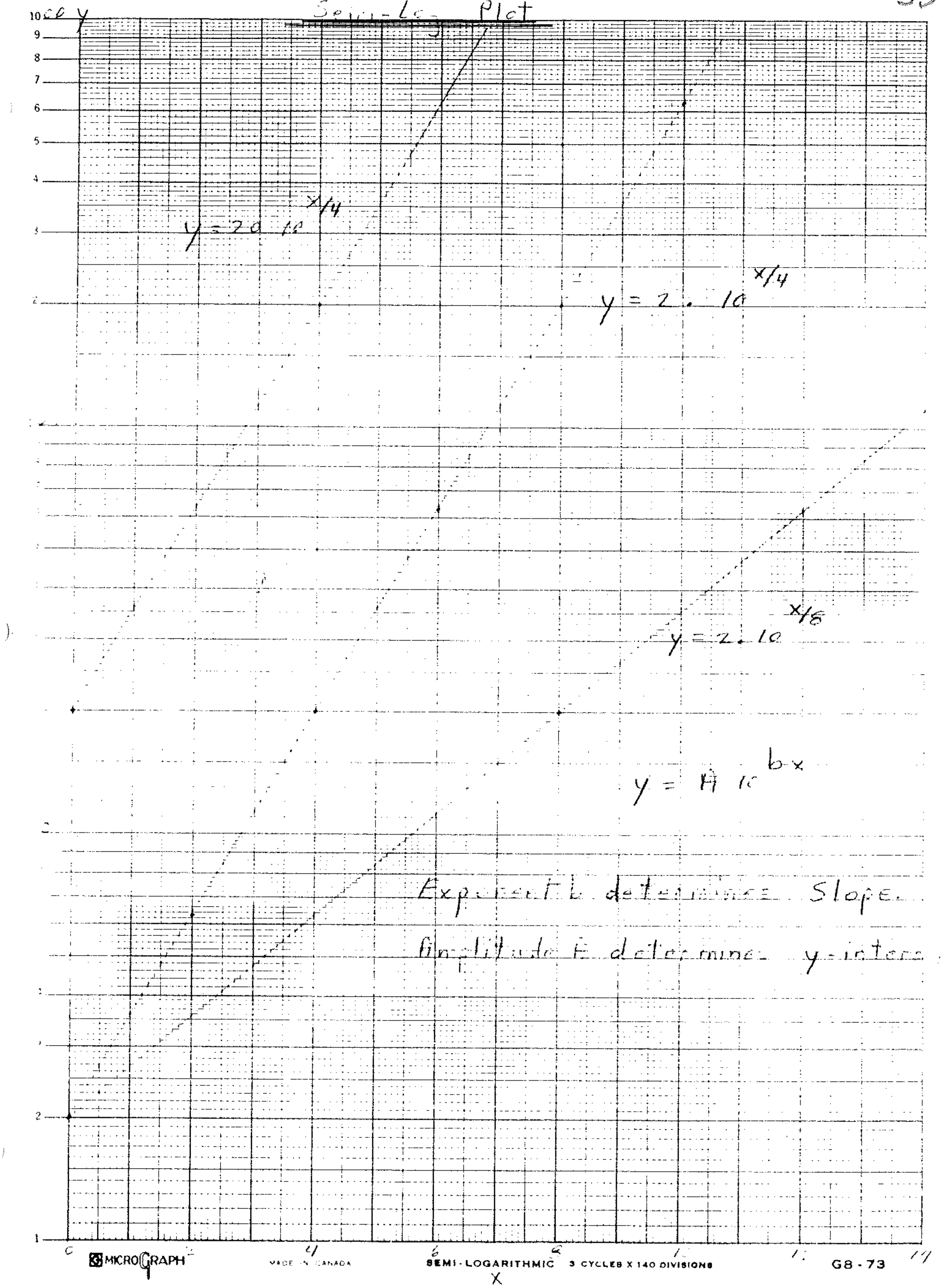


$y = a\, 10^{bx}$

a determines vert. intercept.

b " " slope.

# Semi-Log Plot



Exponent b determines Slope.

Amplitude A determines y-interc

## Trigonometric Functions

Section 2:

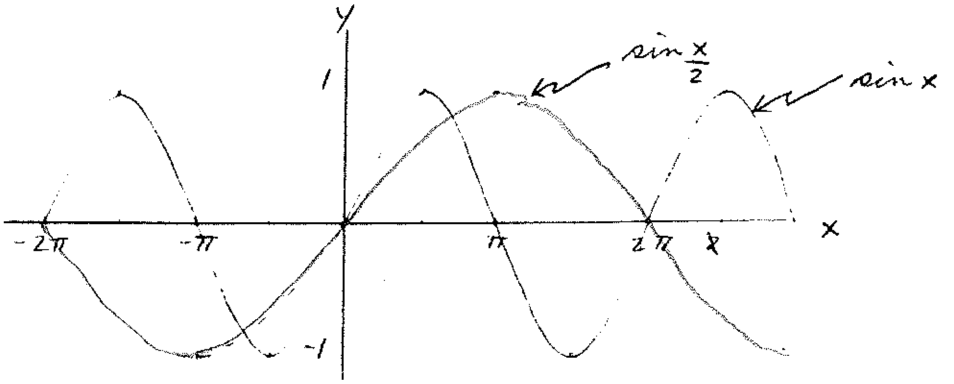
$\sin x$ $\qquad \csc x \equiv \dfrac{1}{\sin x}$

$\cos x$ $\qquad \sec x \equiv \dfrac{1}{\cos x}$

$\tan x \equiv \dfrac{\sin x}{\cos x}$ $\qquad \cot x \equiv \dfrac{\cos x}{\sin x}$

### Sin x

1. Plot $\sin x$ and $\sin \frac{x}{2}$.

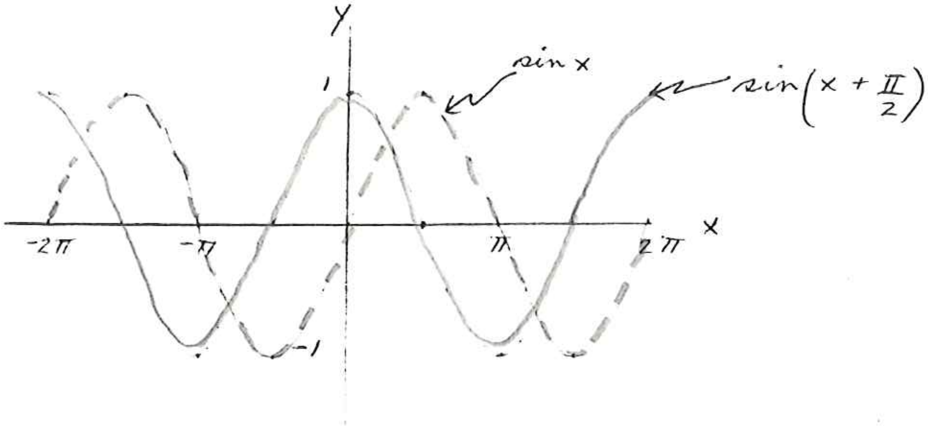


Period of $\sin x$ is $2\pi$.

" " $\sin \frac{x}{2}$ " $2 \times 2\pi$.

" " $\sin bx$ " $\dfrac{2\pi}{b}$.

2. Plot sin x and $\sin\left(x + \frac{\pi}{2}\right)$



$\sin\left(x + \frac{\pi}{2}\right)$ ~~lags behind~~ leads sin x by amount $\frac{\pi}{2}$.

← Feb 19/90

We say $\sin\left(x + \frac{\pi}{2}\right)$ is <u>phase shifted</u> by amount $\frac{\pi}{2}$ with respect to sin x.

## General Form

$y = A \sin(bx + \phi)$

A = amplitude

Period = $\frac{2\pi}{b}$

Phase Shift = $\phi$

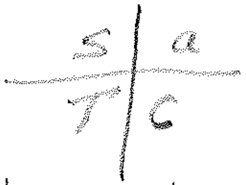
## Algebraic Relations

$$\boxed{\begin{aligned} &\sin(x+y) = \sin x \cos y + \cos x \sin y \\ &\cos(x+y) = \cos x \cos y - \sin x \sin y \\ &\sin^2 x + \cos^2 x = 1 \end{aligned}}$$

MEMORIZE

Examples

also memorize



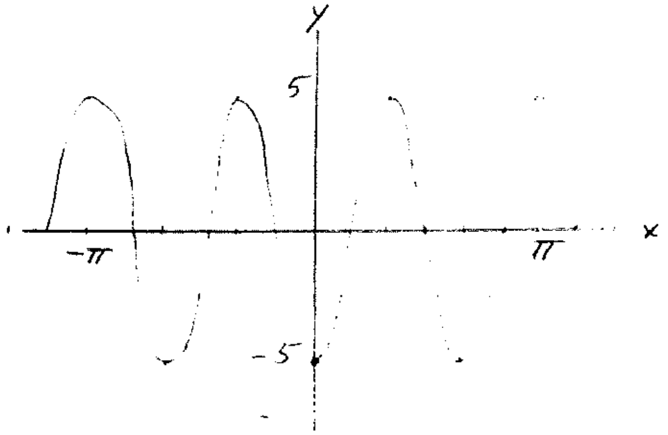
1. $y = 5\cos(3x + \pi)$

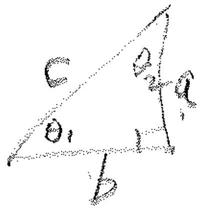
Amplitude $= 5$

Period $= \frac{2\pi}{3}$

& know how to calc. functions for rt. angled △

Phase Shift from $5\cos 3x$ is $\pi$-leading





$c^2 = a^2 + b^2$

$\sin\theta_1 = \frac{a}{c}$

$= \frac{\text{opposite}}{\text{hypotenuse}}$

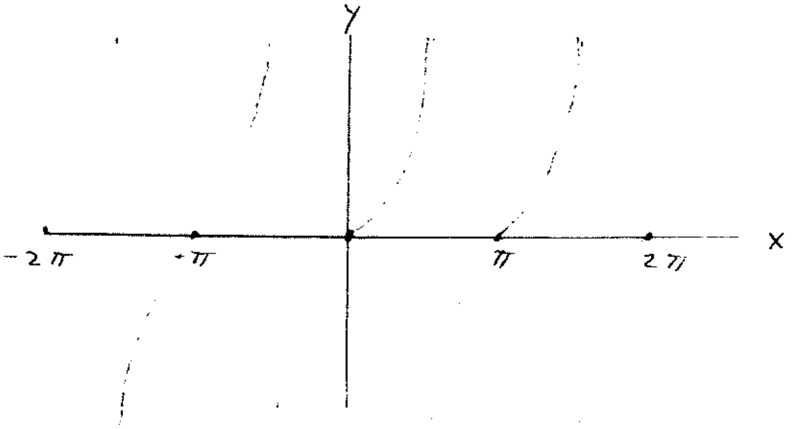
$\cos\theta_1 =$

2. Plot $y = \tan x \equiv \dfrac{\sin x}{\cos x}$

When $x = 0, \pm\pi, \pm 2\pi \cdots \begin{cases} \sin x = 0 \\ \cos x = \pm 1 \end{cases} \Longrightarrow \tan x = 0.$

When $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \cdots \begin{cases} \sin x = \pm 1 \\ \cos x = 0 \end{cases} \Longrightarrow \tan x = \pm\infty.$

When $x = \frac{\pi}{4}, \frac{5\pi}{4} \cdots \quad \tan x = 1.$



$\therefore \; x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \cdots$ are vertical asymptotes.

3. Solve $\sin(2x + \pi) = 1.$

$\sin 2x \cos\pi + \cos 2x \sin\pi = 1.$

$-\sin 2x + 0 = 1.$

$\sin 2x = -1$ so $x = \frac{-\pi}{4}, \frac{-\pi}{4} \pm \pi,$

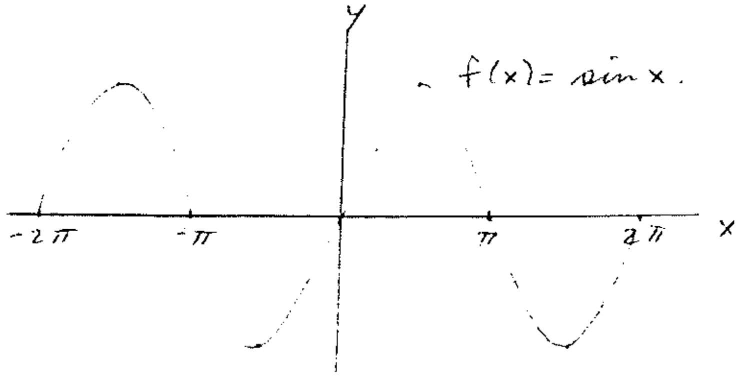
$2x = \frac{-\pi}{2}, -\frac{\pi}{2} \pm 2\pi, \ldots$ $n = 0, \pm 1, \pm 2, \pm 3 \ldots$

$\therefore x = (4n-1)\frac{\pi}{4}$ $\quad = (4n-1)\frac{\pi}{2}$ $\quad n \in \mathbb{Z}$ integers

$= n\pi - \pi/4$

# Inverse Trig Functions

Consider $f(x) = \sin x$. $x \in R$.



$f(x) = \sin x$ $x \in R$ is not 1-1 since $f(x+2\pi) = f(x)$.

But if we restrict the domain to $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $f(x)$ is 1-1 and we can define an inverse function.

The Inverse of $\sin x$ $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is defined to be $\sin^{-1} x$.

Note $\sin^{-1} x \neq \frac{1}{\sin x}$ !!!

Similarly the inverse of $\cos x$ $x \in [0, \pi]$ is $\cos^{-1} x$.
" " $\tan x$ $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is $\tan^{-1} x$.

<u>Examples.</u>

1) $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

2) $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \Rightarrow \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$ ← Bet 20/90

3) $\tan\left(-\frac{\pi}{4}\right) = -1 \Rightarrow \tan^{-1}(-1) = -\frac{\pi}{4}$

4) $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

5) $\tan^{-1}(\infty) = \frac{\pi}{2}$

6) $\sin^{-1}(1) = \frac{\pi}{2}$.

Recall $\Sigma$ angles in triangle = 180°

Cannot use calculators for sin/cos, instead know

## Limits

Section 6

If the value of a function f(x) gets closer and closer to a single real number L as x gets closer and closer to a (from ~~either~~ both the right and left side) we say

$$\lim_{x \to a} f(x) = L$$

Feb 22/90

Examples

1) $\lim_{x \to 2} (x^2 - 3)$

| $x$ | $x^2 - 3$ | $x$ | $x^2 - 3$ |
|---|---|---|---|
| 4 | 13 | 0 | -3 |
| 3 | 6 | 1 | -2 |
| 2.5 | 3.25 | 1.5 | -.75 |
| 2.1 | 1.41 | 1.9 | .61 |
| 2.01 | 1.040 | 1.99 | .960 |
| 2.001 | 1.0040 | 1.999 | .996 |
| ⋮ |  | ⋮ |  |

$$\therefore \lim_{x \to 2} (x^2 - 3) = 1$$

2) $\lim_{x \to 3} \sqrt{4x + 4} = \sqrt{4 . 3 + 4} = \sqrt{16} = 4$

3) $\lim_{x \to \pi} \frac{\cos x}{x} = \frac{\cos \pi}{\pi} = \frac{-1}{\pi}$

4) $\lim_{x \to 0} (2^x - 1) = 2^0 - 1 = 0.$

5) $\lim\limits_{x \to 2} \sqrt[3]{(2x-1)(4x+1)} = \sqrt[3]{(2 \cdot 2 - 1)(4 \cdot 4 + 1)}$ $X = 1.1$

$= \sqrt[3]{3 \times 9}$

$= \sqrt[3]{27}$

$= 3$

6) $\lim\limits_{x \to c} \dfrac{x}{|x|}$

When x approaches 0 from right, $x > 0$, $\dfrac{x}{|x|} = \dfrac{x}{x} = 1$.

" " left, $x < 0$, $\dfrac{x}{|x|} = \dfrac{x}{-x} = -1$.

Two answers disagree!

$\therefore \lim\limits_{x \to 0} \dfrac{x}{|x|}$ doesn't exist.

$\therefore$ be careful when $f(a)$ isn't defined clearly.
i.e. $f(a) = \dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

Insert next page here. ⟶

7) $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1} = \dfrac{0}{0}$ on direct subst.

$$\begin{array}{r|l} & x^2 + x + 1 \\ \cline{2-2} x - 1 & x^3 - 1 \\ & \underline{x^3 - x^2} \\ & x^2 - 1 \\ & \underline{x^2 - x} \\ & x - 1 \\ & \underline{x - 1} \\ & 0 \end{array}$$

$= \lim\limits_{x \to 1} \dfrac{(x-1)(x^2 + x + 1)}{x - 1}$

$= \lim\limits_{x \to 1} (x^2 + x + 1)$

$= 1 + 1 + 1$

$= 3.$

## Procedure to Compute $\lim_{x \to a} f(x)$

1. If $f(a)$ is well defined $\lim_{x \to a} f(x) = f(a)$

2. If $f(a)$ isn't well defined (eg. $f(a) = \frac{0}{0}$ or $\frac{\infty}{\infty}$ ~~denominator is zero~~) then use algebra to simplify $f(x)$ and do step 1.

3. As a last resort one can always substitute values for and estimate $\lim_{x \to a} f(x)$ numerically.

would be $\frac{\infty}{\infty}$ on direct subst.

8) $$\lim_{x \to +\infty} \frac{3x+6}{4x+1} = \lim_{x \to +\infty} \frac{3 + 6/x}{4 + 1/x}.$$

$$= \frac{3}{4}$$

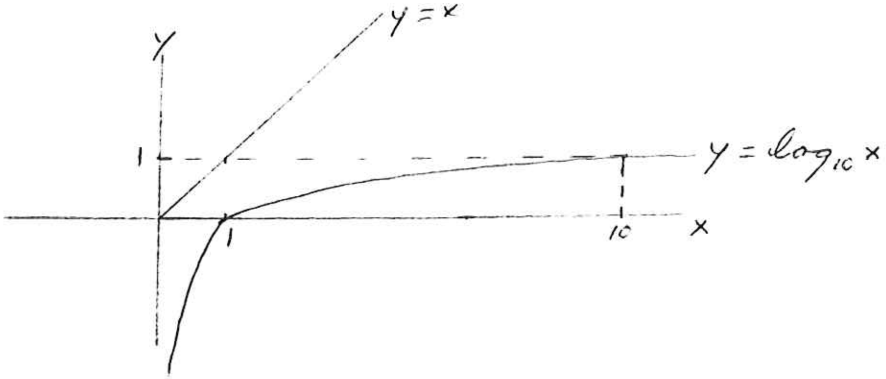
9) $$\lim_{x \to +\infty} \frac{x^2-8}{2x+10000} = \lim_{x \to +\infty} \frac{x - 8/x}{2 + 10000/x}$$

$$= \lim_{x \to +\infty} \frac{x}{2}$$

$$= \infty.$$

← Feb 27/90

10) $$\lim_{x \to +\infty} \frac{\log_{10} x}{x} = 0.$$



$y = x$ gets bigger much faster than $y = \log_{10} x$.

$$\therefore \lim_{x \to +\infty} \frac{\log_{10} x}{x} = 0.$$

$$\log_{10} x = \log_{10} e \times \log_e x$$

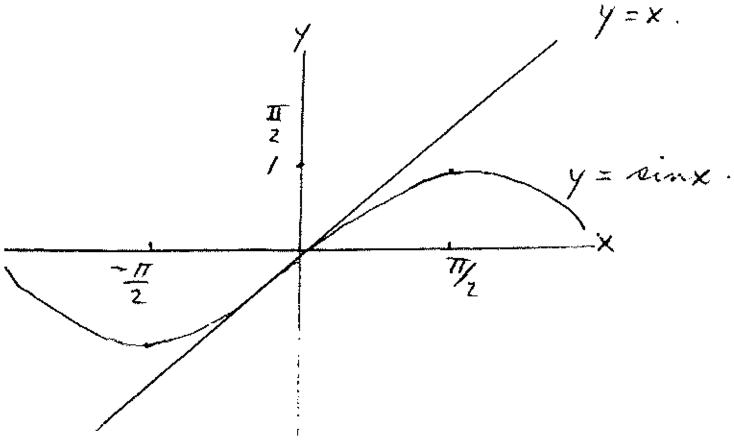
$$\log_e x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \ldots$$

$$= \frac{1-\frac{1}{x}}{1} + \frac{1}{2}\left(\frac{1-\frac{1}{x}}{1}\right)^2 + \frac{1}{3}\left(\frac{1-\frac{1}{x}}{1}\right)^3$$

so $$\frac{\log_{10} x}{x} = k\left[\frac{1-\frac{1}{x}}{x} + \frac{\frac{1}{2}\left(1-\frac{1}{x}\right)^2}{x} + \frac{\frac{1}{3}\left(1-\frac{1}{x}\right)^3}{x} + \ldots\right]$$

$$\rightarrow 0$$

11) $\lim_{x \to 0} \frac{\sin x}{x} = 1$



Case 1: Approach 0 from right hand side.

| x | sin x | $\frac{\sin x}{x}$ |
|---|---|---|
| 1 | .84147 | .84147 |
| .5 | .47943 | .95886 |
| .1 | .099833 | .99833 |
| .01 | .0099998 | .99998 |
| ⋮ | | |

Case 2: Approach 0 from left hand side.

| x | sin x | $\frac{\sin x}{x}$ |
|---|---|---|
| −1 | −.84147 | .84147 |
| −.5 | −.47943 | .95886 |
| −.1 | −.099833 | .99833 |
| −.01 | −.0099998 | .99998 |
| ⋮ | | |

12) $\lim_{x \to c} \frac{\cos x - 1}{x} = 0$ (Homework)

# Origin of $e = 2.718\ldots$

$$\lim_{x \to 0} (1+x)^{1/x}$$

Clearly $(1+x)^{1/x}$ isn't defined at $x=0$. ∴ we shall evaluate numerically.

| $x$ | $(1+x)^{1/x}$ | $x$ | $(1+x)^{1/x}$ |
|---|---|---|---|
| .1 | 2.59374 | −.1 | 2.86797 |
| .01 | 2.70431 | −.01 | 2.73199 |
| .001 | 2.71692 | −.001 | 2.719642 |
| $10^{-4}$ | 2.71814 | $-10^{-4}$ | 2.718417 |
| $10^{-5}$ | 2.71825 | $-10^{-5}$ | 2.71828 |
| ⋮ | | ⋮ | |

∴ $\lim_{x \to 0} (1+x)^{1/x}$ exists.

$$\boxed{\text{We define } e \equiv \lim_{x \to 0} (1+x)^{1/x} = 2.7182\ldots}$$

# Continuous Functions

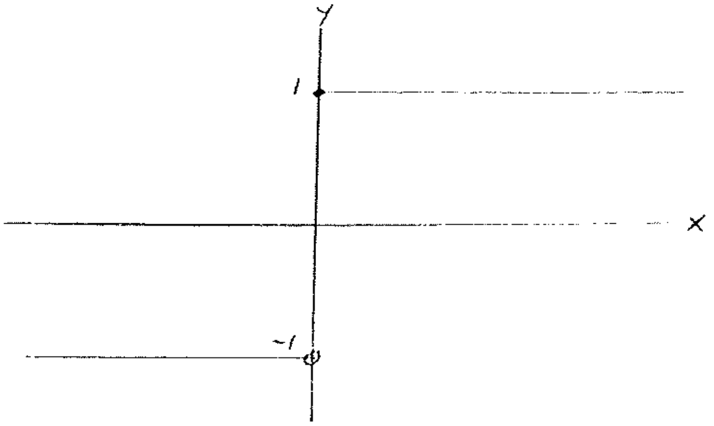
Section 7

A function f(x) is said to be <u>continuous</u> at $x = a$ if the graph of f(x) doesnt $\overset{\text{suddenly jump}}{\text{~~break~~}}$ at the point $(a, f(a))$.

if lim from both sides are equal, finite and [illegible]

Examples

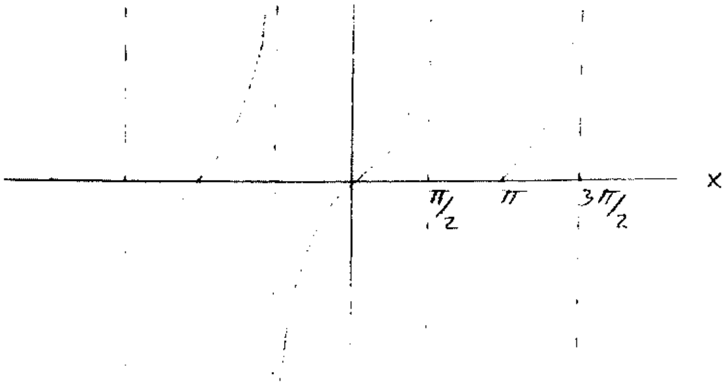
1) $$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$



f(x) is not continuous at $x = 0$.

Recall $\lim\limits_{x \to 0} \frac{|x|}{x}$ doesn't exist.

2) $f(x) = \tan x$



tan x isn't continuous at $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$

Note: $\left(\lim\limits_{x \to \frac{\pi}{2}} \tan x\right)_{x < \frac{\pi}{2}} = +\infty$

$$\left(\lim\limits_{x \to \frac{\pi}{2}} \tan x\right)_{x > \frac{\pi}{2}} = -\infty.$$

$\therefore \lim\limits_{x \to \frac{\pi}{2}} \tan x$ doesn't exist.

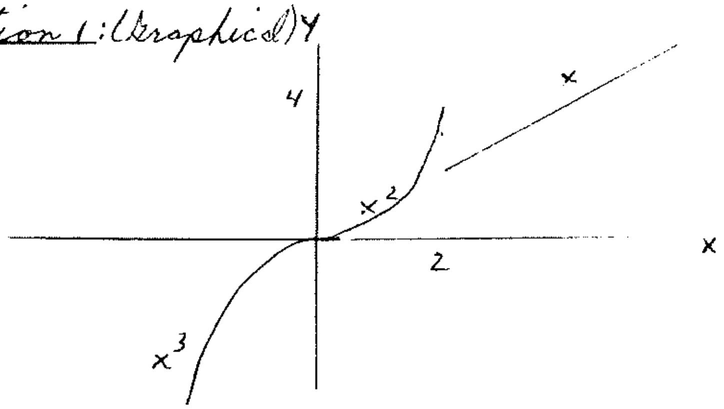
$$\boxed{\therefore\ f(x) \text{ is continuous at } x = a \text{ if and only if } \lim_{x \to a} f(x) = f(a)}$$

Example.

$$f(x) = \begin{cases} x^3 & x < 0 \\ x^2 & 0 \le x \le 2 \\ x & x > 2 \end{cases}$$

Where is f(x) continuous?

Solution 1: (Graphical)



~~$x^3\Big|_{x=0} = x^2\Big|_{x=0} = 0$~~

~~∴ f(x) is continuous at x~~

~~Obviously~~ ∴ f(x) is continuous everywhere but ~~perhaps~~ x = ~~0~~, 2.

**Solution 2** Obviously $f(x)$ is cont. everywhere but perhaps at $x = 0, 2$

$$\left.\begin{aligned}\left(\lim_{x \to 0} f(x)\right)_{x<0} &= \lim_{x \to 0} x^3 = 0 \\ \left(\lim_{x \to 0} f(x)\right)_{x>0} &= \lim_{x \to 0} x^2 = 0\end{aligned}\right\} \Rightarrow \lim_{x \to 0} f(x) = f(0) = 0.$$

$\therefore f(x)$ is continuous at $x =$

$$\left.\begin{aligned}\left(\lim_{x \to 2} f(x)\right)_{x<2} &= \lim_{x \to 2} x^2 = 4 \\ \left(\lim_{x \to 2} f(x)\right)_{x>2} &= \lim_{x \to 2} x = 2\end{aligned}\right\} \begin{aligned}&\lim_{x \to 2} f(x) \text{ doesn't exist.} \\ &\therefore f(x) \text{ is not cont. at } x = 2.\end{aligned}$$

$\therefore f(x)$ is continuous everywhere but at $x = 2$.

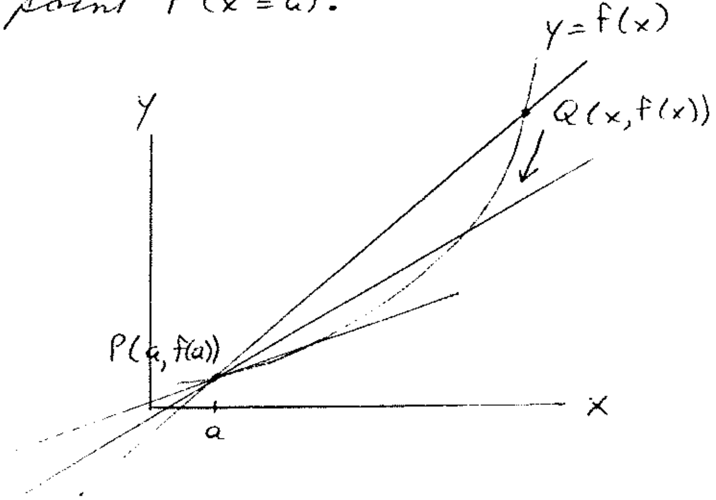
← Mar 1/90

## Derivatives

Section E

### Slope of a Curve

Suppose we wish to find the slope of function $f(x)$ at point $P\ (x = a)$.



$$\text{slope} = \frac{\text{change in } f}{\text{corresponding change in } x}$$

Slope of $f(x)$ at $P$ can be approximated by the slope of line $PQ$. This approximation gets better and better, the closer $Q$ gets to $P$.

$$\therefore \text{slope of } f(x) \text{ at } P = \lim_{Q \to P} \text{slope of line } PQ$$

$$= \lim_{Q \to P} \frac{y_Q - y_P}{x_Q - x_P}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Define $f'(a) \equiv \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

$f'(a)$ is called the <u>derivative</u> of $f(x)$ at $x=a$.

$\therefore$ derivative $f'(a)$ is identical to slope of $f(x)$ at $x=a$.

<u>Examples</u>

1) $f(x) = 5x$.

derivative of $f(x)$ at $x=1$ is:

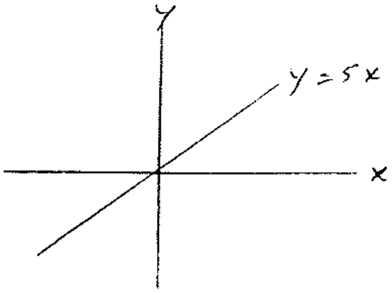
$$f'(1) = \lim_{x \to 1} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x \to 1} \frac{5x-5}{x-1}$$

$$= \lim_{x \to 1} \frac{5(x-1)}{x-1}$$

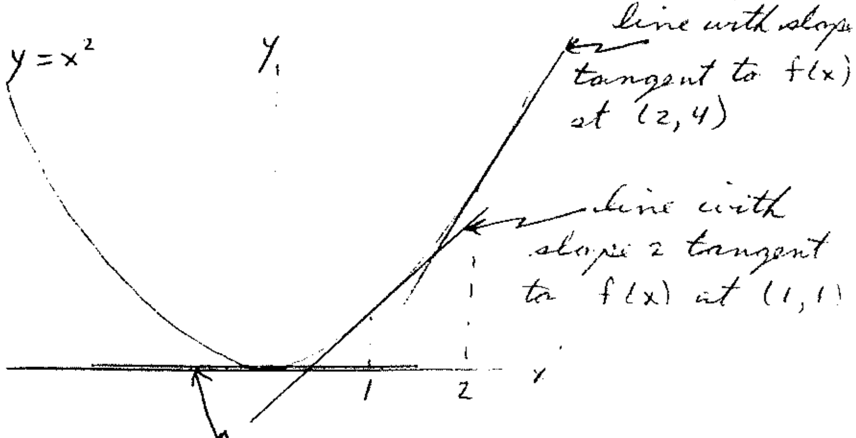
$$= 5$$

$\therefore$ slope of $f(x)=5x$ at $x=1$ is 5.



Indeed slope of $f(x)=5x$ is 5 at all x because it is a line of slope 5.

2) $f(x) = x^2$



line with slope tangent to f(x) at (2,4)

line with slope 2 tangent to f(x) at (1,1)

line with slope 0 tangent to f(x) at (0,0)

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{x^2 - 0^2}{x - 0}$$

$$= \lim_{x \to 0} x$$

$$= 0.$$

∴ slope of $x^2$ at $x = 0$ is 0. This is the slope of the line ~~that touches $f(x) = x^2$ at only one point, i.e. the so-call~~ ~~line~~ tangent to $f(x)$ at $x = 0$.

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x \to 1} x + 1$$

$$= 2$$

∴ slope of $f(x) = x^2$ at $x = 1$ is 2.

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{x^2 - 2^2}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2}$$

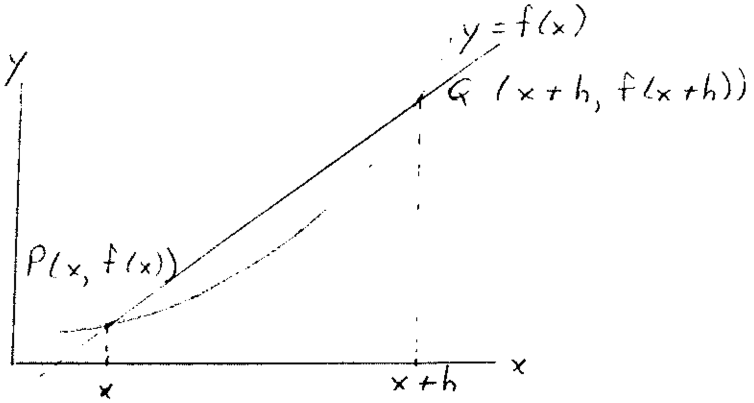
$$= \lim_{x \to 2} x + 2$$

$$= 4$$

$\therefore$ slope of $f(x) = x^2$ at $x = 2$ is 4.

# Derivatives

Suppose we wish to find the slope or derivative of function $f(x)$ at $P\ (x, f(x))$.



slope of $f(x)$ at $P = \lim_{Q \to P}$ slope of line $PQ$
or derivative $f'(x_p)$

$$f'(x_p) = \lim_{Q \to P} \frac{f(Q) - f(P)}{x_Q - x_P}$$

$$= \lim_{x+h \to x} \frac{f(x+h) - f(x)}{x+h-x}$$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

## Examples.

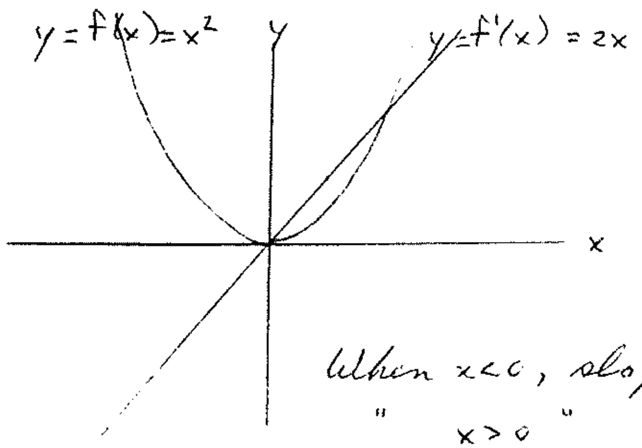
1) $f(x) = x^2$

slope or derivative of $f(x)$ at point $x$ is

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h \to 0} 2x + h \end{aligned}$$

$\therefore f'(x) = 2x$

$$\left.\begin{array}{llll} \text{Slope of } f(x) = x^2 \text{ at} & 0 & \text{is} & f'(0) = 0. \\ \text{''} & 1 & \text{''} & f'(1) = 2 \\ \text{''} & 2 & \text{''} & f'(1) = 4 \end{array}\right\} \text{agreeing with last lecture}$$



When $x<0$, slope of $x^2$, $f'(x)<0$, i e slope is nega
" $x>0$ " " $f'(x)>0$ " " positi

2) $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

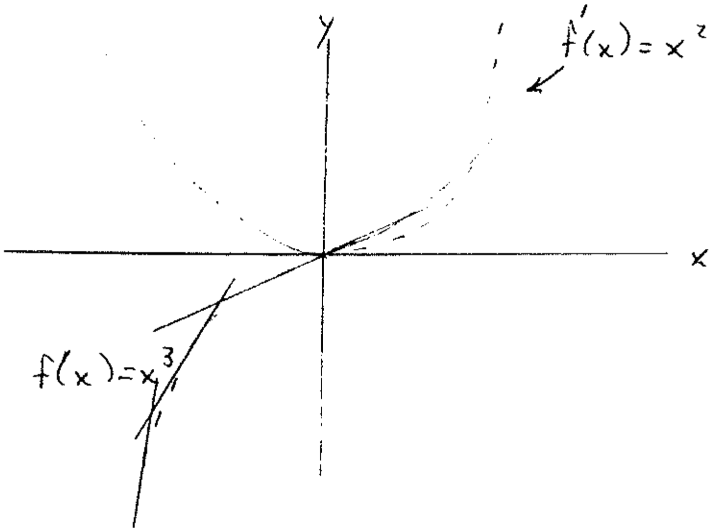
$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2.$$



~~Slope of~~ $f'(x) = x^2 \geq 0$ everywhere.

For negative x, ~~the slope~~ $f'(x)$ decreases as $x \to 0$.

at $x = 0$, ~~the slope~~ $f'(0) = 0$

For positive x, ~~the slope~~ $f'(x)$ increases with increasing x.

3) $f(x) = x^k$ where $k \in R+$ fixed.(i.e. indep. of $x$)

~~$k \neq 0$~~

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^k - x^k}{h}$$

$$= \lim_{h \to 0} \frac{x^k + k x^{k-1} h + \frac{k(k-1)}{2!} x^{k-2} h^2 + \dots + h^k - x^k}{h}$$

using the Binomial Theorem

$$= \lim_{h \to 0} \frac{h\left( k x^{k-1} + \frac{k(k-1)}{2!} x^{k-2} h^{\cancel{1}} + \dots + h^k \right)}{h}$$

$$= \lim_{h \to 0} \left( k x^{k-1} + \frac{k(k-1)}{2!} x^{k-2} h + \dots + h^{k-1} \right)$$

$$= k x^{k-1}$$

$$\boxed{\therefore \text{ for } f(x) = x^k, \ k \in R, \quad f'(x) = k x^{k-1}.}$$

Examples

1) $f(x) = x^{3/4}$ $\qquad f'(x) = \frac{3}{4} x^{3/4 - 1} = \frac{3}{4} x^{-1/4}$

2) $f(x) = \frac{1}{x^3} = x^{-3}$ $\qquad f'(x) = -3x^{-4}$

3) $f(x) = x^{\pi + .0026}$ $\qquad f'(x) = (\pi + .0026)\, x^{\pi + .0026 - 1}$

4) $f(x) = (2x+1)^2 = 4x^2 + 4x + 1$ $\qquad f'(x) = 8x + 4.$

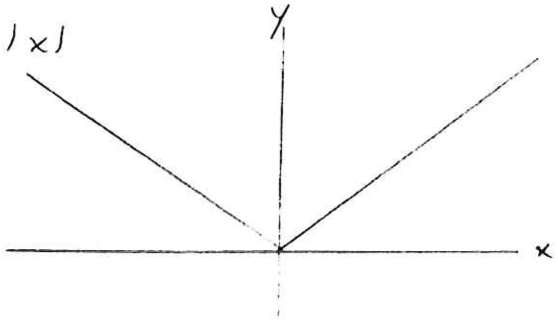
## Existence of Derivatives

Section 8

The derivative of a function at some point is the slope of the function at that point. Hence for the derivative to exist, the slope must be well defined.

### Example

$f(x) = |x|$



Slope of $f(x)$ isn't well defined at $x=0$.

$\therefore$ derivative of $f(x) = |x|$ **isn't defined** at $x=0$.

Conclusion: Derivatives **are** not defined at corners!

### Notation

Derivative of $y = f(x)$, $\quad f'(x) \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Instead of $f'(x)$, one frequently sees: $\frac{df}{dx}$, $f'$, $\frac{dy}{dx}$, $y'$.

text uses $D_x f(x)$

| $f(x)$ | Derivative $f'(x)$ |
|---|---|
| C (a constant) | 0 |
| $x^k$ ($k$ = a constant) | $kx^{k-1}$ |
| $e^x$ | $e^x$ |
| $\ln x$ | $\frac{1}{x}$ |
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\sec^2 x$ |
| $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\cos^{-1} x$ | $\frac{-1}{\sqrt{1-x^2}}$ |
| $\tan^{-1} x$ | $\frac{1}{1+x^2}$ |

## Derivative of $e^x$

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{de^x}{dx} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} e^x \frac{(e^h - 1)}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

Recall $e \equiv \lim_{h \to 0} (1 + h)^{1/h}$.

$\Rightarrow$ for small $h$, $e \simeq (1+h)^{1/h}$

$$\therefore \frac{de^x}{dx} = e^x \lim_{h \to 0} \frac{[(1+h)^{1/h}]^h - 1}{h}$$

$$= e^x \lim_{h \to 0} \frac{1 + h - 1}{h}$$

$$= e^x \lim_{h \to 0} \frac{h}{h}$$

$$\boxed{\therefore \frac{de^x}{dx} = e^x}$$

← Mar 5/90

## Derivative of sin x

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\frac{d\sin x}{dx} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\cos(h) + \cos x\sin(h) - \sin x}{h}$$

$$= \lim_{h\to 0}\left[\sin x\,\frac{(\cos(h)-1)}{h} + \cos x\,\frac{\sin(h)}{h}\right]$$

$$= \sin x \underbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}_{=0 \text{ from H.W. homework}} + \cos x \underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{=1 \text{ from previous lecture}}$$

$$\boxed{\therefore \frac{d\sin x}{dx} = \cos x}$$

## Derivative of cos x (HW)

$$\boxed{\frac{d\cos x}{dx} = -\sin x}$$

# Product Rule

Derivative of a Product of 2 functions $f(x)$ + $g(x)$ is:

$$\boxed{\frac{d\,(f(x)\,g(x))}{dx} = f\,\frac{dg}{dx} + g\,\frac{df}{dx}}$$

Proof: $\displaystyle \frac{d\ f(x)\,g(x)}{dx} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)\,g(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)\,g(x+h) + f(x)g(x+h) - f(x)g(}{h}$$

$$= \lim_{h\to 0} g(x+h)\ \frac{f(x+h) - f(x)}{h}$$

$$+ \lim_{h\to 0} f(x)\ \frac{g(x+h) - g(x)}{h}$$

$$\therefore \frac{d\ f(x)\,g(x)}{dx} = g(x)\ \frac{df(x)}{dx} + f(x)\ \frac{dg(x)}{dx}$$

## Examples

1) $\displaystyle \frac{d}{dx}\ (x+1)(x^3+2x)$

Old Way: $\displaystyle \frac{d}{dx}\ (x+1)(x^3+2x) = \frac{d}{dx}\left[x^4 + 2x^2 + x^3 + 2x\right]$

$$= 4x^3 + 4x + 3x^2 + 2$$

$$= 4x^3 + 3x^2 + 4x + 2.$$

<u>New Way:</u> $\frac{d}{dx}(x+1)(x^3+2x) = (x+1)\frac{d(x^3+2x)}{dx} + (x^3+2x)\frac{d(x+1}{dx}$

$$= (x+1)(3x^2+2) + (x^3+2x) \cdot 1$$

$$= 3x^3 + 2x + 3x^2 + 2 + x^3 + 2x$$

$$= 4x^3 + 3x^2 + 4x + 2.$$

2) $\frac{d}{dx}(e^x \sin x) = e^x \frac{d \sin x}{dx} + \sin x \frac{de^x}{dx}$

$$= e^x \cos x + \sin x \quad e^x$$

$$= e^x (\cos x + \sin x)$$

3) $\frac{d}{dx}\left(x^2 e^x \cos x\right) = x^2 \frac{d}{dx}\left(e^x \cos x\right) + e^x \cos x \frac{d}{dx}(x^2)$

$$= x^2 \left[ e^x \frac{d \cos x}{dx} + \cos x \frac{d e^x}{dx} \right]$$

$$+ e^x \cos x \frac{d x^2}{dx}$$

$$= x^2 \left[ e^x (-\sin x) + \cos x \; e^x \right] + e^x \cos x \quad 2x$$

$$= x e^x \left[ -x \sin x + x \cos x + 2 \cos x \right]$$

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← mar. 6/90

## Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{df}{dx} - f(x)\frac{dg}{dx}}{g^2(x)}$$ proof below

## Reciprocal Rule

$$\boxed{\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-1}{f^2(x)}\frac{df}{dx}}$$

Proof: $$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \lim_{h\to 0}\frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{f(x) - f(x+h)}{f(x+h)f(x)}}{h}$$

$$= \lim_{h\to 0}\frac{-1}{f(x+h)f(x)}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-1}{f^2(x)}\frac{df(x)}{dx}$$

## Derivation of Quotient Rule Using Product + Reciprocal Rules

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{1}{g(x)}\frac{df(x)}{dx} + f(x)\frac{d}{dx}\left(\frac{1}{g(x)}\right)$$

$$= \frac{1}{g(x)}\frac{df(x)}{dx} + f(x)\left(\frac{-1}{g^2(x)}\right)\frac{dg(x)}{dx}.$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\,\frac{df(x)}{dx} - f(x)\,\frac{dg(x)}{dx}}{g^2(x)}$$

Examples

1) $$\frac{d}{dx}\csc x = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= -\frac{1}{\sin^2 x}\,\frac{d\sin x}{dx}.$$

$$= -\frac{\cos x}{\sin^2 x}$$

2) $$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{d\sin x}{dx}\,\frac{1}{\cos x} + \sin x\,\frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \cos x\,\frac{1}{\cos x} + \sin x\left(\frac{-1}{\cos^2 x}\right)\frac{d\cos x}{dx}.$$

$$= 1 - \frac{\sin x}{\cos^2 x}(-\sin x)$$

$$= 1 + \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$\boxed{\therefore \frac{d}{dx}\tan x = \sec^2 x}$$

3) $$\frac{d}{dx}\left(\frac{x e^x}{\cos x}\right) = \frac{d(x e^x)}{dx}\,\frac{1}{\cos x} + x e^x\,\frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \frac{1}{\cos x}\left[x\,\frac{d e^x}{dx} + e^x\,\frac{dx}{dx}\right] + x e^x\,\frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \frac{1}{\cos x}\left[x e^x + e^x\right] + x e^x\,\frac{(-1)}{\cos^2 x}\,\frac{d\cos x}{dx}$$

$$= \frac{e^x}{\cos x}\,(x+1) - \frac{x e^x}{\cos^2 x}\,(-\sin x)$$

$$= \frac{e^x}{\cos^2 x}\left[(x+1)\cos x + x\sin x\right]$$

## Composite Functions

Section 1.

A composite function is a function whose argument is another function. i.e. $f(g(x))$.

### Examples.

1) $\sqrt{2x-1}$ let $u = 2x-1$ $f(\overset{u}{\cancel{x}}) = \sqrt{\cancel{x}u}$

$g(x) = 2x-1.$

2) $\sin 4x^2$ let $u = 4x^2$ $f(u) = \sin u$

$g(x) = 4x^2.$

5) $e^{-4x}$ let $u = -4x$ $f(u) = e^u$

$g(x) = -4x.$

### Chain Rule

$$\boxed{\frac{d\, f(g(x))}{dx} = \frac{d f(g)}{dg} \cdot \frac{d g(x)}{dx}}$$ OR let $u = g(x)$

$$\frac{d f(g(x))}{dx} = \frac{df}{du}\frac{du}{dx}$$

**Proof:** $$\frac{d\, f(g(x))}{dx} = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

as $h \to 0$, $g(x+h) \to g(x)$.

Writing $g(x+h) = g(x) + \delta$ we find that $\delta \to 0$ as $h \to 0$.

$$\therefore \quad \frac{d\, f(g(x))}{dx} = \lim_{\substack{h \to 0 \\ \delta \to 0}} \frac{f(g(x) + \delta) - f(g(x))}{g(x) + \delta - g(x)} \cdot \frac{g(x+h) - g(x}{h}$$

$$= \lim_{\delta \to 0} \frac{f(g + \delta) - f(g)}{\delta} \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$\therefore \quad \frac{d f(g(x))}{dx} = \frac{d f(g)}{dg} \cdot \frac{d\, g(x)}{dx}.$$

## Examples.

1) $\frac{d}{dx} \sqrt{2x-1} = \frac{d\sqrt{g}}{dx}$ $\qquad g = 2x - 1$

$$= \frac{d\sqrt{g}}{dg} \quad \frac{dg}{dx}$$

$$= \frac{1}{2} g^{-1/2} \quad 2$$

$$= \frac{1}{2} (2x-1)^{-1/2} . 2$$

$$= (2x-1)^{-1/2}$$

← Mar 8/90

2) $$\frac{d}{dx}\sin 4x^2 = \frac{d\sin g}{dx} \qquad g = 4x^2$$

$$= \frac{d\sin g}{dg}\ \frac{dg}{dx}$$

$$= \cos g \quad 8x.$$

$$= 8x\cos 4x^2$$

3) $$\frac{d}{dx}(x^2+1)^2$$

Old Way: $$\frac{d}{dx}(x^2+1)^2 = \frac{d}{dx}\left[x^4+2x^2+1\right]$$

$$= 4x^3 + 4x$$

$$= 4x(x^2+1)$$

New Way: $$\frac{d}{dx}(x^2+1)^2 = \frac{du^2}{dX} \qquad u = x^2+1$$

$$= \frac{du^2}{du}\ \frac{du}{dx}.$$

$$= 2u\ ,\ 2x.$$

$$= 4x(x^2+1)$$

4) $$\frac{d}{dx}(x^2+1)^{100} = \frac{du^{100}}{dx} \qquad u = x^2+1$$

$$= \frac{du^{100}}{du}\ \frac{du}{dx}.$$

$$= 100u^{99}\ ,\ 2x.$$

$$= 200x\ (x^2+1)^{99}$$

5) $\frac{d}{dx} e^{4x} \sin x = \frac{de^{4x}}{dx} \sin x + e^{4x} \frac{d \sin x}{dx}.$

$= \frac{de^{u}}{dx} \sin x + e^{4x} \cos x \qquad u = 4x$

$= \frac{de^{u}}{du} \frac{du}{dx} \sin x + e^{4x} \cos x$

$= e^{u} \; 4 \; \sin x + e^{4x} \cos x$

$= e^{4x} ( 4 \sin x + \cos x )$

6) $\frac{d (\sin x^3)^{1/2}}{dx} = \frac{d u^{1/2}}{dx} \qquad u = \sin x^3$

$= \frac{d u^{1/2}}{du} \frac{du}{dx}.$

$= \frac{1}{2} u^{-1/2} \frac{d \sin x^3}{dx}.$ ~~$v = x^3$.~~

$= \frac{1}{2} u^{-1/2} \frac{d \sin v}{dx}. \qquad v = x^3$

$= \frac{1}{2} u^{-1/2} \frac{d \sin v}{dv} \frac{dv}{dx}.$

$= \frac{1}{2} u^{-1/2} \cos v \;\; 3x^2$

$= \frac{3}{2} x^2 (\sin x^3)^{-1/2} \cos x^3$

7) $$\frac{d}{dx}\,(x-1)^2\,(2x+1)^3 = (2x+1)^3\,\frac{d\,(x-1)^2}{dx} + (x-1)^2\,\frac{d\,(2x+1)^3}{dx}.$$

$$= (2x+1)^3\,\frac{du^2}{du}\,\frac{du}{dx} \qquad u = x-1$$

$$+\,(x-1)^2\,\frac{dv^3}{dv}\,\frac{dv}{dx} \qquad v = 2x+1$$

$$= (2x+1)^3\;2u\,.\,1\;+\,(x-1)^2\;3v^2\,.\,2.$$

$$= (2x+1)^3\;2(x-1)\;+\;6\,(x-1)^2\,(2x+1)^2$$

$$= (2x+1)^2(x-1)\,\left[\,2\,(2x+1)\;+\,6\,(x-1)\right]$$

$$= (2x+1)^2\,(x-1)\;(10x-4)$$

$$= 2\,(2x+1)^2\,(x-1)\,(5x-2).$$

## Derivative of ln x

Section 16+29

$$x = e^{\ln x}$$

$$\frac{dx}{dx} = \frac{d\,e^{\ln x}}{dx}$$

$$1 = \frac{d\,e^u}{du}\;\frac{du}{dx} \qquad u = \ln x.$$

$$= e^u \;\frac{du}{dx}$$

$$= e^{\ln x}\;\frac{d\ln x}{dx}$$

$$= x\;\frac{d\ln x}{dx}$$

$$\boxed{\therefore \quad \frac{d\ln x}{dx} = \frac{1}{x}.}$$

## Derivative of $a^x$

$$a^x = e^{\ln a^x} \qquad a = \text{constant}$$

$$= e^{x\ln a}$$

$$\frac{d\,a^x}{dx} = \frac{d\,e^{x\ln a}}{dx}$$

$$= \frac{d\,e^u}{du}\;\frac{du}{dx} \qquad u = x\ln a.$$

$$= e^u\;\ln a$$

$$= e^{x\ln a}\;\ln a.$$

$$\boxed{\therefore \frac{da^x}{dx.} = a^x\;\ln a}$$

## Derivative of $\log_a x$.

Recall $\log_a x = \dfrac{\log_b x}{\log_b a}.$

$$\therefore \log_a x = \frac{\ln x}{\ln a}.$$

$$\therefore \frac{d \log_a x}{dx} = \frac{1}{\ln a} \frac{d \ln x}{dx}$$

$$\boxed{\frac{d \log_a x}{dx} = \frac{1}{x \ln a}}$$

## Derivative of Arcsin x or $\sin^{-1}x$.

Let $y = \sin^{-1} x$.

$$\Rightarrow \quad x = \sin y.$$

take $\frac{d}{dx}$ of both sides

$$\frac{d(x)}{dx} = \frac{d\,\sin y}{dx}$$

$$1 = \frac{d\sin y}{dy}\ \frac{dy}{dx}$$

$$= \cos y\ \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\boxed{\therefore \frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

## Derivative of Arccos x or $\cos^{-1}x$

**Homework:** show $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

← Mar 12/90

# Derivative of Arctan x or $\tan^{-1} x$

Let $y = \tan^{-1} x$

$\Rightarrow \tan y = x.$

$$\frac{d \tan y}{dx} = \frac{dx}{dx}$$

$$\frac{d \tan y}{dy} \frac{dy}{dx} = 1$$

p. 66 $$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{1 + \tan^2 y}$$

$$\boxed{\therefore \frac{d \tan^{-1} x}{dx} = \frac{1}{1 + x^2}}$$

Note: angles must be in radians for calculus formulas to be true.

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

# Example Differentiation Session I & II

1)
$$\frac{d}{dx}\left(2\sqrt{x} + \frac{1}{x^2} + 1\right) = 2\,\frac{d\,x^{1/2}}{dx} + \frac{d}{dx}x^{-2} + \frac{d\,1}{dx}$$
$$= 2\,\frac{1}{2}\,x^{-1/2} + (-2)\,x^{-3} + 0$$
$$= x^{-1/2} - 2\,x^{-3}$$

2)
$$\frac{d\,e^2}{dx} = 0 \quad \text{since } e = \text{constant}$$

3)
$$\frac{d}{dx}\left(\sin x - 2\cos x\right) = \frac{d\sin x}{dx} - 2\,\frac{d\cos x}{dx}$$
$$= \cos x - 2\,(-\sin x)$$
$$= \cos x + 2\sin x$$

4)
$$\frac{d}{dx}\left(2^x - 3^x\right) = \frac{d}{dx}\left(e^{\ln 2^x} - e^{\ln 3^x}\right)$$
$$= \frac{d}{dx}\left(e^{x\ln 2} - e^{x\ln 3}\right)$$
$$= \frac{d\,e^{x.\ln 2}}{dx} - \frac{d\,e^{x.\ln 3}}{dx}$$
$$= \frac{d\,e^u}{du}\,\frac{du}{dx} - \frac{d\,e^v}{dv}\,\frac{dv}{dx} \qquad \begin{array}{l} u = x.\ln 2 \\ v = x\ln 3 \end{array}$$
$$= e^u\,\ln 2 - e^v\,\ln 3$$
$$= e^{x\ln 2}\,\ln 2 - e^{x.\ln 3}\,\ln 3$$
$$= 2^x\ln 2 - 3^x\,\ln 3$$

see also formula on p. 73

5) $$\frac{d}{dx}\left(\sqrt{9x} + 1\right) = \frac{d\sqrt{9x}}{dx} + \frac{d1}{dx}$$

$$= \sqrt{9}\frac{dx^{1/2}}{dx} + 0$$

$$= \sqrt{3}\frac{1}{2}x^{-1/2}$$

$$= \frac{\sqrt{3}}{2}x^{-1/2}$$

6) $$\frac{d}{dx}\left(x^3 - 3\ln x\right) = \frac{dx^3}{dx} - 3\frac{d\ln x}{dx}$$

$$= 3x^2 - \frac{3}{x}$$

$$= 3\left(x^2 - \frac{1}{x}\right)$$

7) $$\frac{d}{dx}(x^2+1)(x^3-1) = (x^3-1)\frac{d(x^2+1)}{dx} + (x^2+1)\frac{d(x^3-1)}{dx}$$

$$= (x^3-1)\ 2x + (x^2+1)\ 3x^2$$

$$= x\left[2(x^3-1) + 3x(x^2+1)\right]$$

$$= x\left[2x^3 - 2 + 3x^3 + 3x\right]$$

$$= x\left[5x^3 + 3x - 2\right]$$

8) $\frac{d}{dx}\left(x+\frac{1}{x}\right)(x^2-x) = (x^2-x)\frac{d\left(x+\frac{1}{x}\right)}{dx} + \left(x+\frac{1}{x}\right)\frac{d(x^2-x)}{dx}$

$= (x^2-x)\left(1-\frac{1}{x^2}\right) + \left(x+\frac{1}{x}\right)(2x-1)$

$= x^2 - 1 - x + \frac{1}{x} + 2x^2 - x + 2 - \frac{1}{x}$

$= 3x^2 - 2x + 1$

9) $\frac{d}{dx}(\sin x \cos x) = \cos x \frac{d\sin x}{dx} + \sin x \frac{d\cos x}{dx}$

can also use:
$\sin 2x = 2\sin x\cos x$

$= \cos x \cos x + \sin x(-\sin x)$

$= \cos^2 x - \sin^2 x$

$= \cos 2x. = \cos(x+x)$ ← Mar 13/90

10) $\frac{d}{dx}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) = \frac{1}{\sqrt{x}-1}\frac{d}{dx}\left(\sqrt{x}+1\right) + (\sqrt{x}+1)\frac{d}{dx}\left(\frac{1}{\sqrt{x}-1}\right)$

$= \frac{1}{\sqrt{x}-1}\frac{1}{2}x^{-1/2} + \frac{(\sqrt{x}+1)(-1)}{(\sqrt{x}-1)^2}\frac{d}{dx}(\sqrt{x}-1)$

$= \frac{1}{\sqrt{x}-1}\frac{1}{2}x^{-1/2} - \frac{(\sqrt{x}+1)}{(\sqrt{x}-1)^2}\frac{1}{2}x^{-1/2}$

$= \frac{1}{2}\frac{x^{-1/2}}{(\sqrt{x}-1)^2}\left[\sqrt{x}-1 - (\sqrt{x}+1)\right]$

$= \frac{1}{2}\frac{x^{-1/2}}{(\sqrt{x}-1)^2}\left[\sqrt{x}-1-\sqrt{x}-1\right]$

$= -x^{-1/2}/(\sqrt{x}-1)^2$

11) $\dfrac{d\cot x}{dx} = \dfrac{d(\cos x/\sin x)}{dx}$

$= \dfrac{1}{\sin x}\dfrac{d\cos x}{dx} + \cos x \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right)$

$= \dfrac{1}{\sin x}(-\sin x) + \cos x \dfrac{(-1)}{\sin^2 x}\dfrac{d\sin x}{dx}$

$= -1 - \dfrac{\cos x}{\sin^2 x}\cos x.$

$= -1 - \dfrac{\cos^2 x}{\sin^2 x}$

$= \dfrac{-1}{\sin^2 x}$

$= -\csc^2 x.$

12) $\dfrac{d\; e^x/(x+1)}{dx} = \dfrac{1}{x+1}\dfrac{de^x}{dx} + e^x\dfrac{d}{dx}\left(\dfrac{1}{x+1}\right)$

$= \dfrac{e^x}{x+1} + e^x\dfrac{(-1)}{(x+1)^2}\dfrac{d(x+1)}{dx}$

$= \dfrac{e^x}{x+1} - \dfrac{e^x}{(x+1)^2} \cdot 1$

$= \dfrac{e^x}{(x+1)^2}(x+1-1)$

$= \dfrac{xe^x}{(x+1)^2}$

13) $$\frac{d}{dx}\frac{(x^2-1)(x^2+3)}{(x-2)^2} = (x^2-1)\frac{d}{dx}\left(\frac{x^2+3}{(x-2)^2}\right) + \frac{x^2+3}{(x-2)^2}\frac{d}{dx}(x^2-1)$$

$$= (x^2-1)\left\{\frac{1}{(x-2)^2}\frac{d}{dx}(x^2+3) + (x^2+3)\frac{d}{dx}\left(\frac{1}{(x-2)^2}\right)\right\}$$

$$+ \frac{x^2+3}{(x-2)^2}\ ,\ 2x.$$

$$-\frac{1}{(x-2)^4}(2(x-2))$$

$$= (x^2-1)\left\{\frac{1}{(x-2)^2}\ ,2x + (x^2+3)(-2)(x-2)^{-3}\right\}$$

$$+ \frac{x^2+3}{(x-2)^2}\ ,\ 2x$$

$$= (x^2-1)\left\{\frac{2x}{(x-2)^2} - \frac{2(x^2+3)}{(x-2)^3}\right\} + 2x\frac{(x^2+3)}{(x-2)^2}$$

$$= \frac{2}{(x-2)^3}\left\{(x^2-1)\left[x(x-2) - (x^2+3)\right] + x(x^2+3)(x-2)\right\}$$

$$= \frac{2}{(x-2)^3}\left\{(x^2-1)\left[x^2-2x-x^2-3\right] + (x^3+3x)(x-2)\right\}$$

$$= \frac{2}{(x-2)^3}\left\{(x^2-1)(-2x-3) + x^4-2x^3+3x^2-6x\right\}$$

$$= \frac{2}{(x-2)^3}\left\{-2x^3-3x^2+2x+3 + x^4-2x^3+3x^2-6x\right\}$$

$$= \frac{2}{(x-2)^3}\left\{x^4-4x^3-4x+3\right\}$$

14) $\frac{d}{dx} \frac{1}{2}\left(e^{x}-e^{-x}\right) = \frac{1}{2}\left(\frac{de^{x}}{dx} - \frac{de^{-x}}{dx}\right)$

$= \frac{1}{2}\left(e^{x} - \frac{de^{u}}{du}\frac{du}{dx}\right)$ $\quad u = -x.$

$= \frac{1}{2}\left(e^{x} - e^{u}(-1)\right)$

$= \frac{1}{2}\left(e^{x} + e^{-x}\right)$

15) $\frac{d}{dx}\left(e^{2x}+1\right)^{5} = \frac{d\,u^{5}}{du}\frac{du}{dx}$ $\quad u = e^{2x}+1.$

$= 5u^{4}\frac{d\left(e^{2x}+1\right)}{dx}$

$= 5u^{4}\left(\frac{de^{v}}{dv}\frac{dv}{dx} + 0\right)$ $\quad v = 2x$

$= 5u^{4}\, e^{v}\, 2.$

$= 10\left(e^{2x}+1\right)^{4} e^{2x}.$

16) $\frac{d}{dx}\ln(x+1) = \frac{d\ln u}{du}\frac{du}{dx}$ $\quad u = x+1.$

$= \frac{1}{u}\quad 1$

$= \frac{1}{x+1}$

17) $\frac{d}{dx}\ln(x+e^{x}) = \frac{d\ln u}{du}\frac{du}{dx}$ $\quad u = x+e^{x}$

$= \frac{1}{u}\cdot\left(1+e^{x}\right) = \frac{1+e^{x}}{x+e^{x}}.$

18) $\frac{d}{dx} 20(1-e^{-2x}) = -20 \frac{de^{-2x}}{dx}$

$= -20 \cdot (-2) e^{-2x}$

$= 40 e^{-2x}$

19) $\frac{d}{dx}(xe^{-x}) = 1 \cdot e^{-x} + x \frac{de^{-x}}{dx}$

$= e^{-x} + x(-1)e^{-x}$

$= e^{-x}(1-x)$

20) ~~$\frac{d}{dx}(xe^{-x}) = 1 \cdot e$~~

$\frac{d}{dx}(x+4\ln x)^3 = 3(x+4\ln x)^2 \cdot \left[1 + 4 \cdot \frac{1}{x}\right]$

$= 3(x+4\ln x)^2 \cdot \left(1+\frac{4}{x}\right)$ ← Mar 15/90

21) $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1}{x} \cdot \frac{1}{x} + \ln x \frac{d}{dx}\left(\frac{1}{x}\right)$

$= \frac{1}{x^2} + \ln x \, (-1)x^{-2}$

$= \frac{1}{x^2}\left(1 - \ln x\right)$

22) $\frac{d}{dx}\left(\frac{e^{-x}}{1-2e^{-x}}\right) = \frac{-e^{-x}}{1-2e^{-x}} + e^{-x}\frac{d}{dx}\left(\frac{1}{1-2e^{-x}}\right)$

$= \frac{-e^{-x}}{1-2e^{-x}} + \frac{e^{-x}(-1)}{(1-2e^{-x})^2}\frac{d}{dx}(1-2e^{-x})$

$$\frac{d}{dx}\left(\frac{e^{-x}}{1-2e^{-x}}\right) = \frac{-e^{-x}}{1-2e^{-x}} - \frac{e^{-x}}{(1-2e^{-x})^2} \quad (-2)(-1)e^{-x}$$

$$= \frac{-e^{-x}}{(1-2e^{-x})^2}\left[1-2e^{-x}+2e^{-x}\right]$$

$$= \frac{-e^{-x}}{(1-2e^{-x})^2}$$

23) $$\frac{d}{dx}\left(e^{-x}\cos 2x\right) = -e^{-x}\cos 2x + e^{-x}(-\sin 2x)\,2.$$

$$= -e^{-x}(\cos 2x + 2\sin 2x)$$

24) $$\frac{d}{dx}x^x = \frac{d}{dx}e^{\ln x^x}$$

$$= \frac{d}{dx}e^{x\ln x}$$

$$= e^{x\ln x}\,\frac{d(x\ln x)}{dx}$$

$$= x^x\cdot\left[\ln x + x\cdot\frac{1}{x}\right]$$

$$= x^x(\ln x + 1)$$

25) $$\frac{d}{dx}\,\arctan(\ln x^2) = \frac{1}{1+(\ln x^2)^2}\,\frac{d\ln x^2}{dx}$$

$$= \frac{1}{1+(\ln x^2)^2}\cdot\frac{1}{x^2}\cdot 2x.$$

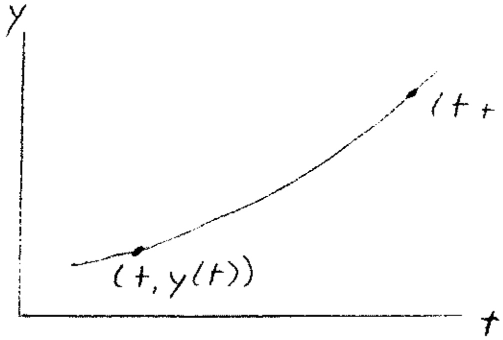
$$= \frac{2}{x}\,\frac{1}{1+(\ln x^2)^2}$$

← Mar 19/90

## Differentiation with respect to Time

Section 11

Consider a function $y(t)$ which is a function of time.



In time $\Delta t$, $y$ changes by amount $y(t+\Delta t) - y(t)$

$\therefore$ $\dfrac{y(t+\Delta t) - y(t)}{\Delta t}$ is the average rate of change of $y$ between times $t$ & $t+\Delta t$.

$$\boxed{\therefore \quad \frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}}$$

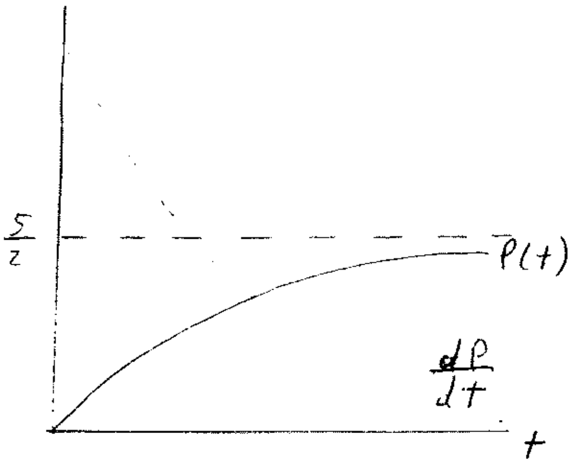
is the instantaneous rate of change of $y(t)$ at time $t$.

### Examples.

1) Suppose the population of grasshoppers $P$ as a function of time $t$ is given by:

$$P = \frac{5t}{1+2t}$$

$t$ measured in days
$P$ " " millions



at small t near 0, P changes quickly. $\Rightarrow$ $\frac{dP}{dt}$ we expect to be large.

at large t, P levels off $\Rightarrow$ $\frac{dP}{dt}$ becomes small.

rate of change of population at time t is

$$\frac{dP}{dt} = \frac{d}{dt}\left(\frac{5t}{1+2t}\right)$$

$$= \frac{5}{1+2t} + 5t\,\frac{d}{dt}\left(\frac{1}{1+2t}\right)$$

$$= \frac{5}{1+2t} + 5t\,\frac{(-1)}{(1+2t)^2}\cdot 2$$

$$= \frac{5(1+2t) - 10t}{(1+2t)^2}$$

$$= \frac{5}{(1+2t)^2}$$

$\frac{dP}{dt}(t=0) = 5$ million/day. $\frac{dP}{dt}(t=2) = \frac{1}{5}$ mill/day

$\frac{dP}{dt}(t=1) = \frac{5}{9}$ mill/day.

2) Temperature T during a summer day is given by

$T = 20 + 10 \sin \frac{\pi t}{12}$

t = measured in hours
t = 0 at 6.00 am
T in °C.

rate of change of T is $\frac{dT}{dt} = 10 \cdot \frac{\pi}{12} \cos \frac{\pi t}{12}$

$= \frac{5\pi}{6} \cos \frac{\pi t}{12}.$

at 6:00 am $\frac{dT}{dt}(t=0) = \frac{5\pi}{6} \simeq 2.5$ °C/hr.

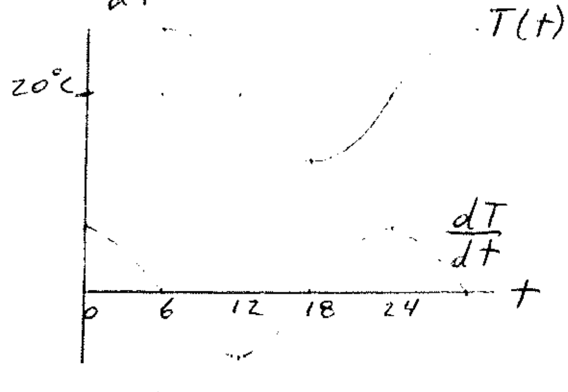
$\frac{dT}{dt}(t=0) > 0$ meaning air is warming up.

at 12:00 am (noon) $\frac{dT}{dt}(t=6) = 0$

meaning air is neither cooling nor warming.

at 6:00 pm. $\frac{dT}{dt}(t=12) = -\frac{5\pi}{6} \simeq -2.5$ °C/hr.

$\frac{dT}{dt}(t=12) < 0$ meaning air is cooling.

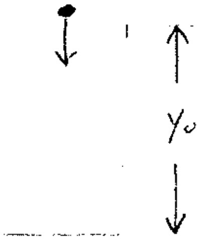


$\left(\frac{dT}{dt} = 0\right)$

∴ max. temp. occurs at noon when neither heating nor cooling occu

## 3) Falling Object.

The height of an object released at height $y_0$ falling to earth is given by:



$$y(t) = y_0 - \frac{1}{2} g t^2$$

$$g = 10 \ m/sec^2$$

rate of change of $y(t)$ is the velocity

$$v = \frac{dy}{dt}$$

$$v(t) = -gt$$

$v(0) = 0$

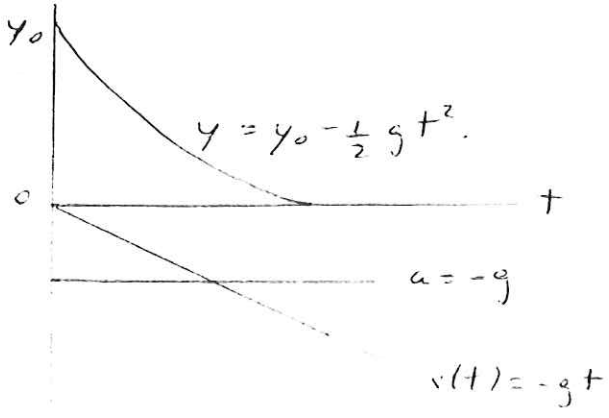
$v(t = 1\ sec) = -g \quad m/sec.$

$v(t = 2\ sec) = -2g \quad m/sec.$

∴ the objects speed increases as it falls.

rate of change of $v(t)$ is the acceleration

$$a = \frac{dv}{dt} = -g.$$



← Mar 20/90

Time rock hits ground is given by:

$$0 = y_0 - \frac{1}{2} g t^2$$

$$t = \sqrt{\frac{2 y_0}{g}}$$

Just before rock hits ground, its velocity is given by:

$$v = -g \sqrt{\frac{2 y_0}{g}} = -\sqrt{2 y_0 g}$$

<u>eg.</u> $y_0 = 500$ meters (CN Tower)

$$t = \left( \frac{2 \times 500 \text{ meters}}{10 \text{ meters/sec}^2} \right)^{1/2} = 10 \text{ sec}.$$

$$v = -(2 \times 500 \text{ meter} \times 10 \text{ m/sec}^2)^{1/2} = -100 \text{ meters/sec}.$$

## Implicit Differentiation

Until now we have differentiated functions that have been written as $y = y(x)$. This isn't always the case.

Examples

1) Find $\frac{dy}{dt}$ if $t^3 + y^2 = 4$.

$$3t^2 + 2y\frac{dy}{dt} = 0$$

$$\therefore \frac{dy}{dt} = -\frac{3t^2}{2y}$$

2) Find $\frac{dy}{dt}$ if $ty^2 + y^3 = 2t$.

$$y^2 + t\,2y\frac{dy}{dt} + 3y^2\frac{dy}{dt} = 2.$$

$$\frac{dy}{dt}\left(2ty + 3y^2\right) = 2 - y^2$$

$$\frac{dy}{dt} = \frac{2 - y^2}{y(2t+3y)}$$

Sometimes we are not given relations $x = x(t)$, $y = y(t)$ but have expressions with all 3 variables.

Examples.

1) Find relation between $\frac{dx}{dt}$ + $\frac{dy}{dt}$ if $x^2 + y^3 = t^2$.

$$\Longrightarrow 2x\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 2t$$

## Word Problems

1) A spherical balloon is being inflated at 20 in³/sec. How fast is diameter increasing when diameter is 10 inches?

Volume of balloon $V = \frac{4}{3}\pi r^3$

$$= \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 \qquad D = \text{diameter}$$

$$= \frac{\pi}{6} D^3.$$

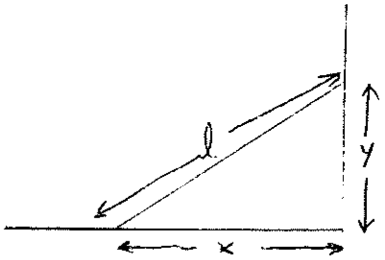
$$\therefore \frac{dV}{dt} = \frac{\pi}{2} D^2 \frac{dD}{dt}$$

$$\frac{dD}{dt} = \frac{2}{\pi D^2} \quad \frac{dV}{dt}$$

When $D = 10$ in, $\frac{dD}{dt} = \frac{2}{\pi \; 10^2} \quad 20$

$$= \frac{2}{\cancel{2}5\pi} \text{ in/sec.}$$

2) A 15 ft. ladder leans against a wall. It begins to fall downward 2 ft/sec. How fast does it move horizontally when it is 10 ft. above ground on wall?



$$l^2 = x^2 + y^2$$

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt} = \frac{-y}{\sqrt{l^2-y^2}}\frac{dy}{dt}$$

$\therefore$ when y is 10 ft. $\frac{dx}{dt} = \frac{-10}{\sqrt{15^2-10^2}} \cdot (-2) = 1.79$ ft/sec.

# Graphing

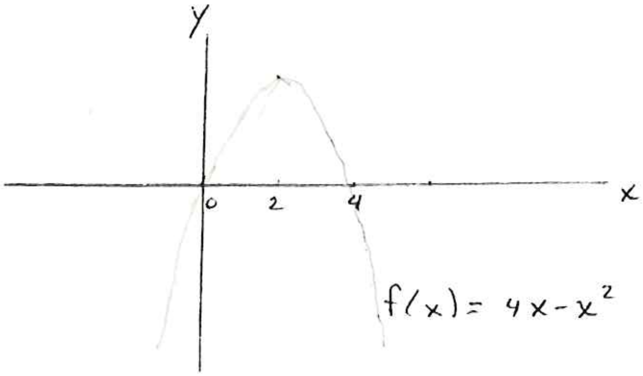
## Critical Point

A critical point of a function $f(x)$ is a solution of $f'(x) = 0$.

### Examples

1) $f(x) = 4x - x^2$
$f'(x) = 4 - 2x$.

$f'(x) = 0 \Rightarrow x = 2$



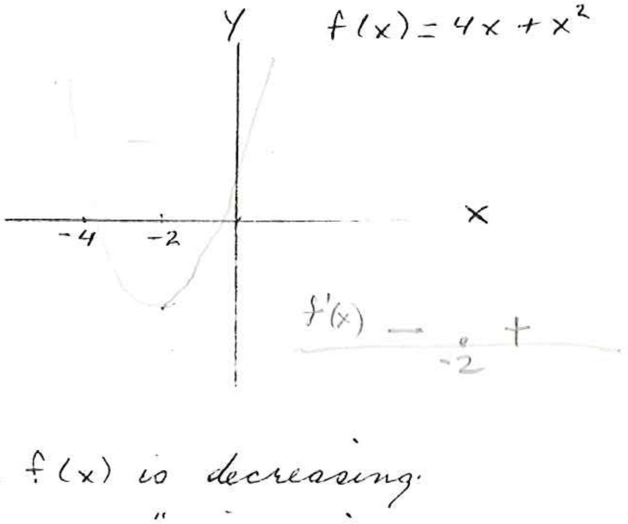
When $x < 2$, $f'(x) > 0$ i.e. $f(x)$ is increasing.
" $x > 2$, $f'(x) < 0$ " " decreasing.

∴ critical point $(2, 4)$ is a so called relative maximum
i.e. top of a hill

← Mar 22/90

2) $f(x) = 4x + x^2$
$f'(x) = 4 + 2x$

$f'(x) = 0 \Rightarrow x = -2$

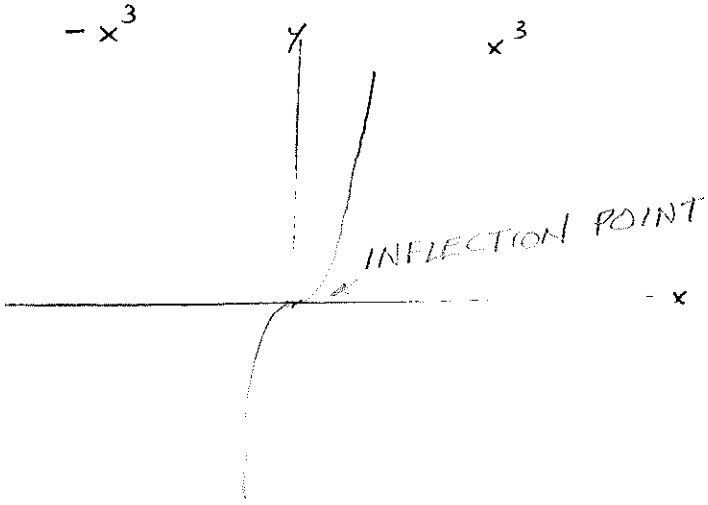


When $x < -2$, $f'(x) < 0$ i.e. $f(x)$ is decreasing.
" $x > -2$ $f'(x) > 0$ " " increasing.
∴ critical pt. $(-2, -4)$ is a so called relative minimum.
i.e. bottom of a valley

3) $f(x) = x^3$

$f'(x) = 3x^2$

$f'(x) = 0 \Rightarrow x = 0$



When $x<0$ $f'(x)>0$, i.e. $f(x)$ is increasing

″ $x>0$ ″ ″ ″

$f'(x)$ $+ \underset{0}{|} +$

4) $f(x) = -x^3$

$f'(x) = -3x^2$

$f'(x) = 0 \Rightarrow x = 0$

$f'(x)$ $- \underset{0}{|} -$

When $x<0$ $f'(x)<0$, i.e. $f(x)$ is decreasing

″ $x>0$ ″ ″ ″

A point where $f'(x) = 0$ and the slope has the same sign on either side of the critical point is called a ~~saddle point.~~ an inflection point.

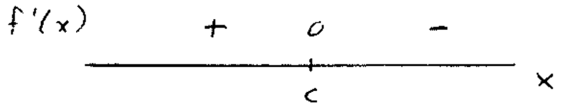
## First Derivative Test

Suppose $f(x)$ has a critical point at $x = c$, i.e. $f'(c) = 0$.

1) A relative maximum (hill) occurs at $x = c$ if

   $f'(x) > 0$ when $x < c$

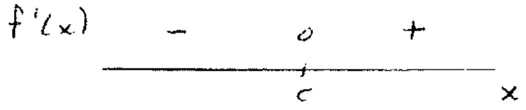
   $f'(x) < 0$ " $x > c$



2) A relative minimum (valley) occurs at $x = c$ if

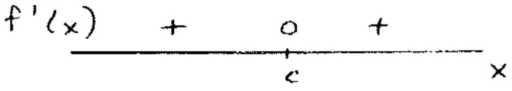
   $f'(x) < c$ when $x < c$

   $f'(x) > 0$ " $x > c$

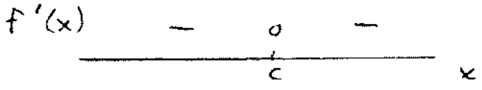


3) A ~~saddle~~ inflection point occurs at $x = c$ if either

   a) $f'(x) > 0$ when $x < c$ & $x > c$



   or b) $f'(x) < 0$ when $x < c$ & $x > c$



We will see later, it is not necessary that $f'(x) = 0$ at inflection point. But $f''(x)$ will have to be zero

Example

Graph $f(x) = x^2(x-1)^3$

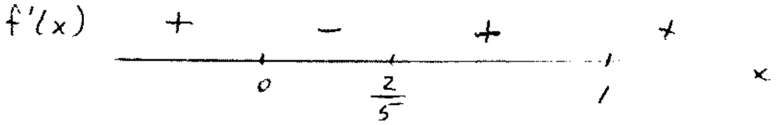
1) x - intercepts

$$f(x) = 0 \Longrightarrow x = 0, 1$$

2) critical points

$$\begin{aligned} f'(x) &= 2x(x-1)^3 + x^2\, 3(x-1)^2 \\ &= x(x-1)^2\left[2(x-1) + 3x\right] \\ &= x(x-1)^2(5x-2) \end{aligned}$$

$$f'(x) = 0 \Longrightarrow x = 0, \tfrac{2}{5}, 1$$



$$f'(-1) = (-1)(-1-1)^2(-5-2) = -4\ (-7) = 28 > 0$$

$$f'(\tfrac{1}{5}) = \tfrac{1}{5}\left(\tfrac{1}{5}-1\right)^2\left(5\cdot\tfrac{1}{5}-2\right) = \tfrac{1}{5}\cdot\tfrac{16}{25}\cdot(-1) = -\tfrac{16}{125} < 0$$

$$f'(\tfrac{1}{2}) = \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)^2\left(5\cdot\tfrac{1}{2}-2\right) = \tfrac{1}{2}\cdot\tfrac{1}{4}\cdot\left(+\tfrac{1}{2}\right) = +\tfrac{1}{16} > 0.$$

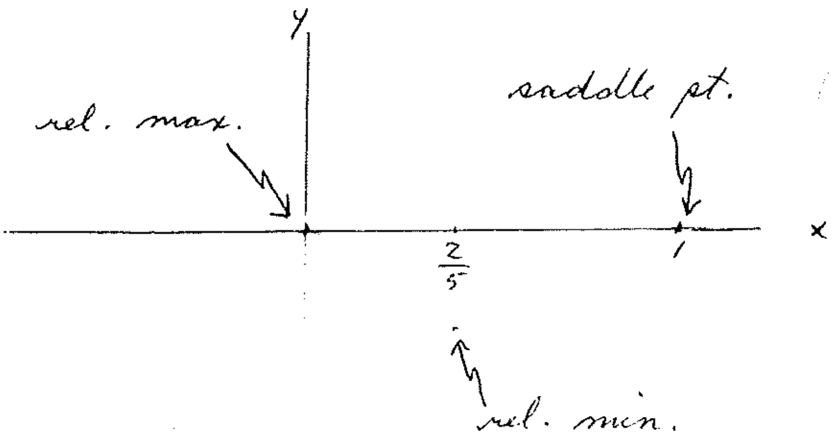
$$f'(2) = 2(2-1)^2(5.2-2) = 2.1.8 = 16 > 0.$$

∴ at $x = 0$ we have a rel. maximum.

" $x = 2/5$ " " " minimum.

" $x = 1$ " " " ~~saddle~~ inflection point.

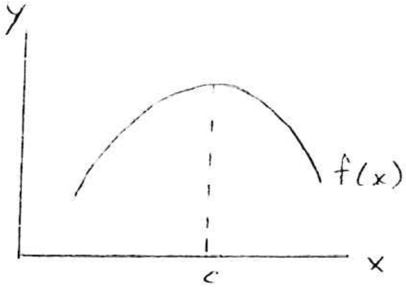
$f(x) = x^2 (x-1)^3$



# Critical Points & The 2nd Derivative

Section 13

Consider a function f(x) having a relative maximum at x = c. i.e. $f'(c) = 0$.



$$\left.\begin{array}{ll} x < c & f'(x) > 0 \\ x = c & f'(x) = 0 \\ x > c & f'(x) < 0 \end{array}\right\} \Longrightarrow$$

$f'(x)$ is decreasing around $x = c$

i.e. $$\frac{f'(x>c) - f'(x<c)}{x_{>c} - x_{<c}} < 0.$$

$\boxed{\therefore\ f''(c) < 0 \text{ for a relative maximum.}}$

← Mar 26/90

Example

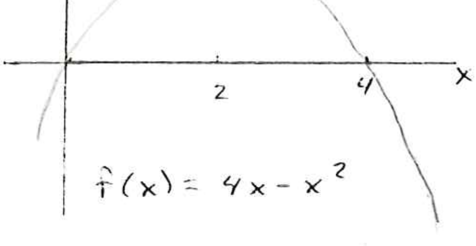
$f(x) = 4x - x^2$

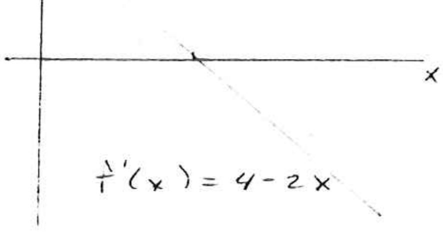
$f'(x) = 4 - 2x$

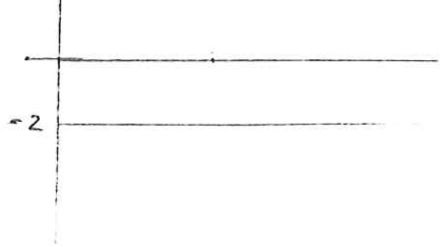
$f''(x) = -2.$

$f'(x) = 0 \Longrightarrow x = 2$

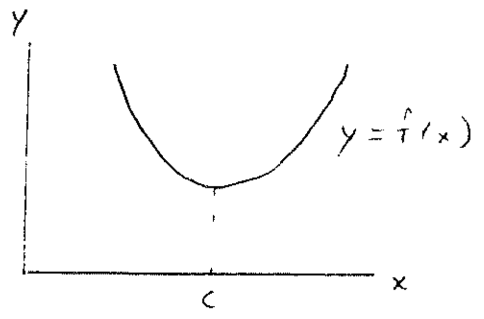
Since $f''(2) < 0$, there is a relative maximum at $x = 2$.







Next consider a function $f(x)$ with a relative minimum at $c$. i.e. $f'(c)=0$.



$$\left.\begin{array}{ll} x<c & f'(x)<0 \\ x=c & f'(c)=0 \\ x>c & f'(x)>0 \end{array}\right\} \Rightarrow$$

$f'(x)$ is increasing near $x=c$.

i.e. $\dfrac{f'(x>c) - f'(x<c)}{x_{>c} - x_{<c}} > 0$

$$\boxed{\therefore\ f''(c) > 0 \text{ for a relative minimum.}}$$

## Second Derivative Test

Suppose $f(x)$ has a critical point at $x=c$, i.e $f'(c)=0$
Then:

1) If $f''(c)>0$ there is a relative minimum at $x=c$.

2) If $f''(c)<0$ " " maximum " ".

## Example

Graph $f(x) = x^3 - 3x$.

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

1) x-intercepts: $f(x) = 0 \Rightarrow x^3 - 3x = 0$

$x(x^2 - 3) = 0$

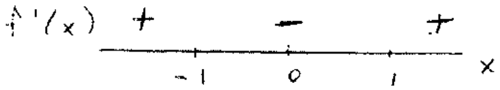
$x = 0, \pm\sqrt{3}$

2) critical points $f'(x) = 0 \Rightarrow 3x^2 - 3 = 0.$

$x^2 - 1 = 0$

$x = \pm 1.$

## Characterization of Critical Points Using First Derivative Test



$f'(-2) = 3(-2)^2 - 3 = 3.4 - 3 = 9 > 0.$

$f'(0) = 3\,0^2 - 3 = -3 < 0.$

$f'(2) = 3(2)^2 - 3 = 9 > 0$

∴ at $x = -1$ there is a relative maximum.

" $x = +1$ " " minimum.

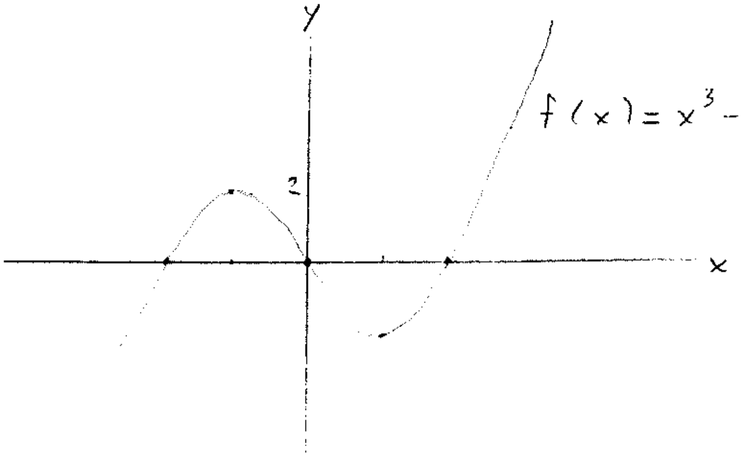
## Characterization of Critical Points Using 2nd Derivative Test

$f''(-1) = 6(-1) = -6 < 0$

$\therefore$ at $x = -1$ there is a relative maximum.

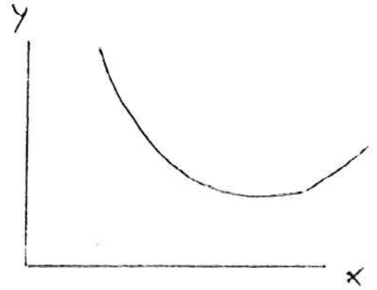
$f''(1) = 6(1) = 6 > 0$

$\therefore$ at $x = 1$ there is a relative minimum.



## Concave Up + Concave Down

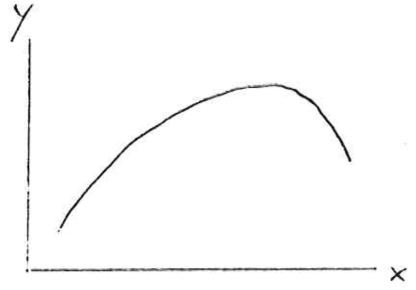
A curve whose slope is increasing is said to be <u>concave up</u> (i.e. valley like)



$f'(x)$ is increasing
$\Rightarrow f''(x) > 0$.

← Mar 27/90

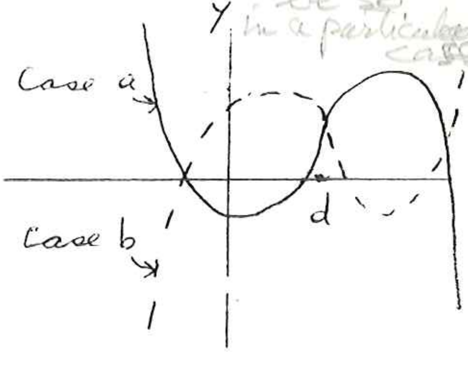
A curve whose slope is decreasing is said to be <u>concave down</u>. (i.e. hill like)



$f'(x)$ is decreasing
$\Rightarrow f''(x) < 0$

## Inflection Point

An inflection point occurs at $x = d$ if the concavity of the function (i.e. $f''(x)$) i.e. $f''(x) = 0$ changes when it passes through $x = d$.
It is not necessary that $f'(x) = 0$ at an inflection point though this may be so in a particular case
Also $f'(x)$ need not be zero.
$\therefore f''(d) = 0$ and either



a) $f''(x<d) > 0$ + $f''(x>d) < 0$

or b) $f''(x<d) < 0$ + $f''(x>d) > 0$.

## Example 1

$$f(x) = \frac{x}{1+x^2}$$

$$f'(x) = \frac{1}{1+x^2} + \frac{x(-1)}{(1+x^2)^2} \cdot 2x$$

$$= \frac{1+x^2 - 2x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

$$f''(x) = \frac{-2x}{(1+x^2)^2} + \frac{(1-x^2)(-2)}{(1+x^2)^3} \cdot 2x$$

$$= \frac{-2x(1+x^2) - 4x(1-x^2)}{(1+x^2)^3}$$

$$= \frac{2x(-3+x^2)}{(1+x^2)^3}$$

x-intercepts: $f(x) = 0 \Rightarrow x = 0.$

critical points: $f'(x) = 0 \Rightarrow \frac{1-x^2}{(1+x^2)^2} = 0 \Rightarrow x = \pm 1.$

$$f''(1) = \frac{2(-3+1)}{(1+1)^3} = \frac{-4}{8} = -\frac{1}{2} < 0.$$

$\therefore$ at $x = 1$ there is a relative maximum.

$$f''(-1) = \frac{2(-1)(-3+1)}{(1+1)^3} = \frac{4}{8} = \frac{1}{2} > 0.$$

$\therefore$ at $x = -1$ there is a relative minimum.

<u>inflection points</u>: $f''(x) = 0 \Longrightarrow \dfrac{2x(-3+x^2)}{(1+x^2)^3} = 0$

$$x = 0, \pm\sqrt{3}$$

In addition to $f''(x) = 0$ at an inflection point we must have a changing concavity. (i.e $f''(x)$ changes sign.)

$f''(x)$   $-$   $+$   $-$   $+$

$-\sqrt{3}$   $0$   $\sqrt{3}$   $x$

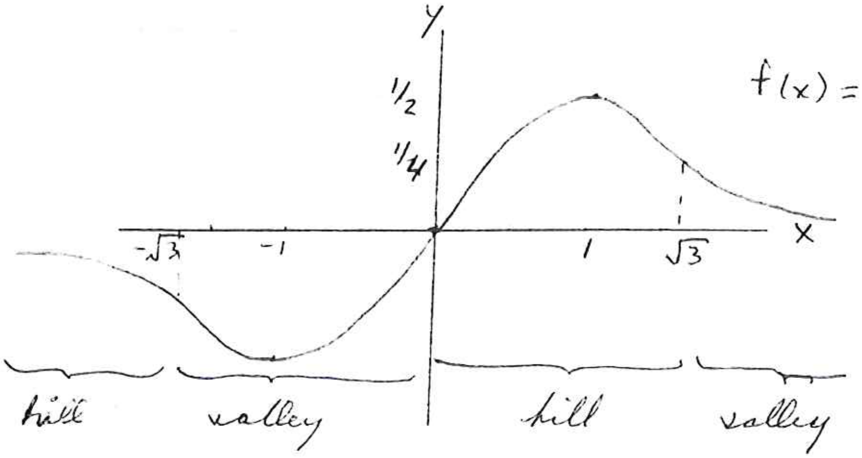
$$f''(-2) = \frac{2(-2)(-3+4)}{(1+4)^3} = \frac{-4}{125} < 0$$

$$f''(-1) = \frac{2(-1)(-3+1)}{(1+1)^3} = \frac{4}{8} = \frac{1}{2} > 0$$

$$f''(1) = \frac{2 \cdot 1(-3+1)}{(1+1)^3} = \frac{-1}{2} < 0$$

$$f''(2) = \frac{2(2)(-3+4)}{(1+4)^3} = \frac{4}{125} > 0$$

$\therefore$ inflection pts. are $(0,0)$, $\left(-\sqrt{3}, \frac{-\sqrt{3}}{4}\right)$ & $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$.



$$f(x) = \frac{x}{1+x^2}.$$

← Mar 29/90

## Example 2.

$$f(x) = x^6 - x^4$$
$$f'(x) = 6x^5 - 4x^3$$
$$f''(x) = 30x^4 - 12x^2$$

x-intercepts: $f(x) = 0 \Rightarrow x^6 - x^4 = 0$

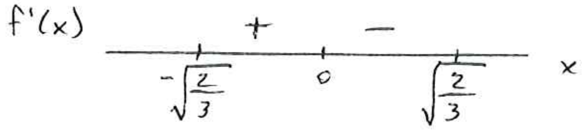
$$x^4(x^2 - 1) = 0$$
$$x = 0, \pm 1$$

critical points: $f'(x) = 0 \Rightarrow 6x^5 - 4x^3 = 0.$

$$2x^3(3x^2 - 2) = 0.$$
$$x = 0, \pm\sqrt{\frac{2}{3}}$$

$$f''(0) = 0$$

∴ 2nd derivative test doesn't tell us anything about this critical point. We therefore use the first derivative test.



$f'\left(-\frac{1}{2}\right) = 6\left(-\frac{1}{2}\right)^5 - 4\left(-\frac{1}{2}\right)^3 = -\frac{6}{32} + \frac{4}{8} = \frac{10}{32} = \frac{5}{16} > 0.$

$f'\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right)^5 - 4\left(\frac{1}{2}\right)^3 = \frac{6}{32} - \frac{4}{8} = \frac{-5}{16} < 0.$

$\therefore$ $(0, 0)$ is a relative maximum.

$f''\left(-\sqrt{\frac{2}{3}}\right) = 30\left(\frac{2}{3}\right)^2 - 12\ \frac{2}{3} = 30 \cdot \frac{4}{9} - \frac{24}{3} = \frac{16}{3} > 0.$

$\therefore$ $\left(-\sqrt{\frac{2}{3}}, -\frac{4}{27}\right)$ is a relative minimum.

$f''\left(+\sqrt{\frac{2}{3}}\right) = 30\left(\frac{2}{3}\right)^2 - 12\ \frac{2}{3} = \frac{16}{3} > 0$

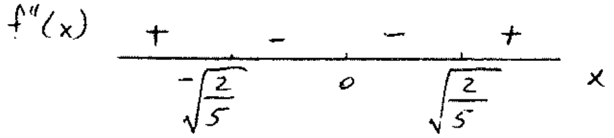
$\therefore$ $\left(\sqrt{\frac{2}{3}}, -\frac{4}{27}\right)$ is a relative minimum.

<u>inflection points</u> $f''(x) = 0 \Longrightarrow 30x^4 - 12x^2 = 0.$

$x^2(5x^2 - 2) = 0.$

$x = 0, \pm\sqrt{\frac{2}{5}}$

In addition to $f''(x) = 0$ at an inflection point we must have a changing concavity or $f''(x)$.



$f''(-1) = 30. - 12 = 18 > 0.$

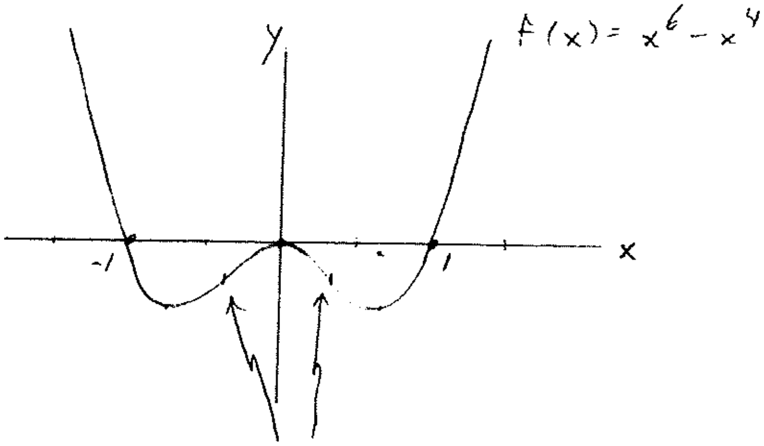
$f''\left(-\frac{1}{2}\right) = 30\left(-\frac{1}{2}\right)^4 - 12\left(-\frac{1}{2}\right)^2 = \frac{30}{16} - \frac{12}{4} = \frac{-3}{8} < 0.$

$f''\left(\frac{1}{2}\right) = -\frac{3}{8} < 0.$

$f''(1) = 18 > 0$

$\therefore$ at $x = \pm\sqrt{\frac{2}{5}}$ there are inflection points.

But $x=0$ isn't an inflection pt.. Previously we showed it is a relative maximum.



inflection pts. $\left(\pm\sqrt{\frac{2}{5}}, \frac{-12}{125}\right)$

Another test for inflection point:
first non vanishing deriv. for $X = \pm\sqrt{\frac{2}{5}}$ is $f'$, i.e. $\pm\sqrt{\frac{2}{5}}$ are not critical points. So they are inflection points.

For $X=0$ $f' = 0$

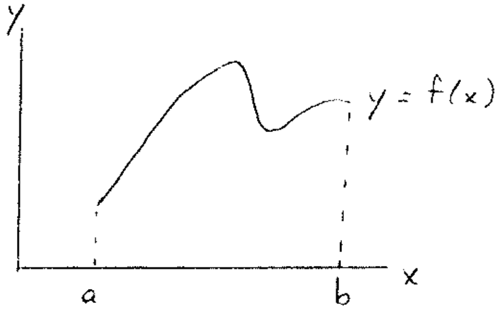
$f'' = 0$

$f''' = 120X^3 - 24X = 0$ at $x = 0$

$f^{IV} = 360X^2 - 24 \neq 0$ at $x=0$ but is even order so 0 is not inflection point. Since even order and negative then $X=0$ is a rel. maximum

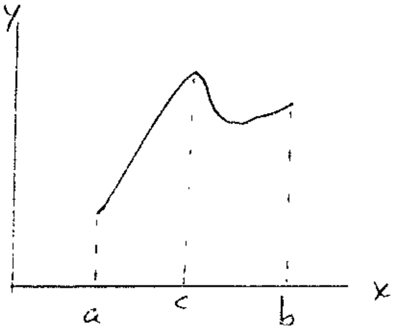
# Optimization Problems

Finding minimum + maximum values of f(x) in Interval [a, b].



The minimum and maximum values of f(x) occur either

1) at an endpoint a or b



f(a) = min. value of f(x) in interval [a, b].

f(c) = max. value of f(x) in [a, b]

2) at a relative max. or min.

Examples.

1) Find largest + smallest value of $f(x) = x - x^3$ when $x \in [0,1]$

Solution

First evaluate $f(x)$ at the endpoints.

$f(0) = 0$

$f(1) = 0$

Next search for a relative min/max.

$f'(x) = 1 - 3x^2$

Critical Points $f'(x) = 0$

$1 - 3x^2 = 0$

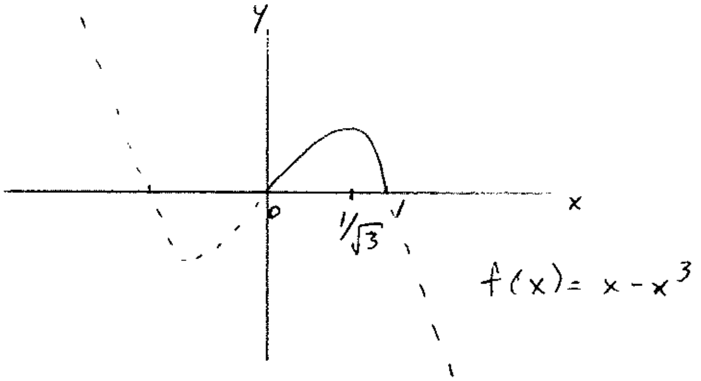
$x = \pm \frac{1}{\sqrt{3}}$

We ignore $\frac{-1}{\sqrt{3}}$ since it is outside $[0,1]$.

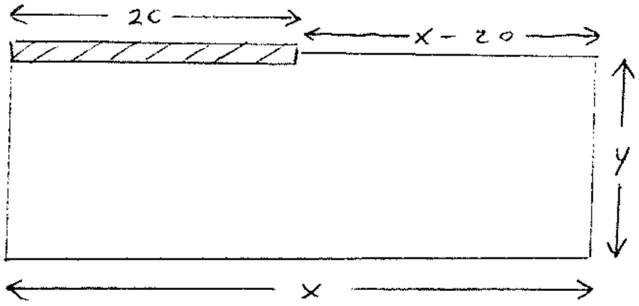
$$f(1/\sqrt{3}) = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}}$$

∴ minimum value of $f(x) = x - x^3$ is $0$ + occurs at $x = 0, 1$.

maximum " " $\frac{2}{3\sqrt{3}}$ " " $\frac{1}{\sqrt{3}}$.



2) A rancher has a 20 ft. stone wall + 400 ft. of fence. What are the dimensions of the largest area field he can enclose?



Let $x$ + $y$ be W-E + N-S dimensions of field.

$400 = 2y + x + x - 20$ = field perimeter fence

$= 2y + 2x - 20$

$420 = 2(x+y)$

$x + y = 210 \quad (1)$

Field area $A = xy \quad (2)$.

Subst. $y = 210 - x$ from (1) in (2) we get:

$A = x(210 - x)$.

$$0 = \frac{dA}{dx}$$

$= 210 - x + x(-1)$

$= 210 - 2x$.

$A'' = -2$ so first nonvanishing deriv. is even & −ve, hence max.

$x = 105$. ft

$\therefore (1) \Rightarrow y = 105$ ft. + $(2) \Rightarrow A = 105^2$ ft$^2$

$\therefore$ a square field 105' × 105' would have the largest area. We neglected testing the endpts. ($x = 0, 400$) since then obviously $A = 0$.

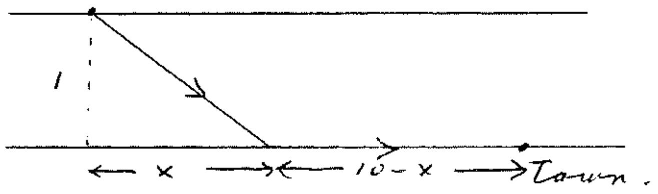
3) A telephone cable is to be laid across a $\overbrace{\text{river}}^{1\,\text{mi.}}$ to a town 10 mi. downriver.

Costs are: \$10,000 /mi on land

\$15,000/mi on water

How is cable to be laid?

<u>Solution</u>



Let x be location cable arrives on town side of river.

$$\text{Total Cost } C = \text{underwater distance} \times \frac{\text{underwater cable price}}{\text{mi.}}$$

$$+ \text{ land distance} \times \frac{\text{land cable price}}{\text{mi.}}$$

$$C(x) = \sqrt{1^2 + x^2} \times 15{,}000 + (10-x) \cdot 10{,}000$$

$$0 = \frac{dC}{dx}$$

$$= \frac{1}{2}(1+x^2)^{-1/2}\, 2x \cdot 15{,}000 - 10{,}000$$

$$x(1+x^2)^{-1/2} = \frac{10{,}000}{15{,}000}$$

$$x = (1+x^2)^{1/2}\ \frac{2}{3}$$

$$\frac{9}{4}x^2 = (1+x^2)$$

$$\frac{5}{4}x^2 = 1.$$

$\therefore\ x = \frac{2}{\sqrt{5}}$ ( Negative distances have no meaning $\therefore$ ignore $-2/\sqrt{5}$ )

$$C(2/\sqrt{5}) = \sqrt{1+\frac{4}{5}}\ \cdot 15,000 + \left(10-\frac{2}{\sqrt{5}}\right)\cdot 10,000$$

$$= \$\ 111,000.$$

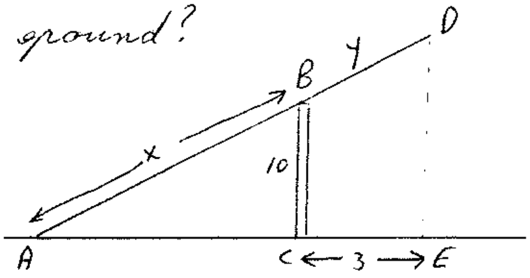
Check endpts: $C(0) = \$115,000$

$C(10) = \$151,000.$

$\therefore$ minimum cost of \$ 111,000 if cable arrives on land $\frac{2}{\sqrt{5}}$ mi. downstream from where it begins.

← apr 2/90

4) A ladder rests on a 10 ft. wall. What is minimum length of ladder if it hangs over wall 3 ft as measured on ground?



Solution

Let $AB = x$
$BD = y$

Ladder length $L = x + y$ (1)

Since triangles ABC & ADE are similar, the sides are proportional.

$$\frac{AB}{BD} = \frac{AC}{CE}$$

$$\frac{x}{y} = \frac{\sqrt{x^2-100}}{3}$$

$$y = \frac{3x}{\sqrt{x^2-100}} \qquad (2).$$

Subst. (2) in (1) $\Rightarrow$ $L(x) = x + \frac{3x}{\sqrt{x^2-100}}.$

$$0 = \frac{dL}{dx}$$

$$= 1 + \frac{3}{\sqrt{x^2-100}} + 3x\left(\frac{-1}{2}\right)(x^2-100)^{-3/2} . 2x .$$

$$0 = 1 + \frac{3}{\sqrt{x^2-100}} \mp \frac{3x^2}{(x^2-100)^{3/2}}$$

$$0 = (x^2-100)^{3/2} + 3(x^2-100) \mp 3x^2$$

$$= (x^2-100)^{3/2} + \cancel{6x^2} - 300$$

$$x^2 - 100 = (300)^{2/3}$$

$$x = \sqrt{100 + (300)^{2/3}}$$

$$= 12 \text{ ft.}$$

$$\therefore (2) \Rightarrow \quad y = \frac{3.12}{\sqrt{144-100}} = 5.4 \text{ ft.}$$

$\therefore$ ladder length $L = 17.4$ ft.

Hence minimum length ladder such that it hang 3 ft. over wall is 17.4 ft.

## Applications of Exponential Functions Section 17

Consider a population of fruit flies in a big isolated cage with lots of food. The population will increase. Assuming the growth rate $\frac{dN}{dt}$ to be directly proportional to the fly population $N$, we have

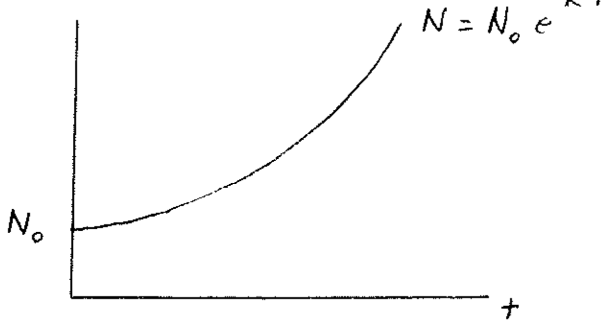
$$\frac{dN}{dt} = k\,N \qquad k = \text{constant}$$

The solution is $N = N_o e^{kt}$ since:

$$\frac{dN}{dt} = k\,N_o e^{kt} = kN.$$

Note $N(t=0) = N_o$. $\therefore$ $N_o$ is the initial population.

$\therefore$ the fly population begins at $N_o$ and grows exponentia



## Example

Suppose $N_0 = 100$

$k = \frac{1}{5}$ days$^{-1}$

$$\therefore \quad N = 100\, e^{t/5}$$

| $t$ | $N$ |
|---|---|
| 0 | 100 |
| 1 | 122 |
| 2 | 149 |
| 5 | 272 |
| 10 | 739 |
| ⋮ | ⋮ |

## Doubling Time

Let us now find the time for the population to double.

i.e. find $T$ such that $N(T) = 2N_0$.

$$N(t) = N_0 e^{kt}$$

$$2N_0 = N_0\, e^{kT}$$

$$2 = e^{kT}$$

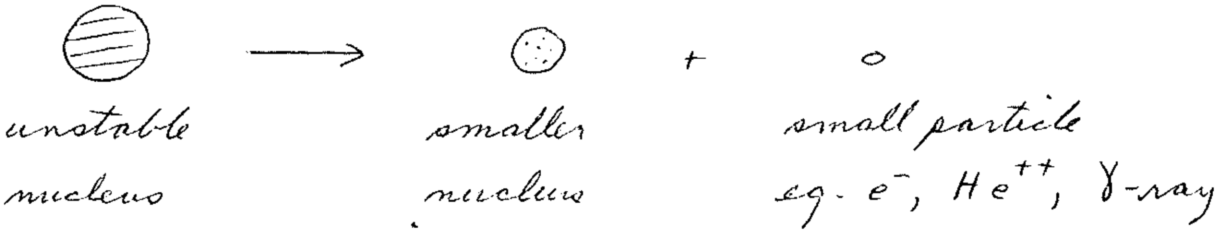
$$\ln 2 = kT$$

$$\boxed{T = \frac{\ln 2}{k}} = \frac{0.693}{k}$$

In the preceding example $k = \frac{1}{5}$. $\Rightarrow$ $T = 5 \ln 2 = 3.47$ days

$\therefore$ every 3.47 days, fruit fly population doubles.

## Radioactive Decay

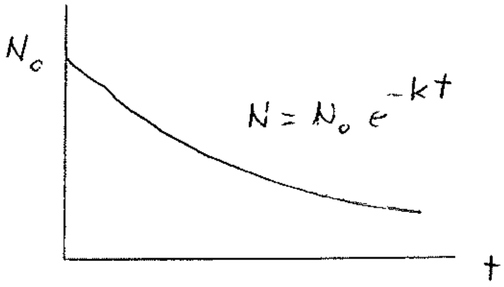


It has been found that the observed decay rate of ~~radioactive~~ unstable nuclei depends on the amount of materia

Let N be the number of unstable nuclei.

$$\Rightarrow \quad \frac{dN}{dt} = \underset{\substack{\uparrow \\ \text{since } N \text{ is decreasing}}}{-} kN$$

Solution is $N = N_0 e^{-kt}$.



## Half-life

The half-life $t_{1/2}$ is the time for radioactivity to be reduced in half.

i.e. $N(t_{1/2}) = \frac{N_0}{2}$.

$$\Rightarrow \quad \frac{N_0}{2} = N_0 e^{-k t_{1/2}}$$

$$\frac{1}{2} = e^{-k t_{1/2}}$$

$$-\ln 2 = -k t_{1/2}$$

$$\boxed{k = \frac{\ln 2}{t_{1/2}}}$$

### Example.

$Sr^{90}$ has a half-life $t_{1/2} = 29$ years

What fraction of $Sr^{90}$ released in an above ground nuclear test in 1964 is left today?

### Solution

$$k = \frac{\ln 2}{t_{1/2}} = \frac{\ln 2}{29} \text{ years}^{-1}$$

$N = N_0 e^{-kt}$

$N$ = amount of $Sr^{90}$ in 1988
$N_0$ = " " 1964
$t$ = 1988 − 1964 = 24 yrs.

$$\frac{N}{N_0} = e^{-kt}$$

$$= e^{-\frac{\ln 2}{29} \times 24}$$

$$= .56$$

∴ 56% of $Sr^{90}$ released in 1964 blast remains today.

← Apr 3/90

## Carbon Dating

← Apr 5/90

Living things absorb carbon which has two forms.

| | |
|---|---|
| $C^{12}$ | $C^{14}$ |
| stable | radioactive |
| | $t_{1/2} = 5760$ yrs. |

When an organism dies, it stops absorbing carbon and the $c^{14}$ decays. ∴ the ratio of $C^{14}/C^{12}$ decreases in time.

$$\frac{N_{C^{14}}}{N_{C^{12}}} = \frac{N_{C^{14}}(t=0)}{N_{C^{12}}} e^{-kt}$$

$R(t) = R_0 e^{-kt}$. where $R(t) \equiv \dfrac{N_{C^{14}}(t)}{N_{C^{12}}}$

## Example

Estimate age of bone having $C^{14}/C^{12}$ ratio 50 times less than that of bone of someone who just died.

## Solution

Old bone. $R(t) = R_0 e^{-kt}$ where $k = \frac{\ln 2}{t_{1/2}}$.

Assuming $C^{14}$ has been constant throughout time, we have that:

$$R_0(\text{old bone}) = R_0(\text{new bone})$$

$$\therefore \frac{R(t)\ \text{old bone}}{R_0\ \text{new bone}} = e^{-kt}$$

$$\frac{1}{50} = e^{-kt}$$

$$\ln 50 = kt$$

$$t = t_{1/2}\frac{\ln 50}{\ln 2}$$

$$= 5760 \times \frac{\ln 50}{\ln 2}$$

$\therefore t = 32,500$ years is the bone's age.

## Cooling

The rate of temperature change of an object is proportional to the difference in temperature of the object and the room.

i.e. $\frac{dT}{dt} = -k\ (T - T_R)$

$T$ = object temperature
$T_R$ = room " "

Solution is $T = T_R + (T_1 - T_R)e^{-kt}$

$T_1$ = objects' initial temperature

## Example

at $t = 0$, coffee temperature $T_1 = 80°C$.
" $t = \frac{1}{2}$ hr. " " $T = 30°C$.

1) If room temp. $T_R = 20°C$, find $k$.
2) What is time for coffee temp to cool to 40°C.

### Solution

1) $$T = T_R + (T_1 - T_R)e^{-kt}$$

$$= 20 + (80-20)e^{-kt}$$

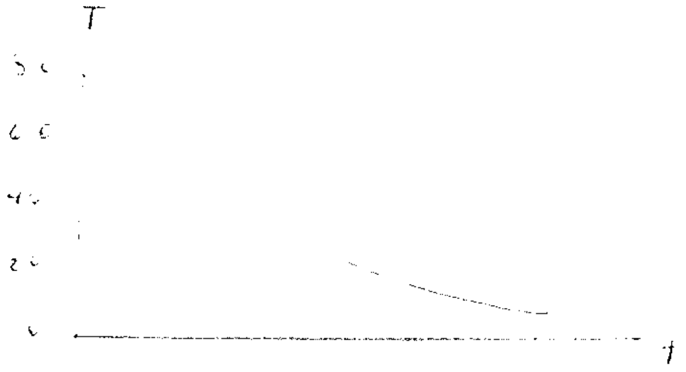
$T(1/2) = 30 \Rightarrow$ $$30 = 20 + 60e^{-k/2}$$

$$\frac{1}{6} = e^{-k/2}$$

$$-\ln 6 = -\frac{k}{2}$$

$$k = 2\ln 6.\ hr^{-1}$$

$\therefore \quad T = 20 + 60e^{-2\ln 6\, t}$



2) $T = 40 \Rightarrow \quad 40 = 20 + 60e^{-2\ln 6\, t}$

$$\frac{1}{3} = e^{-2\ln 6\, t}$$

$$\ln 3 = 2\ln 6\, t$$

$$t = \frac{\ln 3}{2\ln 6} \doteq .31 \text{ hrs.}$$

$\therefore$ coffee has cooled to 40°C in .31 hrs.

## Taylor Polynomial Approximation — Section 51

Since polynomials are easy functions to work with, we shall approximate function f(x) near point x=a as follows.

$$f(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 +$$

### Determination of $b_n$

$$b_0 = f(a)$$

$$f'(x) = b_1 + 2b_2(x-a) + 3b_3(x-a)^2 + \cdots$$

$$\therefore b_1 = f'(a)$$

$$f''(x) = 2b_2 + 3.2\,b_3(x-a) + 4\,3\,b_4(x-a)^2 +$$

$$\therefore b_2 = \frac{f''(a)}{2!}$$

$$f^{(3)}(x) = 3.2\,b_3 + 4.3\,2\,b_4(x-a) + 5\,4\,3\,b_5(x-a)^2 + \cdots$$

$$\therefore b_3 = \frac{f^{(3)}(a)}{3!}$$

$$\vdots$$

$$\therefore b_n = \frac{f^{(n)}(a)}{n!}$$

$$\Longrightarrow \boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \underline{\text{Taylor's Approximation}}}$$

## Examples

1) Expand $e^x$ about $x = 0$.

$a = 0$

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f^{(2)}(x) = e^x \qquad f''(0) = 1$

$\vdots$

$$\therefore \; e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Obviously this expansion is useful ~~only~~ when x is small ~~since~~ ~~then~~ ~~the~~ higher order terms get progressively ~~can be neglected~~ smaller.

2) Expand sin x about $x = 0$

$a = 0$

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f^{(2)}(x) = -\sin x \qquad f^{(2)}(0) = 0$

$f^{(3)}(x) = -\cos x \qquad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$\vdots \qquad \vdots$

$$\therefore \; \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Note that for small $x$, $\sin x \simeq x$.

3) Expand $\ln x$ about $x = 1$

$a = 1$

$f(x) = \ln x$ $\qquad f(1) = 0$

$f'(x) = \frac{1}{x}$ $\qquad f'(1) = 1$

$f^{(2)}(x) = \frac{-1}{x^2}$ $\qquad f^{(2)}(1) = -1$

$f^{(3)}(x) = \frac{(-1)(-2)}{x^3}$ $\qquad f^{(3)}(1) = (-1)(-2)$

$f^{(4)}(x) = \frac{(-1)(-2)(-3)}{x^4}$ $\qquad f^{(4)}(1) = (-1)(-2)(-3)$

$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$ $\qquad f^{(n)}(1) = (-1)^{n-1}(n-1)!$

$$\therefore \ln(x-1) = 0 + (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + - \ldots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + - \ldots$$

$$= \sum_{n=1}^{\infty} \frac{(x-1)^n(-1)^{n+1}}{n}$$

# Introductory Calculus For Biology Students
## Part II

William van Wijngaarden

Spring 1984.

## Course Dates

### Spring Semester

| Monday | Wednesday | Friday |
|---|---|---|
| Jan. 2 | Jan. 4 | Jan. 6 |
| Jan. 9 | Jan. 11 | Jan. 13 |
| Jan. 16 | Jan. 18 | Jan. 20 |
| Jan. 23 | Jan. 25 | Jan. 27 |
| Jan 30 | Feb. 1 | Feb. 3 |
| Feb. 6 | Feb. 8 | Feb. 10 |
| Week | Off | |
| Feb. 20 | Feb. 22 | Feb. 24 |
| Feb. 27 | Mar 1 | Mar. 3 |
| Mar. 6 | Mar. 8 | Mar 10 |
| Mar. 13 | Mar. 15 | Mar. 17 |
| Mar. 20 | Mar. 22 | Good Friday |
| Mar. 27 | Mar. 29 | Mar. 31 |

# Spring Schedule

| Section | Topic | Dates |
|---|---|---|
| 18 | Finding Antiderivatives | Jan 2, 4 |
| 22 | Basic Integration | Jan. 6, 9, 11 |
| 24 | Additional Integ. Techniques | Jan. 13, 16 |
| 30 | Trig. Fct. Integration | Jan. 18 |
| 19 | Summation + Integral | Jan. 20, 23 |
| 20 | Computing Integrals | Jan. 25 |
| 21 | Ave. values, areas, vols. | Jan. 27, 30 |
| Test | Feb. 1 | |
| 33 | Separation of Variables | Feb. 3 |
| 37 | First Order Lin. Eqns. | Feb. 6, 8 |
| 40 | Fcts. of several variables | Feb. 10 |
| 41, 42 | Partial Derivatives | Feb. 20, 22, 24 |
| 43 | Linear Regression + Curve Fitting | Feb. 27, Mar. 1 |
| Test | Mar. 3 | |
| 53 | Vectors + Vector Algebra | Mar. 6, 8 |
| 54 | Free Vectors | Mar. 10 |
| 55 | Matrices + Matrix Algebra | Mar. 13, 15 |
| 56 | Application of Matrices | Mar. 17 |
| Test | Mar. 20 | |
| 57, 58, 59 | Probability | Mar. 22, 27 |
| | Statistics | Mar. 29, 31 |

# Assignments

1) pg. 205 # 1- 25, 31-45 } odd # only
pg. 246 # 1- 19, 35-45 }

2) pg. 258 # 1- 7 odd
10
11-18, 37, 41, 43
pg. 332 # 54, 56, 58

3) pg. 217 # 11, 17
pg. 225 # $\underbrace{1-22}_{\text{odd only}}$, 25, 32, ~~35~~, 36, 38, 39

4) pg. 237 1-13 ~~odd~~
# ~~6~~, 18, 19, 20, 33

# Integration

## Definition

if $\frac{dF(x)}{dx} = f(x)$ then $F(x)$ is called the integral of $f(x)$, and is denoted by: $F(x) = \int f(x)\, dx$.

$\therefore$ integration is the inverse of differentiation.

## Examples

1) $g(x) = \frac{x^2}{2}$

$$\frac{dg}{dx} = x$$

$$\therefore \int x\, dx = \frac{x^2}{2}$$

2) $g(x) = \frac{x^{k+1}}{k+1}$ , $k \neq -1$

$$\frac{dg}{dx} = x^k$$

$$\therefore \int x^k dx = \frac{x^{k+1}}{k+1}$$

3) $g(x) = \frac{e^{ax}}{a}$ $a \neq 0$

$$\frac{dg}{dx} = e^{ax}$$

$$\therefore \int e^{ax} dx = \frac{e^{ax}}{}$$

4) $g(x) = \dfrac{\sin ax}{a} \qquad a \neq 0$

$$\frac{dg}{dx} = \cos ax$$

$$\therefore \int \cos ax \, dx = \frac{\sin ax}{a}$$

5) $g(x) = \ln x \qquad x > 0$

$$\frac{dg}{dx} = \frac{1}{x}$$

$$\therefore \int \frac{1}{x} \, dx = \ln x.$$

## Role of a Constant

Consider the following two functions.

$$g_1(x) = \frac{x^2}{2} \qquad\qquad g_2(x) = \frac{x^2}{2} + 6$$

$$\frac{dg_1}{dx} = x \qquad\qquad \frac{dg_2}{dx} = x$$

$\therefore$ if two functions have the same derivative, they need not be identical, but may differ by a constant.

If one is given a derivative, say $\dfrac{dg}{dx} = x$, then the most general solution is $g(x) = \dfrac{x^2}{2} + C$, $C$ is a constant.

When integrals are listed in tables, the constant is not written, but is understood to be there. In this course, if a constant is not written, marks will be deducted!

## Examples

1) $\int e^{ax} dx = \frac{e^{ax}}{a} + C.$

Check: $\frac{d}{dx}\left( \frac{e^{ax}}{a} + C \right) = \frac{1}{a} a e^{ax} + 0 = e^{ax}$

2) $\int \left( x - \frac{1}{x} \right) dx = \int x \, dx - \int \frac{1}{x} \, dx$

$$= \frac{x^2}{2} + C_1 - \left( \ln x + C_2 \right)$$

$$= \frac{x^2}{2} - \ln x + C \quad \text{where } C = C_1 - C_2$$

$= \text{constant}$

Check: $\frac{d}{dx}\left( \frac{x^2}{2} - \ln x + C \right) = \frac{2x}{2} - \frac{1}{x} + 0 = x - \frac{1}{x}$

3) $\int \left( e^{2x} + e^{-2x} \right) dx = \int e^{2x} dx + \int e^{-2x} dx$

$$= \frac{e^{2x}}{2} + \frac{e^{-2x}}{(-2)} + C$$

$$= \frac{1}{2}\left( e^{2x} - e^{-2x} \right) + C$$

Check: $\frac{d}{dx}\left[ \frac{1}{2}\left( e^{2x} - e^{-2x} \right) + C \right] = \frac{1}{2}\left( 2e^{2x} - (-2)e^{-2x} \right) + 0 = e^{2x} + e^{-2}$

4) $$\int (\pi^2 - x^2)\, dx = \pi^2 \int dx - \int x^2\, dx$$

$$= \pi^2 x - \frac{x^3}{3} + C.$$

Check: $$\frac{d}{dx}\left(\pi^2 x - \frac{x^3}{3} + C\right) = \pi^2 - \frac{3x^2}{3} + C = \pi^2 - x^2.$$

5) $$\int \frac{1}{x+1}\, dx = \ln|x+1| + C$$

↑ emphasize argument of logarithm > 0

# Some Integrals

Handout

| $f(x)$ | $\int f(x)\,dx$ |
|---|---|
| $c$ | $C$ · constant |
| $x^k \quad k \neq -1$ | $\frac{x^{k+1}}{k+1}$ |
| $\frac{1}{x}$ | $\ln\|x\|$ |
| $e^{ax} \quad a \neq 0$ | $\frac{e^{ax}}{a}$ |
| $\sin ax$ | $\frac{-\cos ax}{a}$ |
| $\cos ax$ | $\frac{\sin ax}{a}$ |
| $\sec^2 ax$ | $\frac{\tan ax}{a}$ |
| $\frac{1}{\sqrt{a^2-x^2}}$ | $\sin^{-1}\frac{x}{a}$ |
| $\frac{-1}{\sqrt{a^2-x^2}}$ | $\cos^{-1}\frac{x}{a}$ |
| $\frac{1}{x^2+a^2}$ | $\frac{1}{a}\tan^{-1}\frac{x}{a}$ |

# Integrals of Form $\int f(g(x))\, g'(x)\, dx$

Section 22

Consider $\int f(g(x))\, g'(x)\, dx$.

Let $u = g(x)$.

$$\frac{du}{dx} = g'(x) \quad \text{or} \quad du = g'(x)\, dx.$$

$$\Rightarrow \int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

This final form may be easier to evaluate.

## Examples

i) $\displaystyle\int \frac{1}{\sqrt{1-2x}}\, dx$.

$u = 1 - 2x$.

Solving for x we get: $u - 1 = -2x$

$$x = \frac{1-u}{2}$$

$$\frac{dx}{du} = -\frac{1}{2}$$

$$\therefore \int \frac{1}{\sqrt{1-2x}}\, dx = \int \frac{1}{\sqrt{u}} \frac{dx}{du}\, du$$

$$= \int \frac{1}{\sqrt{u}} \left(-\frac{1}{2}\right) du.$$

$$= \frac{1}{2} \int u^{-1/2}\, du.$$

$$= -\frac{1}{\;}\ 2u^{1/2} + C.$$

$$\int \frac{1}{\sqrt{1-2x}}\,dx = -\sqrt{1-2x} + c$$

2) $$\int \frac{1}{(4-3x)^{10}}\,dx$$

$u = 4-3x$

solving for x we get $u-4=-3x$

$$x = \frac{4-u}{3}$$

$$\therefore \frac{dx}{du} = -\frac{1}{3}$$

$$\int \frac{1}{(4-3x)^{10}}\,dx = \int \frac{1}{u^{10}}\,\frac{dx}{du}\,du.$$

$$= \int \frac{1}{u^{10}}\left(-\frac{1}{3}\right)du.$$

$$= -\frac{1}{3}\int u^{-10}\,du.$$

$$= -\frac{1}{3}\,\frac{u^{-9}}{-9} + C$$

$$= \frac{1}{27}(4-3x)^{-9} + C.$$

3) $\int x e^{-x^2} dx.$

$u = x^2$

$du = 2x\, dx.$

$\frac{du}{2} = x\, dx.$

$$\therefore \int x e^{-x^2} dx = \int e^{-u} \frac{du}{2}$$

$$= \frac{1}{2} \frac{e^{-u}}{-1} + C$$

$$= \frac{-1}{2} e^{-x^2} + C.$$

4) $\int \tan x \sec^2 x\, dx = \int u\, du$

$u = \tan x$

$du = \sec^2 x\, dx$

$$= \frac{u^2}{2} + C$$

$$= \frac{\tan^2 x}{2} + C$$

5) $\int \frac{x \ln(1+x^2)}{1+x^2} dx$

$u = 1 + x^2$

$du = 2x\, dx.$

$$\int \frac{x \ln(1+x^2)\, dx}{1+x^2} = \int \frac{\ln u}{u} \frac{du}{2}$$

$w = \ln u.$

$dw = \frac{1}{u}\, du.$

$$\therefore \int \frac{x \ln(1+x^2)}{1+x^2}\, dx = \frac{1}{2}\int w \, dw.$$

$$= \frac{w^2}{2} + C.$$

$$= \frac{(\ln u)^2}{2} + C.$$

$$= \frac{(\ln(1+x^2))^2}{2} + C.$$

# Integration of Rational Functions

Section 22

Consider $\int \frac{P_1(x)}{P_2(x)}\,dx$ where $P_1(x)$ + $P_2(x)$ are two polynomial functions of degree $n_1$ & $n_2$. ($n_1, n_2 > 0$).

e.g. $P(x) = x^3 + 3x^2 - 1$ has degree 3

$P(x) = (x+2)(x-3)$ has degree 2

We shall consider the case where $P_2(x)$ can be factored as follows.

$$P_2(x) = (x-a_1)^{m_1}(x-a_2)^{m_2} \dots (x-a_k)^{m_k}$$

where $m_1, m_2 \dots m_k \in N$

e.g. $P(x) = x^2 - 3x + 2$

$= (x-1)(x-2)$

Next we assume that the degree of $P_1(x)$ is less than that of $P_2(x)$. ~~i.e. $n_1 \leq n_2$~~ If not we perform a division.

e.g. $$\frac{P_1(x)}{P_2(x)} = \frac{x^3+1}{(x+2)(x+3)}.$$

$$= \frac{x^3+1}{x^2+5x+6}$$

$$
\begin{array}{r|l}
 & x - 5 \\
\hline
x^2 + 5x + 6 & x^3 + 0x^2 + 0x + 1 \\
 & \underline{x^3 + 5x^2 + 6x} \\
 & \quad -5x^2 - 6x + 1 \\
 & \quad \underline{-5x^2 - 25x - 30} \\
 & \qquad\quad 19x + 31
\end{array}
$$

$$\therefore \frac{x^3+1}{(x+2)(x+3)} = \underbrace{x - 5}_{\text{easy to integrate}} + \underbrace{\frac{19x+31}{(x+2)(x+3)}}_{\text{desired form}}$$

We now consider an integral such as $\int \frac{19x+31}{(x+2)(x+3)}\,dx$.

The integrand can be ~~re~~written in the following form.
(partial fractions decomposition)

$$\frac{19x+31}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$ where $A + B$ are to be fou

$$19x + 31 = A(x+3) + B(x+2)$$

$$x = -3 \Longrightarrow \quad -57 + 31 = -B$$
$$B = 26.$$

$$x = -2 \Longrightarrow \quad -38 + 31 = A$$
$$A = -7$$

$$\therefore \int \frac{19x+31}{(x+2)(x+3)}\,dx = \int \left( \frac{-7}{x+2} + \frac{26}{x+3} \right) dx$$

$$= -7 \ln|x+2| + 26 \ln|x+3| + C.$$

## Example 2.

$$\int \frac{x^2}{(x+2)^2(x+3)}\,dx$$

When a factor appears to a power in the denominator, additional terms are needed in the expansion.

$$\frac{x^2}{(x+2)^2(x+3)} = \frac{A_1}{x+2} + \frac{A_2}{(x+2)^2} + \frac{A_3}{x+3}.$$

$$x^2 = A_1(x+2)(x+3) + A_2(x+3) + A_3(x+2)^2$$

$$x = -2 \Rightarrow 4 = A_2$$

$$x = -3 \Rightarrow 9 = A_3$$

$$x = 0 \Rightarrow 0 = 6A_1 + 3A_2 + 4A_3$$

$$= 6A_1 + 12 + 36.$$

$$= 6A_1 + 48.$$

$$\therefore A_1 = -8$$

$$\therefore \int \frac{x^2}{(x+2)^2(x+3)}\,dx = \int \frac{-8}{x+2}\,dx + \int \frac{4}{(x+2)^2}\,dx + \int \frac{9}{x+3}\,dx.$$

$$= -8 \ln|x+2| - 4(x+2)^{-1} + 9 \ln(x+3) + C$$

# Examples of Integration of Rational Functions

1) $$\int \frac{1}{x^3 - 4x}\,dx = \int \frac{1}{x(x^2-4)}\,dx$$

$$= \int \frac{1}{x(x+2)(x-2)}\,dx$$

Let $$\frac{1}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

$$1 = A(x+2)(x-2) + B(x-2)x + C(x+2)x.$$

$x = -2 \Rightarrow \quad 1 = B\,(-4)(-2)$

$B = \frac{1}{8}$

$x = 2 \Rightarrow \quad 1 = C\;4\cdot 2$

$C = \frac{1}{8}$

$x = 0 \Rightarrow \quad 1 = A\,(2)(-2)$

$A = -\frac{1}{4}$

$$\therefore \int \frac{1}{x^3-4x}\,dx = -\frac{1}{4}\int \frac{dx}{x} + \frac{1}{8}\int \frac{dx}{x+2} + \frac{1}{8}\int \frac{dx}{x-2}$$

$$= -\frac{1}{4}\ln|x| + \frac{1}{8}\ln|x+2| + \frac{1}{8}\ln|x-2| + C$$

2) $\displaystyle\int \frac{x^3}{x^2-1}\,dx.$

$$\begin{array}{r|l} & \quad x \\ \cline{2-2} x^2-1 & x^3+0x^2+0x+0 \\ & x^3 \qquad\qquad -x \\ \cline{2-2} & \qquad\qquad\quad x \end{array}$$

$$\therefore \int \frac{x^3}{x^2-1}\,dx = \int\left(x + \frac{x}{x^2-1}\right)dx.$$

Now $\displaystyle\frac{x}{x^2-1} = \frac{x}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}.$

$$x = A(x-1) + B(x+1)$$

$x = 1 \Rightarrow 1 = 2B$

$B = \frac{1}{2}$

$x = -1 \Rightarrow -1 = -2A$

$A = \frac{1}{2}$

$$\therefore \int \frac{x^3}{x^2-1}\,dx = \int\left(x + \frac{1/2}{x+1} + \frac{1/2}{x-1}\right)dx$$

$$= \frac{x^2}{2} + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C.$$

3) $\int \frac{3x}{(x-1)^2} dx.$

Let $\frac{3x}{(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1}$

$3x = A + B(x-1)$

$x = 1 \Longrightarrow 3 = A$

$x = 0 \Longrightarrow 0 = A - B.$

$A = B.$

$\therefore A = B = 3.$

$\therefore \int \frac{3x}{(x-1)^2} dx = \int \left( \frac{3}{(x-1)^2} + \frac{3}{x-1} \right) dx$

$= -3 (x-1)^{-1} + 3 \ln|x-1| + C.$

4) $\int \frac{4x}{(x+1)^2(x-3)} dx.$

Let $\frac{4x}{(x+1)^2(x-3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3}$

$4x = A(x+1)(x-3) + B(x-3) + C(x+1)^2$

$x = -1 \Rightarrow -4 = -4B \Rightarrow B = 1.$

$$x = 3 \implies \quad 12 = 16C.$$

$$C = \frac{3}{4}.$$

$$x = 0 \implies \quad 0 = -3A - 3B + C.$$

$$= -3A - 3 + \frac{3}{4}.$$

$$3A = -\frac{9}{4}$$

$$A = -\frac{3}{4}$$

$$\therefore \int \frac{4x \quad dx}{(x+1)^2(x-3)} = -\frac{3}{4}\int \frac{dx}{x+1} + \int \frac{dx}{(x+1)^2} + \frac{3}{4}\int \frac{dx}{x-3}.$$

$$= -\frac{3}{4} \ln|x+1| - \frac{1}{x+1} + \frac{3}{4} \ln|x-3| + C.$$

# Integration By Parts

## Product Rule of Differentiation

$$\frac{d}{dx}\Big(u(x)\,v(x)\Big) = \frac{du(x)}{dx}\,v(x) + u(x)\,\frac{dv(x)}{dx}$$

$$\int \frac{d}{dx}\big(u(x)\,v(x)\big)\,dx = \int \frac{du(x)}{dx}\,v(x)\,dx + \int u(x)\,\frac{dv(x)}{dx}\,dx.$$

$$u(x)\,v(x) + \underset{\nearrow}{C} = \int v\,\frac{du}{dx}\,dx + \int u\,\frac{dv}{dx}\,dx.$$

integration constant can be omitted ~~neglected~~
if it is included in remaining integrals

$$u(x)\,v(x) = \int v\,du + \int u\,dv.$$

$$\boxed{\therefore \int u\,dv = uv - \int v\,du}$$

**Integration By Parts**

## Examples

1) $$\int \underset{\overset{\uparrow}{u}}{x}\ \underbrace{e^{x}\ dx}_{dv} .$$

Let $u = x \qquad du = dx$

$v = e^{x} \qquad dv = e^{x} dx$

$$\int x e^{x} dx = x e^{x} - \int e^{x}\ dx$$

$$= x e^{x} - e^{x} + C .$$

2) $$\int x \ln x \ dx = \int \underset{\overset{\uparrow}{u}}{\ln x}\quad \underbrace{x\, dx}_{dv}$$

$u = \ln x \qquad du = \frac{1}{x}\ dx .$

$v = \frac{x^2}{2} . \qquad dv = x\, dx$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x}\ dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x\, dx$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C .$$

3) $$\int \underbrace{\ln x}_{u}\ \underbrace{dx}_{dv.} .$$

$u = \ln x \qquad du = \frac{1}{x}\, dx$

$v = x . \qquad dv = dx .$

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} \, dx.$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C.$$

Check: $\frac{d}{dx}\left( x \ln x - x + C \right) = 1 \cdot \ln x + x \cdot \frac{1}{x} - 1 + c.$

$$= \ln x + 1 - 1$$

$$= \ln x.$$

4) $\int x^2 e^{-x} \, dx$

$u = x^2 \qquad du = 2x \, dx$

$v = -e^{-x} \qquad dv = e^{-x} dx.$

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - \int -e^{-x} \, 2x \, dx$$

$$= -x^2 e^{-x} + 2 \int x e^{-x} dx.$$

Consider $\int x e^{-x} dx.$

$u = x \qquad du = dx$

$v = -e^{-x} \qquad dv = e^{-x} dx$

$$\int x e^{-x} dx = -x e^{-x} - \int -e^{-x} \, dx.$$

$$= -x e^{-x} + \int e^{-x} dx.$$

$$\int x e^{-x} dx = -x e^{-x} - e^{-x} + C.$$

$$\therefore \int x^2 e^{-x} dx = -x^2 e^{-x} + 2\left(-x e^{-x} - e^{-x}\right) + C'$$

$$= -e^{-x}\left(x^2 + 2x + 2\right) + C'.$$

## Repeated Integration By Parts

$$\int u \, dv = uv - \int v \, du.$$

$$= uv - \int v \frac{du}{dx} dx.$$

This new integral can be integrated by parts again.

$$u_2 = \frac{du}{dx} \qquad du_2 = \frac{d}{dx}\left(\frac{du}{dx}\right) dx \equiv \frac{d^2u}{dx^2} dx$$

$$v_2 = \int v \, dx. \qquad dv_2 = v \, dx.$$

$$\therefore \int u \, dv = uv - \left\{ \frac{du}{dx} \int v \, dx - \int \underbrace{\int v \, dx}_{v_2} \underbrace{\frac{d^2u}{dx^2}}_{du_2} dx \right\}$$

$$= uv - \frac{du}{dx} \int v \, dx + \int \int v \, dx \frac{d^2u}{dx^2} dx$$

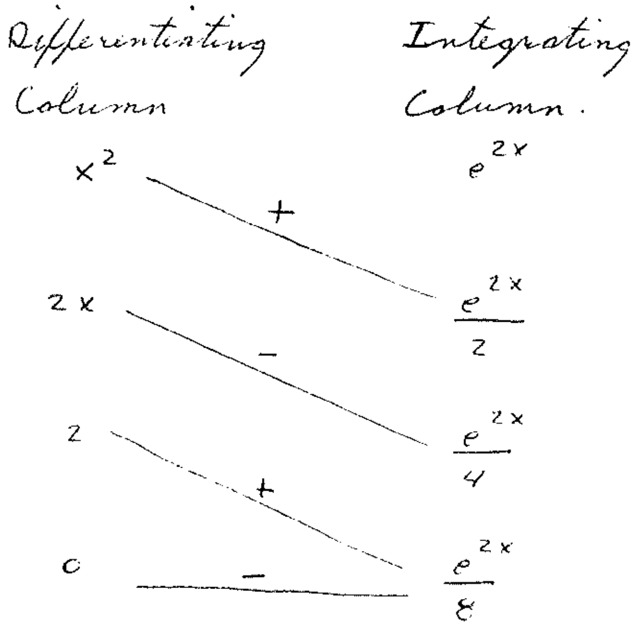
This can be repeated forever.

$$\int u\,dv = uv - \frac{du}{dx}\int v\,dx + \frac{d^2u}{dx^2}\iint v\,dx - \frac{d^3u}{dx^3}\iiint v\,dx$$

$$+ \int \iiint v\,dx\ \frac{d^4u}{dx^4}\,dx.$$

<u>Examples.</u>

1) $\int x^2 e^{2x}\,dx.$



$$\int x^2 e^{2x}\,dx = x^2\frac{e^{2x}}{2} - 2x\frac{e^{2x}}{4} + 2\frac{e^{2x}}{8} + C$$

$$= \frac{e^{2x}}{2}\left(x^2 - x + \frac{1}{2}\right) + C.$$

2) $\int e^{-2x} \sin 3x \, dx$ Diff. Column Integ. Column.

$$\begin{tikzcd}
\sin 3x \arrow[dr, dash, "+"] & e^{-2x} \\
3\cos 3x \arrow[dr, dash, "-"] & \frac{e^{-2x}}{-2} \\
-9\sin 3x \arrow[r, dash, "+"'] & \frac{e^{-2x}}{(-2)^2}
\end{tikzcd}$$

$$\int e^{-2x} \sin 3x \, dx = \frac{e^{-2x}}{-2} \sin 3x - \frac{e^{-2x}}{(-2)^2} 3 \cos 3x$$

$$+ \int -9 \sin 3x \frac{e^{-2x}}{(-2)^2} \, dx$$

$$= e^{-2x} \left( -\frac{1}{2} \sin 3x - \frac{3}{4} \cos 3x \right)$$

$$- \frac{9}{4} \int e^{-2x} \sin 3x \, dx$$

$$\left(1 + \frac{9}{4}\right) \int e^{-2x} \sin 3x \, dx = -\frac{e^{-2x}}{2} \left( \sin 3x + \frac{3}{2} \cos 3x \right)$$

$$\int e^{-2x} \sin 3x \, dx = -\frac{2}{13} e^{-2x} \left( \sin 3x + \frac{3}{2} \cos 3x \right) . + C.$$

3) $\int x^3 \ln x \, dx.$

Diff. Column Integ. Column

$$\begin{tikzcd} \ln x \arrow[dr, dash, "+"] & x^3 \\ \frac{1}{x} \arrow[r, dash, "-"] & \frac{x^4}{4} \end{tikzcd}$$

$$= \frac{x^4}{4} \ln x - \int \frac{1}{x} \frac{x^4}{4} \, dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 \, dx.$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \frac{x^4}{4} + C.$$

$$= \frac{x^4}{4} \left( \ln x - \frac{1}{4} \right) + C.$$

# Some Integration of Trig Functions

The following double angle formulas are useful.

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Examples

1) $\int \sin^2 x \, dx$ .

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - \sin^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

2) $\int \sin x \cos x \, dx = \int \frac{1}{2} \sin 2x \, dx.$

$$= \frac{-1}{4} \cos 2x + C.$$

3) $\int \sin x \cos \frac{x}{2} \, dx = \int \sin\left(\frac{x}{2} + \frac{x}{2}\right) \cos \frac{x}{2} \, dx.$

$$= \int 2 \sin \frac{x}{2} \cos \frac{x}{2} \cos \frac{x}{2} \, dx.$$

$$\int \sin x \cos \frac{x}{2} \, dx = 2 \int \sin \frac{x}{2} \cos^2 \frac{x}{2} \, dx .$$

$u = \cos \frac{x}{2}$

$du = -\frac{1}{2} \sin \frac{x}{2} \, dx$

$$= 2 \int u^2 \, (-2 \, du)$$

$$= -4 \, \frac{u^3}{3} + C .$$

$$= -\frac{4}{3} \cos^3 \frac{x}{2} + C .$$

4) $$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

$$= \int -\frac{du}{u}$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\ln |u| + C$$

$$= -\ln |\cos x| + C$$

## Integration of Inverse Trig Functions

$$\int \frac{dx}{\sqrt{1-x^2}} = \text{Arcsin } x$$

$$\int \frac{-dx}{\sqrt{1-x^2}} = \text{Arccos } x$$

$$\int \frac{dx}{1+x^2} = \text{Arctan } x$$

Examples.

1) $$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{4-4u^2}}\,2\,du.$$

$u = \frac{x}{2}$ or $x = 2u$.

$dx = 2\,du$

$$= \frac{2}{\sqrt{4}} \int \frac{du}{\sqrt{1-u^2}}$$

$$= \text{Arcsin}\, u + C.$$

$$= \text{Arcsin}\, \frac{x}{2} + C.$$

2) $$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{du}{1+u^2}$$

$u = e^x$

$du = e^x\,dx.$

$$= \text{Arctan}\, u + C$$

$$= \text{Arctan}\, e^x + C.$$

3) $$\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx$$

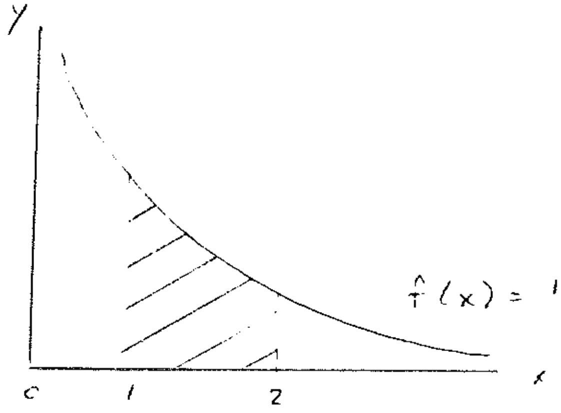
$$= \int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= x - \text{Arctan}\, x + C.$$

# Finding the Area Beneath a Curve

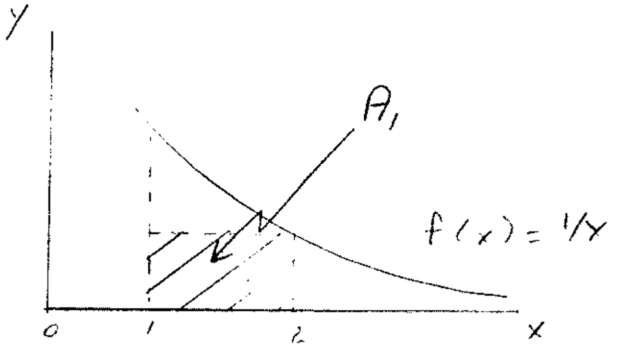
Section 14

Let's find the area between the function $f(x) = 1/x$ and the x axis in the interval $[1, 2]$.

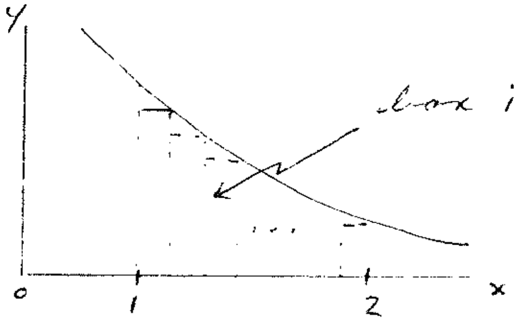


The area (shaded) can be crudely estimated by evaluating $f(x)$ at $x=2$ and multiplying by the interval width $(2-1) =$

$\therefore$ very crude area estimate $A_1 = f(\frac{2}{2}) \cdot (2-1) = \frac{1}{2}$.



To improve this estimate, divide $[1, 2]$ into subintervals.



For convenience divide the interval into $n=10$ boxes of equal width $\Delta x = \frac{2-1}{10} = 0.1$.

Box $i$ begins at $1+(i-1)\Delta x$ + ends at $x_i = 1 + i\,\Delta x$, $i=1,2\ldots$

Area of box $i$ is $f(x_i)\,\Delta x$.

Area of 10 boxes $A_{10} = \sum_{i=1}^{10} f(x_i)\,\Delta x$.

$$= \sum_{i=1}^{10} \frac{1}{x_i} \cdot 0.1$$

| $i$ | $x_i$ | $f(x_i)=1/x_i$ | $f(x_i)\,\Delta x$ |
|---|---|---|---|
| 1 | 1.1 | .909 | .0909 |
| 2 | 1.2 | .833 | .0833 |
| 3 | 1.3 | .769 | .0769 |
| 4 | 1.4 | .714 | .0714 |
| 5 | 1.5 | .667 | .0667 |
| 6 | 1.6 | .625 | .0625 |
| 7 | 1.7 | .588 | .0588 |
| 8 | 1.8 | .556 | .0556 |
| 9 | 1.9 | .526 | .0526 |
| 10 | 2.0 | .500 | .0500 |

$A_{10} = \sum_{i=1}^{10} f(x_i)\,\Delta x = .6688$

The estimate gets better, as the number of boxes increases.

i.e. as $n \to \infty$, $A_n \to$ exact area.

Using the computer we find the following results

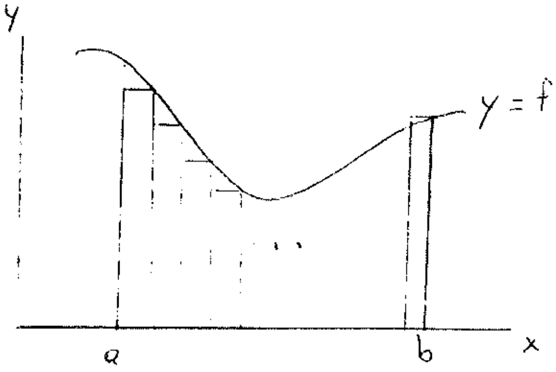
| # of boxes $n$ | $A_n$ |
|---|---|
| 1 | .50<br>~~.6667~~ |
| 10 | .6688 |
| 100 | .6907 |
| 1000 | .6929 |
| 10,000 | .6931 |

## Theorem (Fundamental Theorem of Calculus)

If $F(x) = \int f(x)\,dx$, then the area under curve $f(x)$ in interval $[a, b]$ is $F(b) - F(a)$.

This is denoted by $F(b) - F(a) = \int_a^b f(x)\,dx$.

## Proof



To find the area, under $f(x)$, we shall divide $[a, b]$ into $n$ subintervals of equal width $\Delta x = \dfrac{b-a}{n}$.

Box $i$ begins at $a + (i-1)\Delta x$ and ends at $x_i = a + i\Delta x$, $i = 1, 2, 3, \ldots n$.

$$\therefore \text{area of } n \text{ boxes } A_n = \sum_{\substack{i=1 \\ (x=a)}}^{\substack{(x=b) \\ i=n}} f(x_i)\,\Delta x.$$

$$\text{True area } A = \lim_{\substack{n\to\infty \\ \Delta x \to 0}} A_n$$

$$A = \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \sum_{\substack{i=1 \\ (x=a)}}^{\substack{(x=b) \\ n}} f(x_i)\ \Delta x .$$

But $f(x) = \dfrac{dF}{dx}$

$$= \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

$\therefore$ exact area $A = \lim\limits_{\substack{n \to \infty \\ \Delta x \to 0}} \sum\limits_{i=1}^{n} \dfrac{F(x_i + \Delta x) - F(x_i)}{\Delta x} \cdot \Delta x$

$$= \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \sum_{i=1}^{n} \left[ F(x_i + \Delta x) - F(x_i) \right]$$

$$= \lim_{n \to \infty\ \Delta x \to 0} \sum_{i=1}^{n} \left[ F(x_{i+1}) - F(x_i) \right]$$

$$= \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \Big\{ F(x_1 + \Delta x) - F(x_1) + F(x_2 + \Delta x) - F(x_2)$$

$$+ F(x_3 + \Delta x) - F(x_3) + \cdots + F(x_n + \Delta x) - F(x_n)$$

$$= \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \Big\{ \cancel{F(x_2)} - F(x_1) + \cancel{F(x_3)} - \cancel{F(x_2)}$$

$$+ F(x_4) - \cancel{F(x_3)} + \cdots + F(x_{n+1} \cancel{+ \Delta x}) - \cancel{F(x_n)} \Big]$$

$$= \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \left[ F(x_{n+1} \cancel{+ \Delta x}) - F(x_1) \right]$$

$\therefore$ exact area $A = F(b) - F(a)$.

## Examples.

1) Find area under $1/x$ in interval $[1, 2]$.

$$\text{Area} = \int_1^2 f(x)\,dx$$

$$= \int_1^2 \frac{1}{x}\,dx.$$

$$= \Big[\ln x + c\Big]_1^2$$

$$= \ln 2 + c - (\ln 1 + c)$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

$$\therefore \text{Area} = .6931$$

Note that constant $c$ drops out. $\therefore$ when evaluating a so called definite integral $\int_a^b f(x)\,dx$ one can forget about the constant.

2) Find area under $x^2 + e^{-x}$ in interval $[-1, 2]$.

$$\text{Area} = \int_{-1}^{2} (x^2 + e^{-x})\,dx.$$

$$= \left[\frac{x^3}{3} + \frac{e^{-x}}{(-1)}\right]_{-1}^{2}$$

$$\text{Area} = \frac{2^3}{3} - e^{-2} - \left(\frac{(-1)^3}{3} - e^{-(-1)}\right)$$

$$= \frac{8}{3} - e^{-2} - \left(\frac{-1}{3} - e\right)$$

$$= 3 - e^{-2} + e.$$

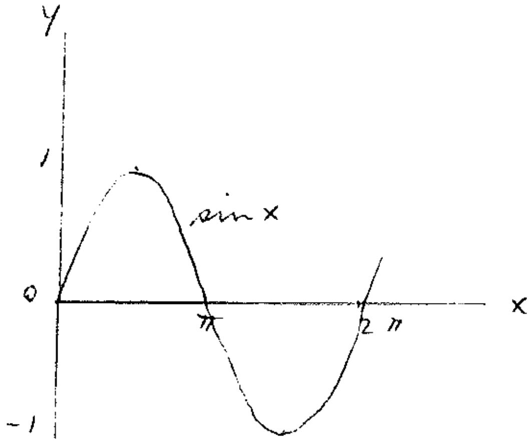
3) Find area under sin x in interval $[0, 2\pi]$.

$$\text{Area} = \int_0^{2\pi} \sin x \, dx$$

$$= \Big[-\cos x\Big]_0^{2\pi}$$

$$= (-\cos 2\pi + \cos 0)$$

$$= 0.$$



4) Find area under $f(x) = \begin{cases} 3 - x^2 & 0 \le x \le 1 \\ 4 - x & 1 < x \le 2 \\ \frac{4}{x} & x > 2. \end{cases}$

in the interval $[0, 3]$.

$$\text{Area} = \int_0^3 f(x)\, dx.$$

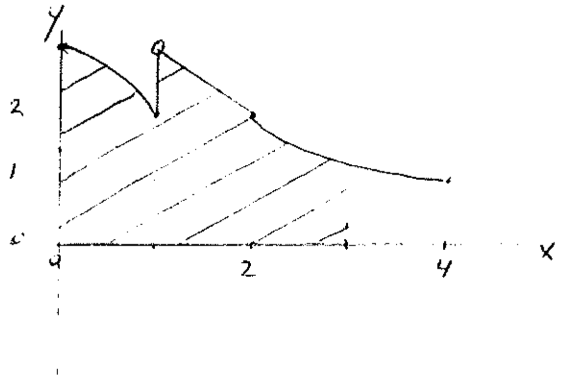
$$= \int_0^1 (3 - x^2)\, dx + \int_1^2 (4 - x)\, dx + \int_2^3 \frac{4}{x}\, dx.$$

$$= \left[3x - \frac{x^3}{3}\right]_0^1 + \left[4x - \frac{x^2}{2}\right]_1^2 + \left[4 \ln x\right]_2^3$$

$$= 3 - \frac{1}{3} + \left(8 - \frac{4}{2}\right) - \left(4 - \frac{1}{2}\right) + 4 \ln 3 - 4 \ln 2.$$

$$= \frac{8}{3} + 6 - \frac{7}{2} + 4 \ln\left(\frac{3}{2}\right)$$

$$\therefore \text{Area} = \frac{31}{6} + 4 \ln\left(\frac{3}{2}\right)$$



5) A falling stone has a velocity $v = -10t$. m/sec.
$t$ = time measured from when stone was dropped.

How far does stone fall during 1) first second
2) next second.

$$\frac{dy}{dt} = -10t$$

$$y = \int -10t \ dt.$$

In first second $y = \int_0^1 -10t \ dt = -5t^2 \Big|_0^1 = -5$ meters

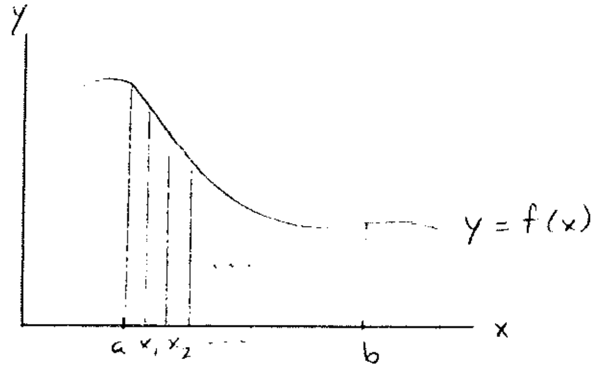
∴ in first second, stone falls 5 meters.

During next second $y = \int_1^2 -10t \ dt = -5t^2 \Big|_1^2 = -5(4-1) = -15$

∴ in 2nd. second, stone falls 15 meters.

# Average Value

Consider a function $f(x)$ on interval $[a, b]$.



Find the average value of $f(x)$ in $[a, b]$.

## Solution

Divide $[a, b]$ into $n$ subintervals of width $\Delta x = \frac{b-a}{n}$.

SubInterval $i$ starts at $a + (i-1)\Delta x$ and ends at $x_i = i\Delta x + a$,
$i = 1, 2, \dots n$

Ave. value of $f(x)$ on $[a, b]$ is

$$\bar{f} = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} f(x_i)}{n}$$

$$\bar{f} = \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \frac{\sum_{i=1}^{n} f(x_i)\,\Delta x}{b-a}$$

$$= \frac{1}{b-a} \underbrace{\lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(x_i)\,\Delta x}_{\text{area beneath } f(x) \text{ \& } x \text{ axis in } [a,b].}$$

$$\therefore \boxed{\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx}$$

Examples

1) Find average of $f(x) = x^2$ in $[1, 5]$.

$$\bar{f} = \frac{1}{5-1} \int_1^5 x^2\, dx.$$

$$= \frac{1}{4} \left. \frac{x^3}{3} \right|_1^5$$

$$= \frac{1}{4} \frac{1}{3} (125 - 1)$$

$$\therefore \bar{f} = \frac{31}{3}$$

2) World population is increasing according to $N(t) = 5 e^{.02t}$ billion. What is average population over next 50 years?

$$\bar{N} = \frac{1}{50} \int_0^{50} N(t)\, dt$$

$$= \frac{1}{50} \int_0^{50} 5 e^{.02t}\, dt$$

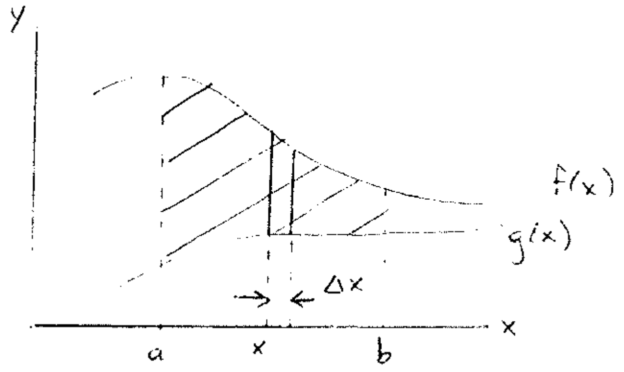
$$= \frac{1}{10} \left. \frac{e^{.02t}}{.02} \right|_0^{50}$$

$$= \frac{1}{10} \frac{1}{.02} \left( e^{.02 \times 50} - 1 \right)$$

$$\therefore \bar{N} = 11.6 \text{ billion}$$

# Area Between Two Functions

Consider 2 curves f(x) & g(x).



Find area between f & g on interval [a, b].

## Solution

Consider a vertical strip of width Δx at position x.

Height of strip = $f(x) - g(x)$

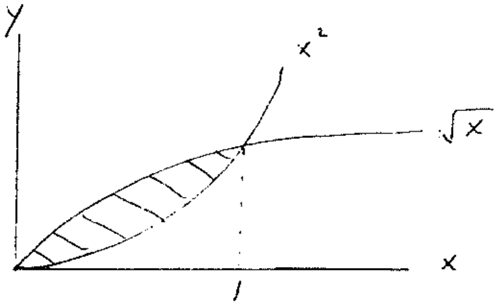
Area of strip = $(f(x) - g(x))\,\Delta x$.

$\therefore$ Area between curves = $\displaystyle\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} (f(x) - g(x))\,\Delta x$

$$\boxed{\therefore \text{ area } A = \int_a^b (f(x) - g(x))\,dx.}$$

## Examples

1) Find area between $x^2$ & $\sqrt{x}$ in $[0, 1]$.



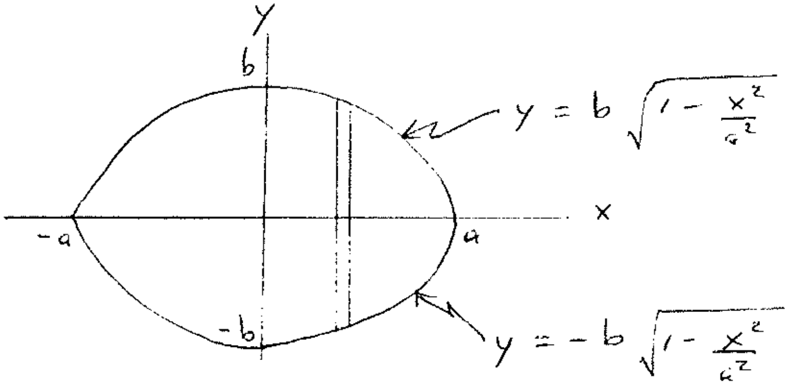
$$\text{area} = \int_0^1 (\sqrt{x} - x^2)\, dx$$

$$= \left(\frac{2}{3} x^{3/2} - \frac{x^3}{3}\right)_0^1$$

$$= \frac{2}{3} - \frac{1}{3}$$

$$= \frac{1}{3}$$

2) Find area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



$$\text{Area} = \int_{-a}^{a} \left( b\sqrt{1-\frac{x^2}{a^2}} - (-b)\sqrt{1-\frac{x^2}{a^2}} \right) dx.$$

$$= 2b \int_{-a}^{a} \sqrt{1-\frac{x^2}{a^2}} \; dx$$

$$= \frac{4b}{a} \int_{0}^{a} \sqrt{a^2-x^2} \; dx.$$

$$= \frac{4b}{a} \left[ \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} \right]_0^a$$ using Table on pg. 262

$$= \frac{4b}{a} \quad \frac{a^2}{2} \sin^{-1} \frac{a}{a}$$

$$= 2ab \sin^{-1} 1$$

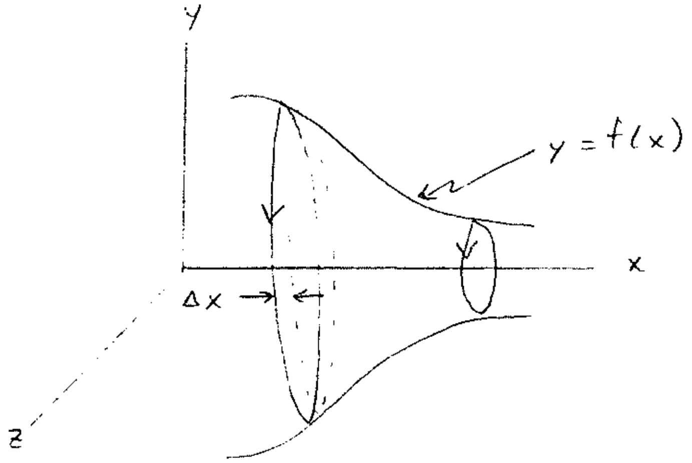
$$= \pi ab$$

$\therefore$ area of ellipse is $\pi ab$.

If $a = b$, area $= \pi a^2$. (i.e. a circle)

# Volume of a Solid of Revolution

Suppose we spin a curve f(x) about the x axis.



We wish to find the enclosed volume for $x \in [a, b]$.

Consider a slice or disk at location x of width Δx

Radius of circular disk at x is f(x).

area of " " $\pi f^2(x)$

volume of " " $\pi f^2(x) \Delta x$.

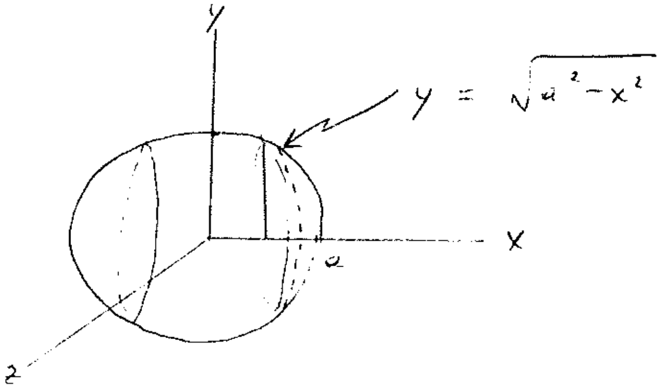
∴ volume enclosed by spinning f(x) in [a, b] is

$$\boxed{V = \int_a^b \pi f^2(x)\, dx .}$$

## Example.

1) Find the volume of a sphere.

A sphere is obtained by rotating a $\overbrace{\text{circle}}^{\text{in the xy plane}}$ about the x axis.



$$\text{Volume} = \int_{-a}^{a} \pi \left(\sqrt{a^2 - x^2}\right)^2 \, dx.$$

$$= \pi \int_{-a}^{a} (a^2 - x^2) \, dx$$

$$= 2\pi \int_{0}^{a} (a^2 - x^2) \, dx$$

$$= 2\pi \left(a^2 x - \frac{x^3}{3}\right)_0^a$$

$$= 2\pi \left( a^3 - \frac{a^3}{3}\right)$$

$\therefore V = \dfrac{4\pi a^3}{3}$ is volume of a sphere with radius

# Differential Equations

A differential equation has the form $\frac{dy}{dt} = f(t, y)$.

e.g. $\frac{dy}{dt} = \frac{t}{y^2}$

We shall solve some of them using the so called method of separation of variables.

## Separation of Variables Method

Write the differential equation in the form

$$h(y)\,\frac{dy}{dt} = g(t).$$

i.e. all y variables on left side
" t " " right "

and integrate it.

### Examples

1)
$$\frac{dy}{dt} = \frac{t}{y^2}$$

$$y^2\,\frac{dy}{dt} = t$$

$$\int y^2\,dy = \int t\,dt$$

$$\therefore \; y^3/3 = t^2/2 + C$$

2)
$$\frac{dy}{dt} = y \sin t$$

$$\int \frac{dy}{y} = \int \sin t \ dt$$

$$\ln |y| = -\cos t + C.$$

Sometimes boundary conditions are given allowing us to solve for C. When t denotes time, ~~these are also referred to as an~~ one is sometimes given the so calle initial condition ~~if one is given~~ $y(t=0)$.

3)
$$\frac{dy}{dt} = 1+y \quad \text{where} \quad y(0) = 0.$$

$$\frac{dy}{1+y} = dt.$$

$$\ln (1+y) = t + C.$$

$$(1+y) = e^{t+C}$$

$$y(0) = 0 \Longrightarrow \quad 1 = e^{C}$$

$$\therefore (1+y) = e^{t}$$

4) At high temperatures $NO_2 \rightarrow NO + \frac{1}{2} O_2$

It is found that $\frac{dy}{dt} = -.05\, y^2$ where y is the $NO_2$ concentration in moles/l, and t is time. Solve for y if $y(t=0) = y_0$.

$$\frac{dy}{dt} = -.05\, y^2.$$

$$\int \frac{dy}{y^2} = -.05 \int dt$$

$$-\frac{1}{y} = -.05t + C.$$

$$\frac{1}{y} = .05t - C.$$

$$y(t=0) = y_0 \Rightarrow \quad \frac{1}{y_0} = -C.$$

$$\therefore \quad \frac{1}{y} = .05t + \frac{1}{y_0}$$

$$\frac{1}{y} - \frac{1}{y_0} = .05t$$

# First Order Linear Differential Equations

This is an equation having the form

$$y' = \alpha(t)\, y + \beta(t)$$

where $\alpha(t)$ + $\beta(t)$ are functions.

## Name

First Order: Only the first derivative of y appears.
If $y''$ appeared it would be called 2nd order.

Linear: Only y appears on the right side.
If $y^2$ or $y^{1/3}$ appeared, equation would be nonlinear.

If $\beta = 0$, $y' = \alpha(t)\, y$. This is called a homogeneous equation.

## Examples

1) $\dfrac{dy}{dt} = -yt^2 + 6$ is a first order linear nonhomogeneo
diff. eqn.

2) $\dfrac{dy}{dt} = -yt^2$ is a first order lin. hom. diff. eqn.

3) $\frac{dy}{dt} = -y^2 t^2$ is a first order nonlinear diff. eqn.

4) $\frac{d^2y}{dt^2} + \frac{dy}{dt} = yt^2$ is a 2nd order linear diff. eqn.

## First Order Linear Homogeneous D.E.

$$y' = \alpha(t) y$$

This can be solved using separation of variables.

$$\int \frac{dy}{y} = \int \alpha(t)\, dt$$

$$\ln y = A(t) + c_1 \qquad \text{where } A(t) \equiv \int \alpha(t)\, dt.$$

$$y = e^{A(t) + c_1}$$

$$\boxed{y = C e^{A(t)}}$$ where C can be found using a boundary condition

Example.

1) Solve $\frac{dy}{dt} = 3y t^2$ where $y(0) = 5$.

$$\int \frac{dy}{y} = \int 3t^2\, dt$$

$$\ln y = t^3 + c_1$$

$$\longrightarrow y = e^{t^3 + c_1}$$

$$y(0) = 5 \Rightarrow 5 = e^{c_1}$$

$$\therefore y = 5e^{t^3}.$$

# First Order Linear Nonhomogeneous D. E.

$$y' = \alpha y + \beta \qquad (1)$$

Note that if $y_1$ + $y_2$ are solutions of (1), then $y = y_2 - y_1$ is a solution of the corresponding homogeneous D. E. $y' = \alpha y$ since

$$y' = y_2' - y_1'$$

$$= \alpha y_2 + \beta - (\alpha y_1 + \beta) \quad \text{using (1)}$$

$$= \alpha (y_2 - y_1)$$

$$\therefore y' = \alpha y$$

But we just found the general solution of $y' = \alpha y$ to be

$$y = c e^{A(t)} \quad \text{where} \quad A(t) = \int \alpha(t)\, dt.$$

$\therefore$ any solution $y_2$ of $y' = \alpha y + \beta$ can be expressed as

$$y_2 = c e^{A(t)} + y_1$$

where $y_1$ is a particular solution of $y' = \alpha y + \beta$.

## Finding a particular solution of $y' = \alpha y + \beta$.

Let $y_1 = u(t)\, e^{A(t)}$ where $u(t)$ is as yet unspecified.

$$\frac{dy_1}{dt} = \frac{du}{dt} e^{A(t)} + u(t)\, e^{A(t)} \frac{dA}{dt}$$

$$= \frac{du}{dt} e^{A(t)} + y_1\, \alpha(t) \quad \text{since } A = \int \alpha\, dt$$

Substituting $y_1'$ into $y' = \alpha y + \beta$ gives:

$$\frac{du}{dt} e^{A(t)} + y_1\, \alpha(t) = \alpha y_1 + \beta$$

$$\frac{du}{dt} e^{A(t)} = \beta.$$

$$\therefore u(t) = \int \beta(t)\, e^{-A(t)}\, dt.$$

Therefore the general solution of $y' = \alpha y + \beta$ is

$$y = u(t)\, e^{A(t)} + c\, e^{A(t)}$$

where: $A(t) = \int \alpha\, dt$

$$u(t) = \int \beta(t)\, e^{-A(t)}\, dt.$$

## Recipe For Solving $y' = \alpha y + \beta$

1) First solve $y' = \alpha y$.

2) Find particular solution $u(t) e^{A(t)}$
where $A(t) = \int \alpha \, dt$ , $u(t) = \int \beta(t) e^{-A(t)} dt$ .

$$\Rightarrow \; y = u(t) e^{A(t)} + C e^{A(t)}$$

3) Find constant $C$ using boundary condition.

### Examples

1) Solve $y' = -2y - e^{3t}$ with $y(0) = -1$.

This eqn. has form $y' = \alpha y + \beta$ with $\alpha = -2$, $\beta = -e^{3t}$.

a) First solve corresponding homogeneous equation.

$$y' = -2y$$

$$\int \frac{dy}{y} = -2 \int dt$$

$$\ln y = -2t + C_1$$

$$y = e^{-2t + c_1}$$

$$\therefore \; y = C e^{-2t} .$$

b) Find a particular solution of $y' = -2y - e^{3t}$.

This was found to be $u(t)\,e^{A(t)}$ where:

$$A(t) = \int \alpha\,dt = \int -2\,dt = -2t$$

$$u(t) = \int \beta e^{-A}\,dt = \int -e^{3t}\,e^{+2t}\,dt = -\int e^{5t}\,dt = -\frac{e^{5t}}{5}$$

We have not included constants here, since that is already taken into account by $C$, and we only need a particular solution.

$\therefore$ a particular soln. is $-\frac{e^{5t}}{5}\,e^{-2t} = -\frac{e^{3t}}{5}$.

$\therefore$ general soln. of $y' = -2y - e^{3t}$ is

$$y = -\frac{e^{3t}}{5} + C\,e^{-2t}$$

c) $y(0) = -1 \Rightarrow -1 = -\frac{1}{5} + C$

$$C = -\frac{4}{5}$$

$$\therefore y = -\frac{e^{3t}}{5} - \frac{4}{5}\,e^{-2t}$$

Check that $y' = -2y - e^{3t}$: $L.S. = y'$

$$= -\frac{3}{5}\,e^{3t} + \frac{8}{5}\,e^{-2t}$$

$$R.S. = -2\left(-\frac{e^{3t}}{5} - \frac{4}{5}\,e^{-2t}\right) - e^{3t} = \frac{-3}{5}\,e^{3t} + \frac{8}{5}\,e^{-2t} = L.S.$$

2) Solve $(1+t^2)y' = 1 - 2ty$ with $y(0)=5$.

$$y' = \frac{-2t}{1+t^2}y + \frac{1}{1+t^2}$$

$\therefore$ eqn. has form $y' = \alpha y + \beta$ with $\alpha = \frac{-2t}{1+t^2}$ + $\beta = \frac{1}{1+t^2}$.

a) First solve $y' = \frac{-2t}{1+t^2}y$.

$$\int \frac{dy}{y} = -\int \frac{2t}{1+t^2}\,dt$$

$$\ln y = -\ln(1+t^2) + C_1$$

$$y = e^{C_1 - \ln(1+t^2)}$$

$$y = \frac{C}{1+t^2}$$

b) Find a particular solution of $y' = \frac{-2t}{1+t^2}y + \frac{1}{1+t^2}$.

It is $u(t)e^{A(t)}$ where:

$$A(t) = \int \alpha\,dt = \int \frac{-2t}{1+t^2}\,dt = -\ln(1+t^2)$$

$$u(t) = \int \beta e^{-A}\,dt = \int \frac{1}{1+t^2}e^{\ln(1+t^2)}\,dt = \int \frac{1+t^2}{1+t^2}\,dt$$

$$= \int dt$$

$$= t$$

$\therefore$ a particular soln. is $y = t\, e^{-\ln(1+t^2)} = \dfrac{t}{1+t^2}$ .

$\therefore$ general solution of $y'(1+t^2) = 1 - 2ty$ is:

$$y = \frac{t}{1+t^2} + \frac{c}{1+t^2}$$

c) $y(0) = 5 \implies 5 = 0 + c$

$$c = 5.$$

$$\therefore \quad y = \frac{t+5}{t^2+1}$$

<u>Check</u> that $(1+t^2)\,y' = 1 - 2ty$

$$L.S. = (1+t^2)\left\{ \frac{1}{t^2+1} + (t+5)(-1)(t^2+1)^{-2} \cdot 2t \right\}$$

$$= \frac{1+t^2}{(1+t^2)^2} \left\{ t^2+1 - 2t(t+5) \right\}$$

$$= \frac{1}{1+t^2} \left\{ -t^2 - 10t + 1 \right\}$$

$$R.S. = 1 - 2t \; \frac{(t+5)}{t^2+1}$$

$$= \frac{1}{1+t^2} \left\{ 1 + t^2 - 2t^2 - 10t \right\}$$

$$= \frac{1}{1+t^2} \left\{ -t^2 - 10t + 1 \right\}$$

$\therefore R.S. = L.S.$

# Functions of Several Variables

$$\begin{array}{ccccc} \text{Inputs} & & & & \text{Output} \\ (x_1, x_2, x_3, \dots) & \longrightarrow & \boxed{f} & \longrightarrow & f(x_1, x_2, \dots) \end{array}$$

## Examples

1) Temperature on earth is a function of latitude $y$ + longitude $x$. i.e. $T = T(x, y)$

2) Pressure of an ideal gas depends on the gas volume $V$, temperature $T$ + mole number $n$. i.e $P = P(V, T, n)$

## Domain

This is the set of inputs for which $f$ is defined.

## Examples

1) $f(x, y) = \sqrt{9 - x^2 - y^2}$

$f$ is well defined if $9 - x^2 - y^2 \geq 0$.

$x^2 + y^2 \leq 9$.

$\therefore$ domain of $f = \{ (x, y) \mid x^2 + y^2 \leq 9 \}$.

$\therefore$ $(4, 0)$ is not in the domain since $4^2 + 0^2 \nleq 9$.

2) $f(x,y) = \ln(x+y)$

f is defined if $x + y > 0$

$\therefore$ domain of $f = \{(x,y) \mid x + y > 0\}$.

## Level Curves

A level curve is the resulting curve when one variable is kept fixed.

eg Suppose we wish to plot temperature at latitude 20° N.

$\therefore \; y = 20^\circ N \Rightarrow$ level curve $T = T(x, 20^\circ N)$

Level curves of functions having 2 $\underbrace{\text{variables}}_{}^{\text{i.e. } z = f(x,y)}$ are planar slices of the general curve. These are useful when determining a 3 dimensional plot of $z = f(x,y)$.

## Example

Graph $f(x,y) = x^2 + y^2$.

i.e. $z = x^2 + y^2$.

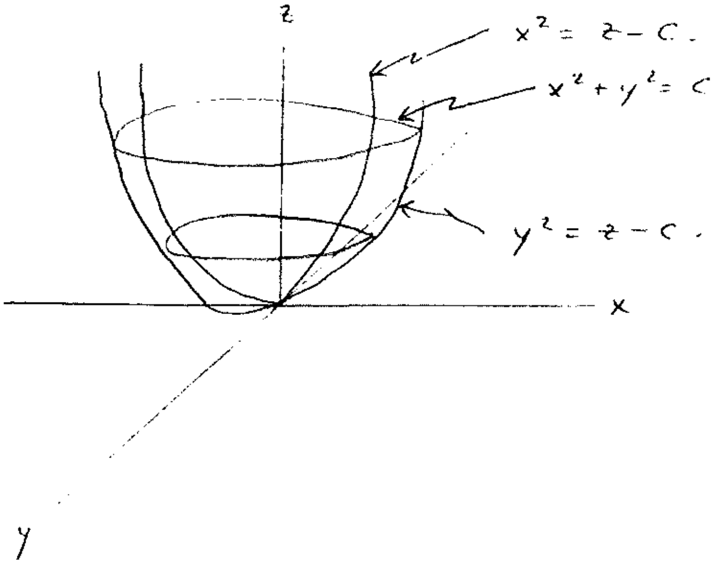
Suppose we set $z =$ const. (i.e. take a slice through plane $z = c$)

$\Rightarrow \quad x^2 + y^2 = c \quad$ i.e. a circle of radius $\sqrt{c}$, $c > 0$.

Next take a slice through the plane $x = \text{const}$.

$$\Rightarrow \quad z = c + y^2$$

$$y^2 = z - c \quad \text{i.e a parabola.}$$



## Partial Derivatives

Section 41

Consider the function $f(x, y)$, where $y$ doesn't depend on $x$ and $x$ doesn't depend on $y$. When taking the derivative of f we must specify with respect to which variable we are differentiating. We therefore introduce the partial derivative which is denoted by:

$$f_x \equiv \frac{\partial f(x,y)}{\partial x} \equiv \text{differentiate } f(x,y) \text{ w.r.t. } x \text{ holding } y \text{ constan}$$

Similarly $f_y \equiv \dfrac{\partial f(x,y)}{\partial y} \equiv$ " " $y$ " $x$ "

### Examples

1) $w = xy + y^2$

$$\frac{\partial w}{\partial x} = y$$

$$\frac{\partial w}{\partial y} = x + 2y$$

2) $f(x,y) = \ln(x + 2y)$

$$\frac{\partial f}{\partial x} = \frac{1}{x+2y}$$

$$\frac{\partial f}{\partial y} = \frac{2}{x+2y}$$

3) $f(x,y) = x \cos xy$

$$\frac{\partial f}{\partial x} = \cos xy \quad + x(-\sin xy)\, y$$

$$= \cos xy - xy \sin xy$$

$$\frac{\partial f}{\partial y} = -x \sin xy \quad x$$

$$= -x^2 \sin xy$$

4) $f(x,y,z) = \ln(xy^2z)$

$$\frac{\partial f}{\partial x} = \frac{1}{xy^2z} \quad y^2 z = \frac{1}{x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{xy^2z} \quad x z\, 2y = \frac{2}{y}$$

$$\frac{\partial f}{\partial z} = \frac{1}{xy^2z} \quad x y^2 = \frac{1}{z}$$

## Second Derivatives

One can also compute second order partial derivatives.

i.e. $f_{xx} \equiv \frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad f_{xy} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ etc.

## Example

$$f(x,y) = 3xy - xy^2 - \ln xy$$

$$f_x = \frac{\partial f}{\partial x} = 3y - y^2 - \frac{1}{xy} \cdot y = 3y - y^2 - \frac{1}{x}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{1}{x^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 3 - 2y$$

$$f_y = \frac{\partial f}{\partial y} = 3x - 2xy - \frac{1}{xy} \cdot x = 3x - 2xy - \frac{1}{y}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = -2x + \frac{1}{y^2}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 3 - 2y$$

$$\boxed{\text{Note that } f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}}$$

# Functions of 2 Variables

Section 42

A critical point of a function $z = f(x,y)$ is a solution to the equations $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$.

## Example

$$f(x,y) = x^2 + y^2$$

$$0 = \frac{\partial f}{\partial x} = 2x. \Rightarrow x = 0$$

$$0 = \frac{\partial f}{\partial y} = 2y \Rightarrow y = 0.$$

$\therefore$ a critical point is $(0,0)$.

## Second Derivative Test

Suppose $(a,b)$ is a critical point. i.e. $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$.

$\Delta(x,y) \equiv f_{xx} f_{yy} - (f_{xy})^2$ is called the discriminant

1) If $\Delta(a,b) = 0$, this test yields no information.

2) If $\Delta(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(x,y)$ has a relative minimum at $(a,b)$.

3) If $\Delta(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(x,y)$ has a relative maximum at $(a,b)$.

4) If $\Delta(a,b) < 0$, then $f(x,y)$ has a saddle point at $(a,b)$.

## Example

$f(x,y) = x^2 + y^2$

$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y$

$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial y^2} = 2 \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0.$

$\therefore \Delta(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

$= 4.$

$\therefore$ at critical point $(0,0)$, $\Delta > 0$
$f_{xx}(0,0) = 2 > 0$

$\therefore$ $(0,0)$ is a relative minimum.

# Chain Rule For Partial Derivatives

Consider the function $f(x,y)$ where $x = x(t)$ and $y = y(t)$.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Examples

1) $w = x\sqrt{y}$

$x = t^3 + 3t + 1$

$y = 3e^{-t}$

$$\frac{\partial w}{\partial x} = \sqrt{y}$$

$$\frac{\partial w}{\partial y} = x\,\frac{1}{2}\,y^{-1/2}$$

$$\frac{dx}{dt} = 3t^2 + 3$$

$$\frac{dy}{dt} = -3e^{-t}$$

$$\therefore \frac{dw}{dt} = \sqrt{y}\,(3t^2+3) + \frac{xy^{-1/2}}{2}\,(-3e^{-t})$$

$$= 3\sqrt{y}\,(t^2+1) - \frac{3}{2}\,\frac{xe^{-t}}{\sqrt{y}}$$

2) What is the rate at which a cylinder's volume is increasing if a) its radius increases at 2 cm/sec.
b) its height decreases at 1 cm/sec.
when its radius is 10 cm. and its height 20 cm.

$\therefore \frac{dr}{dt} = 2, \frac{dh}{dt} = -1$

Cylinder volume $V = \pi r^2 h$.

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

$$= 2\pi \cdot (10)(20)(2) + \pi (10)^2 (-1)$$

$$= \pi (800 - 100)$$

$$= 700 \pi \text{ cm}^3/\text{sec}.$$

$\therefore$ volume is expanding at $700\pi$ cm$^3$/sec.

# Estimating Errors in Measurement

The derivative of a function $f(x)$ is

$$\frac{df}{dx}(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\simeq \frac{\Delta f}{\Delta x} \quad \text{for small } \Delta x.$$

$$\boxed{\Delta f = \frac{df}{dx}(x)\, \Delta x.}$$

## Example

The radius of a sphere is measured to be 10 ± 0.1 cm. What is the uncertainty in the volume.

## Solution

$$V = \frac{4}{3} \pi R^3$$

$$\Delta V = \frac{dV}{dR} \Delta R$$

$$= 4\pi R^2 \Delta R$$

$$= 4\pi (10\,\text{cm})^2\ 0.1\,\text{cm}.$$

$$= 40\pi\ \text{cm}^3$$

$\therefore$ the uncertainty in volume is $40\pi\ \text{cm}^3$.

Consider now a function of 2 variables $f(x,y)$.

When $x \longrightarrow x + \Delta x \implies f \longrightarrow f + \dfrac{\partial f}{\partial x} \Delta x$

" $y \longrightarrow y + \Delta y \implies f \longrightarrow f + \dfrac{\partial f}{\partial y} \Delta y$

$\therefore$ when $\begin{cases} x \longrightarrow x + \Delta x \\ \& \, y \longrightarrow y + \Delta y \end{cases} \implies f \rightarrow f + \dfrac{\partial f}{\partial x} \Delta x + \dfrac{\partial f}{\partial y} \Delta y.$

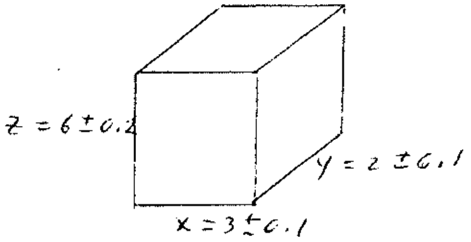
$$\boxed{\therefore \quad \Delta f = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y}$$

## Example

What is the uncertainty of volume for a box whose 3 sides are measured to be 3 ± 0.1 cm, 2 ± 0.1 cm and 6 ± 0.2 cm.?

## Solution

box volume $V = x y z$.



$$\Delta V = \frac{\partial V}{\partial x} \Delta x + \frac{\partial V}{\partial y} \Delta y + \frac{\partial V}{\partial z} \Delta z.$$

$$= y z \, \Delta x + x z \, \Delta y + x y \, \Delta z.$$

$$\frac{\Delta V}{V} = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \frac{\Delta z}{z}.$$

$$\therefore \Delta V = V\left\{ \frac{\Delta x}{x} + \frac{\Delta y}{y} + \frac{\Delta z}{z} \right\}$$

$$= 3 \times 2 \times 6 \left\{ \frac{0.1}{3} + \frac{0.1}{2} + \frac{0.2}{6} \right\}$$

$$= 4.2 \text{ cm}^3.$$

$\therefore$ uncertainty in box volume is 4.2 cm$^3$.

# Fitting a Line To Data

Frequently we like to draw a line best fitting data. Today we shall show how to find that line.

Consider a set of n data points $(x_i, y_i)$ $i = 1, 2, \ldots n$. Let $y = mx + b$ be the fitted line. We shall find m + b using the so called method of least squares. This method minimizes the following quantity.

Let $$D = \sum_{i=1}^{n} \Big( \underset{\substack{\uparrow \\ \text{observed} \\ \text{data}}}{y_i(x_i)} - \underset{\substack{\uparrow \\ \text{theoretical} \\ \text{data from fitted line}}}{y(x_i)} \Big)^2$$

$$D(m, b) = \sum_{i=1}^{n} ( y_i - m x_i - b )^2$$

i.e. The line best fitting the data corresponds to the m + b values minimizing $D(m, b)$.

At the minimum of $D(m, b)$ we have $\frac{\partial D}{\partial m} = 0$ + $\frac{\partial D}{\partial b} = 0$.

$$0 = \frac{\partial D}{\partial m} = \sum_{i=1}^{n} 2(y_i - m x_i - b)(-x_i).$$

$$= \sum_i x_i y_i - m \sum_i x_i^2 - b \sum_i x_i \qquad (1)$$

$$0 = \frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2(y_i - m x_i - b)(-1)$$

$$= \sum_{i=1}^{n} y_i - m \sum_i x_i - b n \qquad (2)$$

(1) + (2) can be solved to yield:

$$\boxed{\begin{aligned} m &= \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \left(\sum x_i\right)^2} \\[2em] b &= \frac{\sum y_i \sum x_i^2 - \sum x_i y_i \sum x_i}{n \sum x_i^2 - \left(\sum x_i\right)^2} \end{aligned}}$$

<u>Examples</u>

1) Heart rate y (beats/min) varies with age x (years).

| x | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 |
|---|---|---|---|---|---|---|---|---|
| y | 108 | 102 | 92 | 88 | 90 | 86 | 82 | 79 |

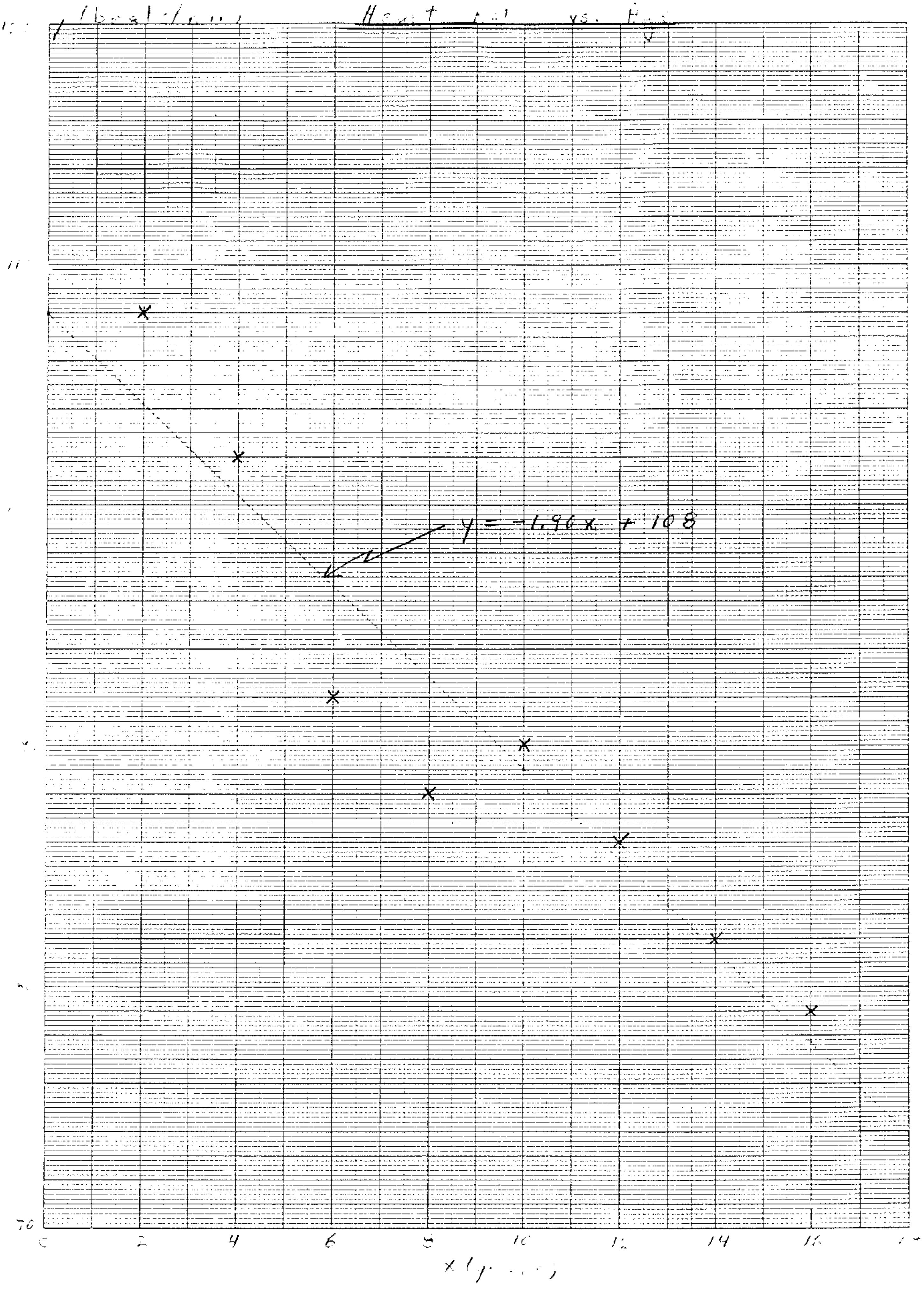
Fit a line through the data and graph the results.

| $i$ | $x_i$ | $y_i$ | $x_i^2$ | $x_i y_i$ |
|---|---|---|---|---|
| 1 | 2 | 108 | 4 | 216 |
| 2 | 4 | 102 | 16 | 408 |
| 3 | 6 | 92 | 36 | 552 |
| 4 | 8 | 88 | 64 | 704 |
| 5 | 10 | 90 | 100 | 900 |
| 6 | 12 | 86 | 144 | 1032 |
| 7 | 14 | 82 | 196 | 1148 |
| 8 | 16 | 79 | 256 | 1264. |
|  | $\sum x_i = 72$ | $\sum y_i = 727$ | $\sum x_i^2 = 816$ | $\sum x_i y_i = 6224$ |

$$m = \frac{8\,(6224) - (72)(727)}{8\,(816) - (72)^2} = -1.90$$

$$b = \frac{(727)(816) - (6224)(72)}{8\,(816) - (72)^2} = 108.$$

∴ line of best fit is $y = -1.90\,x + 108$.


y (beats/min)
Heart rate vs. Age
y = -1.46x + 108
120
110
100
90
80
70
0
2
4
6
8
10
12
14
16
18
x (years)

2) Let $y$ be percentage of light at ocean's surface that penetrates to a depth of $x$ meters. Fit an exponential to the data.

$$\text{i.e. } y = k_1 e^{-k_2 x} \qquad k_1, k_2 \text{ are unknown.}$$

$$\ln y = \ln k_1 - k_2 x .$$

$$= mx + b \quad \text{where} \quad m = -k_2 ,\ b = \ln k_1 .$$

∴ $\ln y$ replaces $y$ in our formulae

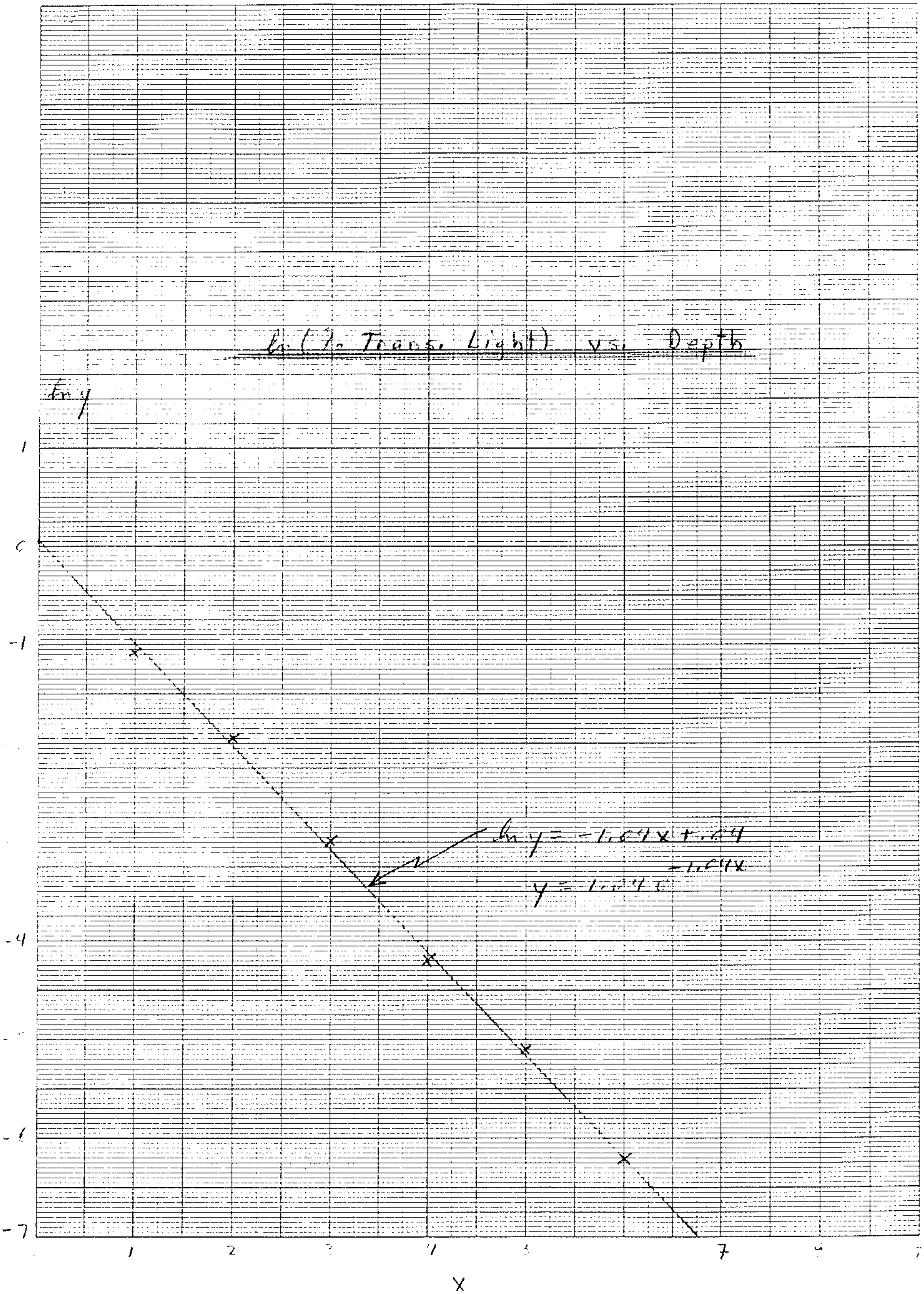
| $i$ | $x_i$ (m.) | $y_i$ | $\ln y_i$ | $x_i^2$ | $x_i \ln y_i$ |
|---|---|---|---|---|---|
| 1 | 1 | .34 | −1.079 | 1 | −1.079 |
| 2 | 2 | .14 | −1.966 | 4 | −3.932 |
| 3 | 3 | .05 | −2.996 | 9 | −8.988 |
| 4 | 4 | .015 | −4.200 | 16 | −16.800 |
| 5 | 5 | .006 | −5.116 | 25 | −25.580 |
| 6 | 6 | .002 | −6.215 | 36 | −37.290 |
|  | $\Sigma x_i = 21$ |  | $\Sigma \ln y_i = -21.57$ | $\Sigma x_i^2 = 91$ | $\Sigma x_i \ln y_i = -93.67$ |

$$m = \frac{6\,(-93.67) - 21\,(-21.57)}{6\,(91) - (21)^2} = -1.04$$

$$b = \frac{(-21.57)(91) - (-93.67)(21)}{6\,(91) - (21)^2} = .04.$$

$$\therefore \ln y = -1.04\,x + .04$$

$$y = e^{.04 - 1.04x} = 1.04\,e^{-1.04x} .$$


ln (% Trans. Light) vs. Depth
ln y
1
0
-1
-4
-
-6
-7
1
2
3
4
5
7
X
ln y = -1.64x + .64
y = 1.64 e^-1.64x

# Vectors

An n dimensional vector is an ordered group of n real numbers denoted by

$$\vec{x} = (x_1, x_2, \cdots, x_n)$$

By ordered we mean $(x_1, x_2, \cdots, x_n) \neq (x_2, x_1, \cdots, x_n)$.

## Examples

1) $\vec{x}$ = ( # women in math, # men in math)

2) $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$

$x_1$ = # people aged 0-20
$x_2$ " " 21-40
$x_3$ " " 41-60
$x_4$ " " 61-80
$x_5$ " " 81-100

## Definition of Vector Addition

If $\vec{x} = (x_1, x_2, \cdots, x_n)$ + $\vec{y} = (y_1, y_2, \cdots, y_n)$ then

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n).$$

# Definition of Scalar Multiplication of a Vector

If $\vec{x} = (x_1, x_2, \dots, x_n)$ and $a \in R$ then

$$a\vec{x} = (ax_1, ax_2, \dots, ax_n).$$

## Properties

1) $\vec{x} + \vec{o} = \vec{o} + \vec{x} = \vec{x}$ where $\vec{o} = (o, o, \dots, o)$

2) $\vec{x} + (-\vec{x}) = \vec{o}$

3) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$

4) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

5) $o . \vec{x} = \vec{o}$ and $1 . \vec{x} = \vec{x}$

6) $(a + b)\vec{x} = a\vec{x} + b\vec{x}$ $a, b \in R.$

7) $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$

## Proof of 2

$$\begin{aligned}\vec{x} + (-\vec{x}) &= (x_1, x_2, \dots, x_n) + (-x_1, -x_2, \dots, -x_n)\\ &= (x_1 - x_1, x_2 - x_2, \dots, x_n - x_n)\\ &= (o, o, \dots, o)\\ &= \vec{o}\end{aligned}$$

## Examples

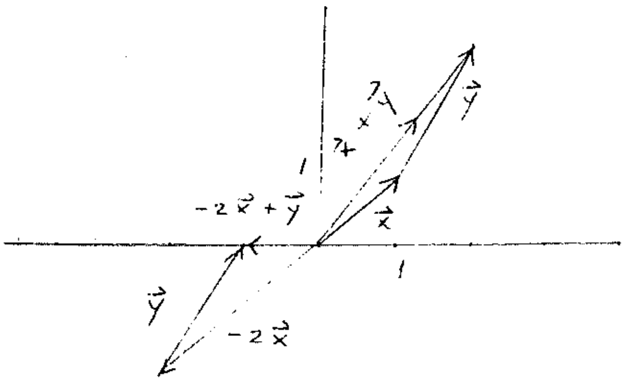
1) $\vec{x} = (1,1)$

$\vec{y} = (1,2)$

Find $\vec{x} + \vec{y}$ & $-2\vec{x} + \vec{y}$.

$$\vec{x} + \vec{y} = (1,1) + (1,2) = (2,3)$$

$$-2\vec{x} + \vec{y} = -2(1,1) + (1,2) = (-2,-2) + (1,2) = (-1,0)$$



2) Solve for $a$ & $b$ if $\vec{x} = 2\vec{y} - 3\vec{z}$ where $\vec{x} = (2,3)$, $\vec{y} = (4, a)$, $\vec{z} = (b, 1)$.

$$\vec{x} = 2\vec{y} - 3\vec{z}$$

$$(2,3) = 2(4,a) - 3(b,1)$$

$$= (8, 2a) + (-3b, -3)$$

$$= (8-3b, 2a-3)$$

$\therefore\ 2 = 8 - 3b$ and $3 = 2a - 3$

$-6 = -3b$ $\qquad 6 = 2a$

$\therefore\ b = 2$ & $a = 3$.

## Dot Product (Scalar Product)

If $\vec{x} = (x_1, x_2, \ldots, x_n)$ & $\vec{y} = (y_1, \ldots y_n)$ then

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

### Example

$\vec{N}$ = ( # horses, # cows, # pigs)
$\vec{P}$ = (horse value, cow value, pig value)

$\therefore$ $\vec{N} \cdot \vec{P}$ = total value of livestock

### Properties of Dot Product

1) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$

2) $(a\vec{x}) \cdot (b\vec{y}) = ab \; \vec{x} \cdot \vec{y}$

3) $\vec{x} \cdot (\vec{y} + \vec{w}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{w}.$

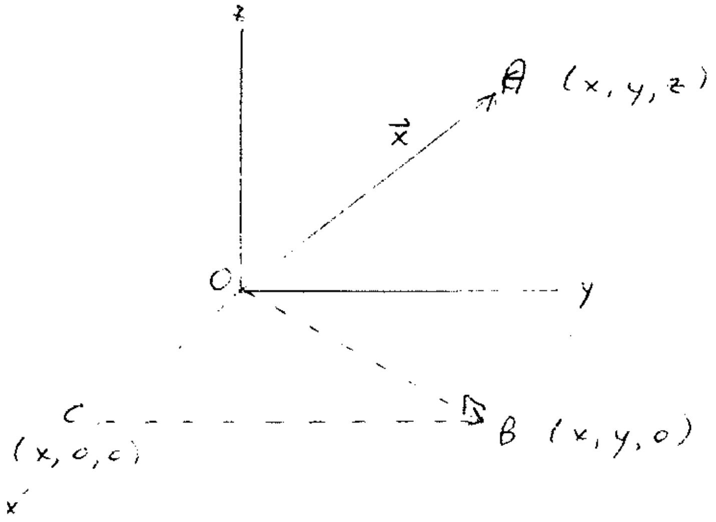
## Length of $\vec{x}$

The length of a vector denoted by $|\vec{x}|$ is defined to be

$$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}}$$

$$= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

For a vector in 3 dimensions $|\vec{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. This agrees with the result of the Pythagorean Theorem.



$$OB = \sqrt{OC^2 + CB^2}$$
$$= \sqrt{x^2 + y^2}$$

length of vector $\vec{x}$, $|\vec{x}| = OA$

$$= \sqrt{OB^2 + BA^2}$$

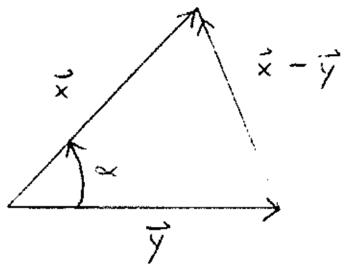
$$\therefore |\vec{x}| = \sqrt{x^2 + y^2 + z^2}$$

### Example

$\vec{x} = (1, -3, 4)$ has length $|\vec{x}| = \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{26}$

## A Useful Result

Consider vectors $\vec{x}$ + $\vec{y}$ intersecting at angle $\alpha$.



$$(\vec{x} - \vec{y})^2 = (\vec{x} - \vec{y}).(\vec{x} - \vec{y})$$

$$|\vec{x} - \vec{y}|^2 = \vec{x} \quad \vec{x} - 2\vec{x}.\vec{y} + \vec{y}.\vec{y}$$

$$= |\vec{x}|^2 + |\vec{y}|^2 - 2\vec{x}.\vec{y}. \qquad (1)$$

Using the cosine law for the above triangle, we have that

$$|\vec{x} - \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}||\vec{y}|\cos\alpha. \quad (2).$$

$$\therefore (1) + (2) \Longrightarrow \boxed{\vec{x}.\vec{y} = |\vec{x}||\vec{y}|\cos\alpha.}$$

**Note:** When $\vec{x}.\vec{y} = 0 \Rightarrow \cos\alpha = 0$ assuming $|\vec{x}| + |\vec{y}| \neq 0$

$\therefore \alpha = \frac{\pi}{2}$

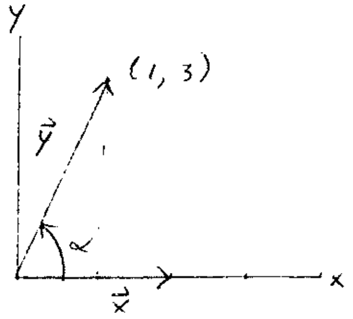
Hence we say two vectors $\vec{x}$ + $\vec{y}$ are <u>orthogonal</u> or perpendicular if $\vec{x}.\vec{y} = 0$.

## Examples

1) Show $\vec{x} \cdot \vec{y} = |\vec{x}|\,|\vec{y}| \cos\alpha$ for $\vec{x} = (2, 0)$
$\vec{y} = (1, 3)$.

$$\vec{x} \cdot \vec{y} = (2, 0) \cdot (1, 3) = 2$$

$$|\vec{x}| = \sqrt{2^2 + 0^2} = 2$$
$$|\vec{y}| = \sqrt{1^2 + 3^2} = \sqrt{10}$$



$$\therefore \cos\alpha = \frac{1}{\sqrt{1^2 + 3^2}} = \frac{1}{\sqrt{10}}$$

$$\therefore |\vec{x}|\,|\vec{y}| \cos\alpha = 2 \cdot \sqrt{10} \cdot \frac{1}{\sqrt{10}} = 2$$

$$\Rightarrow \vec{x} \cdot \vec{y} = |\vec{x}|\,|\vec{y}| \cos\alpha .$$

2) Find vector orthogonal to (3, 4) having unit length.

Let vector be $(x_1, x_2)$

Two vectors are orthogonal $\Rightarrow$ $(x_1, x_2) \cdot (3, 4) = 0$.

$$3x_1 + 4x_2 = 0$$
$$x_1 = -\frac{4}{3} x_2 \qquad (1)$$

$(x_1, x_2)$ has unit length $\Rightarrow$ $\sqrt{x_1^2 + x_2^2} = 1$

$$x_1^2 + x_2^2 = 1 \qquad (2)$$

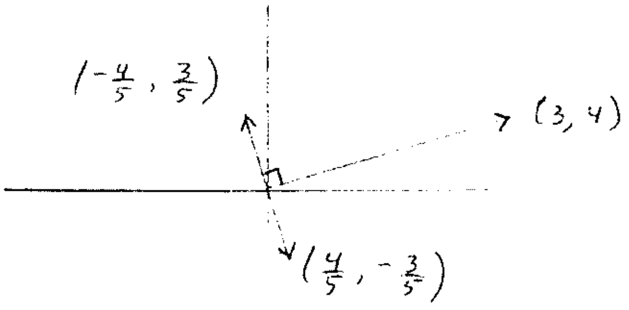
Subst. (1) into (2) $\Rightarrow$ $\frac{16}{9} x_2^2 + x_2^2 = 1.$

$$\frac{25}{9} x_2^2 = 1$$

$$x_2 = \pm \frac{3}{5}.$$

Subst. $x_2$ in (1) $\Rightarrow$ $x_1 = \mp \frac{4}{5}$.

Hence there are two desired vectors $\left(-\frac{4}{5}, \frac{3}{5}\right)$ + $\left(\frac{4}{5}, -\frac{3}{5}\right)$.



# Cross Product

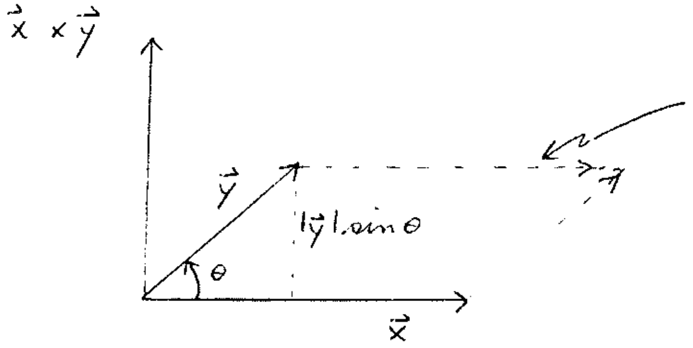
For two 3 dimensional vectors $\vec{x} = (x_1, x_2, x_3)$, $\vec{y} = (y_1, y_2, y_3)$ the cross product denoted by $\vec{x} \times \vec{y}$ is defined by

$$\vec{x} \times \vec{y} = (x_2 y_3 - x_3 y_2, -x_1 y_3 + x_3 y_1, x_1 y_2 - x_2 y_1)$$

## Properties

1) $$\left.\begin{array}{l} \vec{x} \cdot (\vec{x} \times \vec{y}) = 0 \\ \vec{y} \cdot (\vec{x} \times \vec{y}) = 0 \end{array}\right\} \Longrightarrow \quad \vec{x} \times \vec{y} \text{ is } \perp \text{ to } \vec{x} \,\&\, \vec{y}.$$

2) $|\vec{x} \times \vec{y}| = |\vec{x}|\,|\vec{y}| \sin\theta$ where $\theta$ is the angle between the 2 intersecting vectors $\vec{x}$ & $\vec{y}$.



area of parallelogram is $|\vec{x} \times \vec{y}|$.

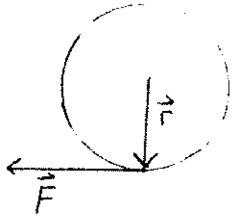
## Right Hand Rule.

a) Point fingers along $\vec{x}$. (of right hand!!)
b) Move fingers toward $\vec{y}$.

$\Longrightarrow$ Thumb points along $\vec{x} \times \vec{y}$.

Example.

Consider a force $\vec{F}$ acting on the edge of a wheel of radius $\vec{r}$.



$\vec{r} \times \vec{F}$ points into the page
= torque applied to wheel.

# Applications of Vectors

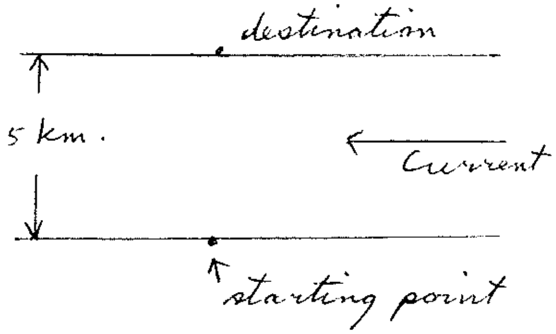
1) A boat crosses a 5 km. wide river having a 10 km/hr. current. In still water it has a top speed of 20 km/hr.

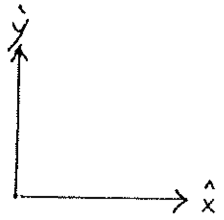
a) What direction should captain steer so as to arrive in minimum time?

b) What is resultant speed of boat w.r.t. land?

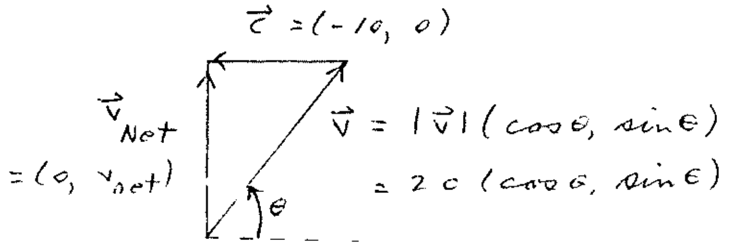
c) What is $\overbrace{\text{time}}^{\text{minimum}}$ of trip?

Solution





a) Fastest trip occurs when captain steers against current, such that current pushes him back on course.



$$\therefore \quad \vec{V}_{Net} = \vec{V} + \vec{c}$$

$$(0, v_{net}) = (20\cos\theta, 20\sin\theta) + (-10, 0)$$

$\Rightarrow \quad 0 = 20 \cos\theta - 10$

or $\cos\theta = \frac{1}{2}$

$\theta = 60°$

∴ captain steers a course 60° away from x axis toward y.

b) and $v_{net} = 20 \sin\theta + 0.$

$= 20 \sin 60°$

$= 20 \frac{\sqrt{3}}{2}$

$= 10\sqrt{3}$ km/hr.

∴ boat's net speed is $10\sqrt{3}$ km/hr.

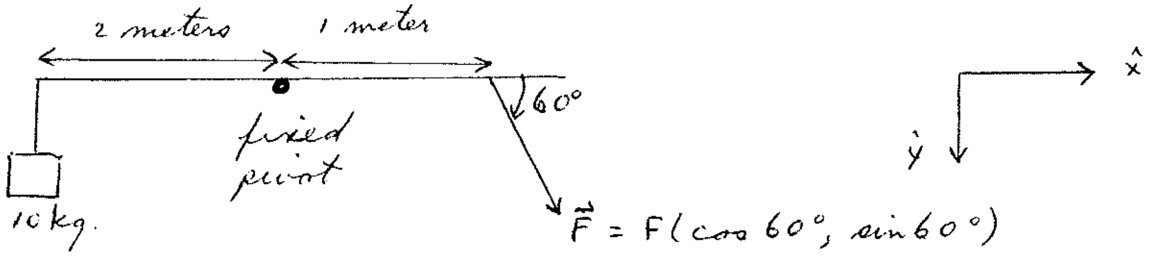
c) Minimum time of trip $= \frac{5 \text{ km.}}{10\sqrt{3} \text{ km/hr.}} = .29$ hr. $= 17$ min.

## 2) Lever Principle

A lever balances when $\sum_i F_{i\perp} \cdot r_i = 0$ where

$F_{i\perp}$ is component of force $F_i$ exerted ⊥ to lever.
$r_i$ is distance from pivot point to position where $F_i$ is exerted.

### Example.



How large must F be to balance the 10 kg. mass?

### Solution

Horizontal component of $\vec{F}$, $F \cos 60°$ does nothing to balance 10 kg.
Downward component of $\vec{F}$ is $F \sin 60°$.

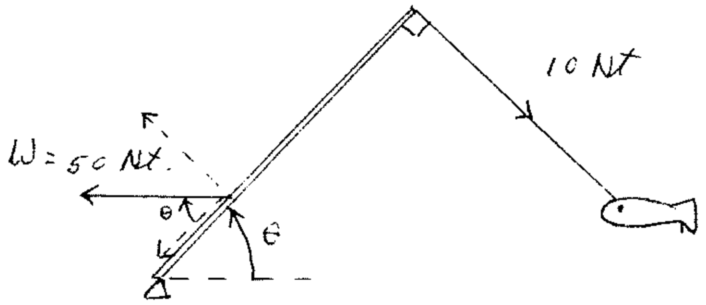
$$\therefore \text{ for balance: } \quad 10\,kg. \times \underset{\substack{\uparrow \\ \text{accel. due to} \\ \text{gravity } 10\,m/sec^2}}{g} \times 2 \text{ meters} = F \sin 60° \times 1 \text{ mete}$$

$$\therefore F = \frac{10 \times 10 \times 2}{1 \times \sin 60^\circ}$$

$$= \frac{400}{\sqrt{3}}$$

$$\therefore F = \frac{400\sqrt{3}}{3} \text{ Nts.}$$

3) A fish exerts a 10 Nt. force on a 2 meter fishing rod stuck in mud. The fisherman has his hands 1/2 meter from the pivot end and pulls with a force of 50 Nt.



Find θ so that the rod remains stationary.

Solution

Component of $\vec{W}$ along rod, $W\cos\theta$ does nothing to balance the force of the fish. This is done by component of W perpendicular to rod, $W\sin\theta$.

$$\therefore W\sin\theta \times \frac{1}{2} = 2 \times 10$$

$$\sin\theta = \frac{2 \times 2 \times 10}{50} = .8.$$

$$\Rightarrow \theta = 53^\circ$$

# Matrices

An $n \times m$ matrix is a rectangular array of numbers arranged in $n$ rows and $m$ columns. The element in the $i$th row and $j$th column of matrix $A$ is denoted $a_{ij}$.

## Examples

1) $A = \begin{pmatrix} 4 & -2 & 6 & 1 \\ 3 & 0 & 8 & 4 \end{pmatrix}$ is a $2 \times 4$ matrix

$a_{11} = 4 \quad a_{12} = -2 \quad a_{13} = 6$

$a_{21} = 3 \quad a_{22} = 0 \quad \ldots$

2) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a $3 \times 1$ matrix

## Matrix Addition

Let $A$ + $B$ be $n \times m$ matrices.
Then $A + B$ is the $n \times m$ matrix whose element in the $i$th row and $j$th column is $a_{ij} + b_{ij}$.

Note: Matrix addition is defined only for matrices having the same number of rows + columns.

eg. $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \end{pmatrix}$ makes no sense!!

## Scalar Multiplication of a Matrix

Let $A$ be an $n \times m$ matrix.
Then $cA$, $c \in R$ is an $n \times m$ matrix whose element in the ith row & jth column is $c\,a_{ij}$.

### Example

$$A = \begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -4 & 0 \end{pmatrix}$$

$$\begin{aligned} A - 2B &= \begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 & 3 \\ -1 & -4 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 3 \end{pmatrix} + \begin{pmatrix} -2 & -4 & -6 \\ 2 & 8 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -6 & -5 \\ 4 & 8 & 3 \end{pmatrix} \end{aligned}$$

## Laws of Matrix Algebra

Let $A, B$ & $C$ be $n \times m$ matrices and $a, b \in R$.

1) $A + \underset{\sim}{O} = \underset{\sim}{O} + A = A$ where $\underset{\sim}{O}$ is $n \times m$ matrix with all elements equal 0.

2) $A + (-A) = (-A) + A = \underset{\sim}{O}$.

3) $A + (B + C) = (A + B) + C$

4) $A + B = B + A$

5) $(a + b)A = aA + bA$

6) $a(A + B) = aA + aB$

7) $0 \cdot A = \underset{\sim}{0}$ and $1 \cdot A = A$.

## Matrix Multiplication

Let $A$ be an $n \times r$ matrix and $B$ an $r \times m$ matrix. Then $C = AB$ is an $n \times m$ matrix with elements

$$c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}$$

$$= \text{row } i \text{ of } A \times \text{column } j \text{ of } B$$

### Examples

1) $A = \begin{pmatrix} 4 & 0 & 2 \\ -3 & 1 & -2 \end{pmatrix}$ $\qquad B = \begin{pmatrix} 4 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 6 & 3 \end{pmatrix}$

$$AB = \begin{pmatrix} 4 & 0 & 2 \\ -3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 4 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 6 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 4\cdot 4 + 0\cdot(-1) + 2\cdot 0 & 4\cdot 2 + 0\cdot 3 + 2\cdot 6 & 4\cdot 1 + 0\cdot 2 + 2\cdot 3 \\ -3\cdot 4 + 1\cdot(-1) - 2\cdot 0 & -3\cdot 2 + 1\cdot 3 - 2\cdot 6 & -3\cdot 1 + 1\cdot 2 - 2\cdot 3 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 20 & 10 \\ -13 & -15 & -7 \end{pmatrix}$$

Note $BA = \begin{pmatrix} 4 & 2 & 1 \\ -1 & 3 & 2 \\ 0 & 6 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 & 2 \\ -3 & 1 & -2 \end{pmatrix}$ isn't defined!

2) $C = \begin{pmatrix} 4 & 2 \\ -1 & 3 \\ 0 & 6 \end{pmatrix}$

$$AC = \begin{pmatrix} 4 & 0 & 2 \\ -3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ -1 & 3 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 16 & 20 \\ -13 & -15 \end{pmatrix}$$

$$CA = \begin{pmatrix} 4 & 2 \\ -1 & 3 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 4 & 0 & 2 \\ -3 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 10 & 2 & 4 \\ -13 & 3 & -8 \\ -18 & 6 & -12 \end{pmatrix}$$

$\therefore\ AC \neq CA$.

## More Laws of Matrix Algebra

Let $A, B$ + $C$ be matrices and $a, b \in R$. Then provided $A, B$ + $C$ have dimensions for which multiplication is defined, we have:

1) $A(BC) = (AB)C$

2) $A(B+C) = AB + AC$

3) $(B+C)A = BA + CA$

4) $(aA)(bB) = (ab)AB$

## Square Matrices

A square matrix has an equal number of rows and columns.

### Examples

1) $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -3 & 6 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -2 & 7 \end{pmatrix}$$

$\therefore$ even for square matrices $AB \neq BA$.

2) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A\,I = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} = A.$

$\therefore\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the $2 \times 2$ identity matrix.

The general <u>$k \times k$ identity matrix</u> has 1's as the diagonal elements and all others 0.

3) $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ $\qquad C = \frac{1}{6}\begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$

$A\,C = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore$ C is called the <u>inverse of A and is denoted by $A^{-1}$</u>.

$\Rightarrow\ A\,A^{-1} = A^{-1}A = I$

## Inverse of a 2x2 Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then one can show $A^{-1} = \dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (1).

## Determinants

The denominator of (1) is called the determinant of A which is denoted as:

$$\det A \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc$$

If $\det A = 0$, $A^{-1}$ doesn't exist. Hence the determinant determines whether $A^{-1}$ exists.

For a 3×3 matrix, the determinant is defined as follows.

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \equiv a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{2} \\ a_{31} & a_{3} \end{vmatrix}$$

# Finding the Inverse of a Matrix

The method of finding an inverse will be demonstrated with an example.

Find $A^{-1}$ of $A = \begin{pmatrix} 4 & -6 \\ 2 & -8 \end{pmatrix}$.

## Solution

$\det A = 4(-8) - 2.(-6) = 20 \neq 0.$

$\therefore$ inverse exists.

$$\left(\underbrace{\begin{array}{cc} 4 & -6 \\ 2 & -8 \end{array}}_{A} \middle| \underbrace{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}}_{I}\right) \quad \text{row } 1 \div 4 \rightarrow \text{row } 1$$

We perform operations to change $A$ to $I$ by adding rows & multiplying rows by appropriate numbers as follows.

$$\left(\begin{array}{cc|cc} 1 & -3/2 & 1/4 & 0 \\ 2 & -8 & 0 & 1 \end{array}\right) \quad 2 \times \text{row } 1 - \text{row } 2 \rightarrow \text{row } 2$$

$$\left(\begin{array}{cc|cc} 1 & -3/2 & 1/4 & 0 \\ 0 & +5 & 1/2 & -1 \end{array}\right) \quad \text{row } 2 \div 5 \rightarrow \text{row } 2$$

$$\left(\begin{array}{cc|cc} 1 & -3/2 & 1/4 & 0 \\ 0 & 1 & 1/10 & -1/5 \end{array}\right) \quad \text{row } 1 + \frac{3}{2} \text{ row } 2 \rightarrow \text{row } 1$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 2/5 & -3/10 \\ 0 & 1 & 1/10 & -1/5 \end{array}\right)$$

$$\therefore \; A^{-1} = \begin{pmatrix} 2/5 & -3/10 \\ 1/10 & -1/5 \end{pmatrix}$$

Check: $$A A^{-1} = \begin{pmatrix} 4 & -6 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} 2/5 & -3/10 \\ 1/10 & -1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## Example 2.

Find inverse of $A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 0 & 3 \\ -1 & 0 & 0 \end{pmatrix}$.

### Solution

$$\det \begin{pmatrix} 4 & -1 & 6 \\ 2 & 0 & 3 \\ -1 & 0 & 0 \end{pmatrix} = 4(0-0) + 1(0+3) + 6(0-0) = 3 \neq 0.$$

$\therefore$ inverse of matrix exists.

$$\left(\begin{array}{ccc|ccc} 4 & -1 & 6 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad (1) \div 4 \rightarrow (1).$$

$$\left(\begin{array}{ccc|ccc} 1 & -1/4 & 3/2 & 1/4 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} (2) - 2(1) \rightarrow (2) \\ (1) + (3) \rightarrow (3) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1/4 & 3/2 & 1/4 & 0 & 0 \\ 0 & 1/2 & 0 & -1/2 & 1 & 0 \\ 0 & -1/4 & 3/2 & 1/4 & 0 & 1 \end{array}\right) \quad (2) \times 2 \rightarrow (2)$$

$$\left(\begin{array}{ccc|ccc} 1 & -1/4 & 3/2 & 1/4 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & -1/4 & 3/2 & 1/4 & 0 & 1 \end{array}\right) \quad \begin{array}{l} (1) + \frac{1}{4}(2) \rightarrow (1) \\ \\ (3) + \frac{1}{4}(2) \rightarrow (3) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 3/2 & 0 & 1/2 & 1 \end{array}\right) \quad \frac{2}{3}(3) \rightarrow (3)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1/3 & 2/3 \end{array}\right) \quad (1) - \frac{3}{2}(3) \rightarrow (1)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1/3 & 2/3 \end{array}\right)$$

$$\therefore\ A^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix}$$

Check $A A^{-1} = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 0 & 3 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

## Solving Equations

Frequently one has to solve a set of n equations having n unknowns, $x_1, x_2 \cdots x_n$.

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = y_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = y_2$$
$$\vdots$$
$$a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = y_n$$

Using matrices we can rewrite this as follows.

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{Y}$$

$$A X = Y.$$

If we know $A^{-1}$ then $A^{-1} A X = A^{-1} Y$

$$X = A^{-1} Y.$$

## Examples.

1) Solve $4x_1 - 6x_2 = 1$

$2x_1 - 8x_2 = 0.$

New-Way. $\begin{pmatrix} 4 & -6 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Previously we found inverse of $A = \begin{pmatrix} 4 & -6 \\ 2 & -8 \end{pmatrix}$ to be $A^{-1} = \left( \begin{matrix} 2/5 & -3/1 \\ 1/10 & -1/5 \end{matrix} \right.$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/5 & -3/10 \\ 1/10 & -1/5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2/5 \\ 1/10 \end{pmatrix}$$

Check: $4x_1 - 6x_2 = 4 \cdot \frac{2}{5} - 6 \cdot \frac{1}{10} = 1$

$2x_1 - 8x_2 = 2 \cdot \frac{2}{5} - 8 \cdot \frac{1}{10} = 0$

Old Way $4x_1 - 6x_2 = 1$ eqn 1 ÷ 4 → eqn. 1

$2x_1 - 8x_2 = 0.$

$x_1 - \frac{3}{2}x_2 = \frac{1}{4}$ −eqn. 2 + 2 x eqn. 1 → eqn. 2

$2x_1 - 8x_2 = 0$

$x_1 - \frac{3}{2}x_2 = \frac{1}{4}$

eqn. 2 ÷ 5 → eqn. 2.

$0x_1 + 5x_2 = \frac{1}{2}$

$$x_1 - \frac{3}{2} x_2 = \frac{1}{4} \qquad \text{eqn.1} + \frac{3}{2}\,\text{eqn.2} \rightarrow \text{eqn.1}$$

$$0x_1 + x_2 = \frac{1}{10}$$

$$x_1 + 0x_2 = \frac{2}{5}$$

$$0x_1 + x_2 = \frac{1}{10}$$

Hence steps taken when finding matrix inverse are exactly the same as those taken when solving equations, except x's are not written!

2) Solve

$$4x_1 - x_2 + 6x_3 = 1$$

$$2x_1 + 3x_3 = 4$$

$$-x_1 = 2.$$

In matrix form

$$\underbrace{\begin{pmatrix} 4 & -1 & 6 \\ 2 & 0 & 3 \\ -1 & 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

Previously we found inverse of $A$ to be $A^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix}$.

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 8/3 \end{pmatrix}$$

3) Solve $x_1 + x_2 = 4$

$2x_1 + 2x_2 = 8$

In matrix form $\underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

$\det A = 2 - 2 = 0 \Rightarrow A^{-1}$ doesn't exist.

The problem is that the second equation is the first multiplied by 2. <u>Hence $\det A = 0$ means not enough information is given for there to be a single answer.</u>
All we can say is that the solution $= \{(x_1, x_2) \mid x_1 + x_2 = 4\}$.

# Probability

## Sample Space

The sample space S is the set of all possible outcomes to an experiment.

### Examples

1) When a dice is rolled once, sample space $S = \{1, 2, 3, 4, 5, 6\}$.

2) When a dice is rolled twice
$$S = \{ (1,1), (1,2), (1,3) \ldots (1,6) \\ (2,1), (2,2) \ldots \\ \vdots \\ (6,1), (6,2) \ldots (6,6) \}$$

3) When a coin is flipped twice $S = \{HH, HT, TH, TT\}$
H = heads, T = tails

4) When a child is born to carriers of Tay Sachs disease (Carrier = T+, Tay Sachs sufferer = ++)
T - normal gene
+ - defective gene

$S = \{TT, T+, +T, ++\}$.

$$\boxed{\text{Probability of event } A = \frac{\#\text{ of events } A \text{ in sample space}}{\text{total }\#\text{ of events in sample space}}}$$

$\Rightarrow 0 \le P(A) \le 1$ and $P(S) = 1$.

## Examples.

1) Probability of rolling a 7 with 2 dice is

$$= \frac{\text{\# ways of rolling 7}}{\text{\# of all possible rolls with 2 dice}}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}.$$

2) Probability of having at least 1 boy in 2 children

$= \frac{3}{4}$ since $S = \{GG, GB, BG, BB\}$.

3) Two Tay Sachs carriers have a child. What is probability it is a) sick with Tay Sachs.
b) a carrier

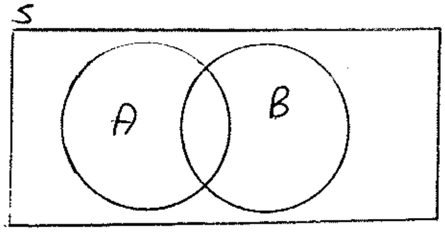
| | T | + |
|---|---|---|
| T | TT | T+ |
| + | T+ | ++ |

probability child is a carrier $= \frac{2}{4} = \frac{1}{2}$

" " has Tay Sachs $= \frac{1}{4}$.

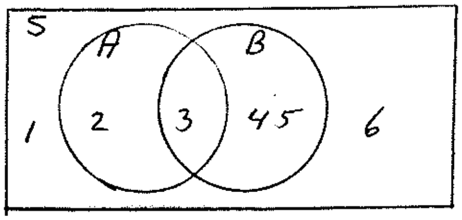
# Venn Diagrams

These are useful to picture sets.
$S$ = sample space
$A, B$ events in sample space $S$



## Examples

1) If we roll 1 dice: $S = \{1, 2, 3, 4, 5, 6\}$.
Let $A = \{2, 3\}$, $B = \{3, 4, 5\}$.



From Venn Diagram we see that $P(A \cup B) = P(A) + P(B) - P(A \cap$

Check for Example: $P(A \cup B) = P(2, 3, 4, 5) = \frac{4}{6} = \frac{2}{3}$

$$\left.\begin{aligned} P(A) &= P(2,3) = \frac{2}{6} = \frac{1}{3} \\ P(B) &= P(3,4,5) = \frac{3}{6} = \frac{1}{2} \\ P(A \cap B) &= P(3) = \frac{1}{6} \end{aligned}\right\} \Longrightarrow P(A) + P(B) - P(A \cap B) = \frac{2}{3} = P(A \cup B.$$

2) What is the probability of rolling at least one 12 in 2 rolls of the dice?

Method 1

Let A be event that a 12 is rolled on first roll.
" B " " " second ".
∴ $A \cap B$ " " " both rolls.

Prob. of at least one 12 being rolled in 2 rolls

$$= P(A \cup B)$$

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{36} + \frac{1}{36} - \frac{1}{36} \cdot \frac{1}{36}$$

$$= .055$$

Method 2

Prob. of at least one 12 rolled in 2 rolls

= 1 − prob. that no 12 is rolled in 2 rolls

$$= 1 - \frac{35}{36} \cdot \frac{35}{36}.$$

$$= \left(1 - \frac{35}{36}\right)\left(1 + \frac{35}{36}\right)$$

$$= \frac{1}{36}\left(1 + 1 - \frac{1}{36}\right)$$

$$= \frac{1}{36} + \frac{1}{36} - \frac{1}{36 \cdot 36}$$

3) How many times should dice be rolled for prob. of a 12 appearing to be better than 50%?

$$P(\text{at least one 12 in } n \text{ rolls}) = 1 - P(\text{no 12 in } n \text{ rolls})$$

$$= 1 - \left(\frac{35}{36}\right)^n$$

We want $n$ such that $1 - \left(\frac{35}{36}\right)^n > .50$

$$\left(\frac{35}{36}\right)^n < .50.$$

$$n \ln\left(\frac{35}{36}\right) < \ln .50.$$

$$n > \frac{\ln .50}{\ln\left(\frac{35}{36}\right)} = 24.6.$$

∴ dice should be rolled 25 times or more.

# Statistics

Reference: A First Course in Probability & Statistics
By Malik & Mullen
Published By Addison-Wesley

Statistics summarizes and analyzes data for the purpose of learning information about an entire population rather than about a particular individual.

## Definitions

Consider a set of N items of data denoted by $x_i$, $i = 1, 2, \dots$

1) Average $\bar{x} \equiv \dfrac{\sum_{i=1}^{N} x_i}{N}$

The average is also called the mean.

2) If the data is ordered from the lowest to the highest, then one can define the median as follows.

$$x_{Med.} = \begin{cases} x_{\frac{N+1}{2}} & \text{if } N \text{ is odd} \\ \dfrac{x_{N/2} + x_{N/2+1}}{2} & \text{if } N \text{ is even} \end{cases}$$

## Example

People in a room have the following ages. $\{10, 12, 13, 14, 11, 95\}$

Average age $\bar{x} = 25.8$.

To find median, we order data $\Rightarrow \{10, 11, 12, 13, 14, 95\}$

Median Age $= \dfrac{12 + 13}{2} = 12.5.$

3) <u>Standard Deviation</u>

standard deviation $s \equiv \sqrt{\dfrac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N-1}}$

$s$ is a measure of the spread of data from the average value $\bar{x}$.

<u>Examples.</u>

1) Consider data $\{20, 23, 24, 22, 29, 21, 23, 24, 25, 22\}$

$\bar{x} = 23.3.$

$$s = \left( \frac{\sum_{i=1}^{10} (x_i - 23.3)^2}{10-1} \right)^{1/2} = 2.50.$$

2) Consider data $\{5, 6, 32, 40, 50, 2, 1, 7, 25, 65\}$

$\bar{x} = 23.3$

$s = 22.8.$

# Example of a Frequency Table

The weights of 97, 40 day old rats was recorded as follows.

**TABLE 7.2.1**

The weights (in grams) of 97 forty-day-old male rats

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 143 | 133 | 156 | 123 | 143 | 136 | 125 | 173 | 163 | 127 |
| 127 | 139 | 164 | 149 | 163 | 174 | 159 | 181 | 167 | 158 |
| 132 | 156 | 167 | 162 | 171 | 128 | 155 | 164 | 155 | 142 |
| 135 | 128 | 155 | 144 | 129 | 154 | 139 | 145 | 145 | 131 |
| 164 | 155 | 163 | 132 | 159 | 163 | 177 | 180 | 153 | 147 |
| 137 | 149 | 133 | 148 | 133 | 150 | 149 | 170 | 147 | 145 |
| 141 | 187 | 130 | 160 | 165 | 155 | 136 | 149 | 167 | 152 |
| 130 | 180 | 147 | 117 | 154 | 145 | 145 | 145 | 158 | |
| 139 | 159 | 128 | 149 | 144 | 169 | 164 | 133 | 149 | |
| 143 | 157 | 139 | 130 | 131 | 173 | 181 | 152 | 132 | |

A frequency table is a plot of rat number versus weight. The weight axis is divided into a convenient number of intervals as follows.

\# data points $N = 97$

lowest data point = 117

highest " " = 187

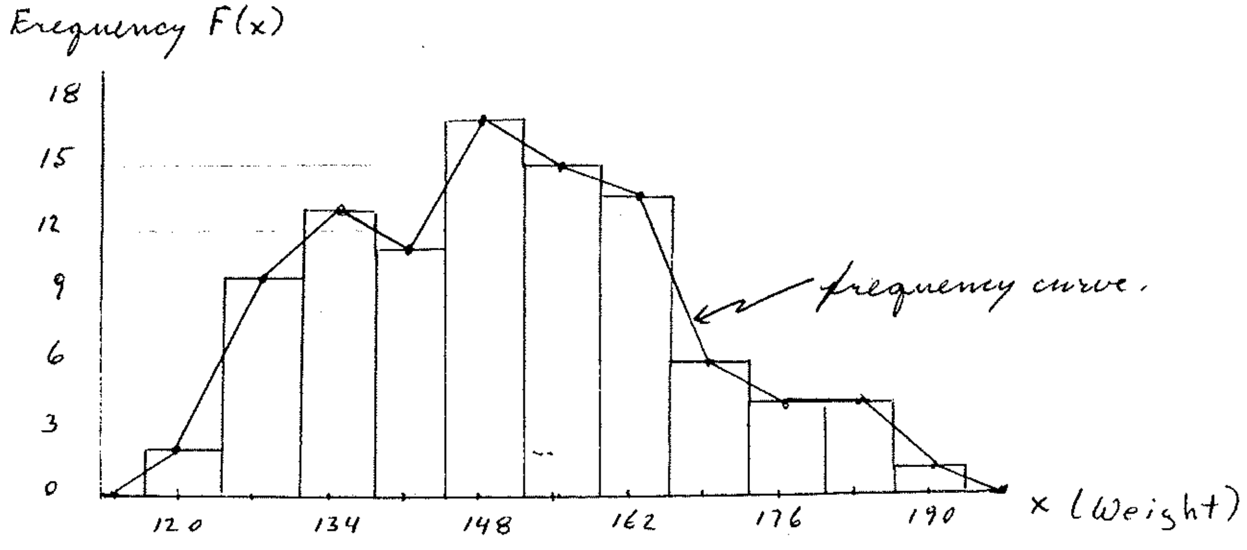
$\Rightarrow$ 10 intervals sounds good ( $N/_{10} \sim 10$ pts / interval)

$$\text{interval width} = \frac{187 - 117}{10} = 7$$

Let first interval begin at 116.5. Hence one additional interval is needed for highest point.

| Interval # | Interval Limits | Interval Center | Tally Marks | Tally Total |
|---|---|---|---|---|
| 1 | 116.5 - 123.5 | 120 | \|\| | 2 |
| 2 | 123.5 - 130.5 | 127 | 𝍸 𝍸 | 10 |
| 3 | 130.5 - 137.5 | 134 | 𝍸 𝍸 \|\|\| | 13 |
| 4 | 137.5 - 144.5 | 141 | 𝍸 𝍸 \| | 11 |
| 5 | 144.5 - 151.5 | 148 | 𝍸 𝍸 𝍸 \|\| | 17 |
| 6 | 151.5 - 158.5 | 155 | 𝍸 𝍸 𝍸 | 15 |
| 7 | 158.5 - 165.5 | 162 | 𝍸 𝍸 \|\|\|\| | 14 |
| 8 | 165.5 - 172.5 | 169 | 𝍸 \| | 6 |
| 9 | 172.5 - 179.5 | 176 | \|\|\|\| | 4 |
| 10 | 179.5 - 186.5 | 183 | \|\|\|\| | 4 |
| 11 | 186.5 - 193.5 | 190 | \| | 1 |
|  |  |  |  | Total = 97 |

The above information can be plotted as a histogram as follows.



Alternatively one can plot points just at the interval center and connect with lines. —

The area under the frequency curve F(x) = # of data points N.

$$\text{i.e.} \int_{\substack{\text{sum over}\\ \text{all } x.}} F(x)\, dx = N.$$

Probability Density Function or Probability Distribution

$$\boxed{f(x) \equiv \frac{F(x)}{N}.}$$

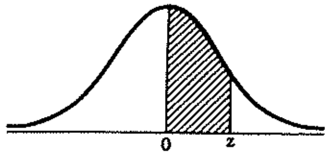
## Properties of Prob. Density Function f(x)

1) $$\int_{\text{all } x} f(x)\, dx = 1.$$

2) $$\text{Prob. } x \in [a, b] = \int_a^b f(x)\, dx.$$

## Appendix Table 2: Areas Under the Normal Probability Curve*

*A* denotes the area between the line of symmetry (i.e. $z = 0$) and the given $z$-value.



$$\frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-z^2/2} \, dz.$$

| z | A | z | A | z | A | z | A |
|---|---|---|---|---|---|---|---|
| 0.00 | 0.0000 | 0.47 | 0.1808 | 0.94 | 0.3264 | 1.41 | 0.4207 |
| .01 | .0040 | .48 | .1844 | .95 | .3289 | 1.42 | .4222 |
| .02 | .0080 | .49 | .1879 | .96 | .3315 | 1.43 | .4236 |
| .03 | .0120 | .50 | .1915 | .97 | .3340 | 1.44 | .4251 |
| .04 | .0160 | .51 | .1950 | .98 | .3365 | 1.45 | .4265 |
| .05 | .0199 | .52 | .1985 | .99 | .3389 | 1.46 | .4279 |
| .06 | .0239 | .53 | .2019 | 1.00 | .3413 | 1.47 | .4292 |
| .07 | .0279 | .54 | .2054 | 1.01 | .3438 | 1.48 | .4306 |
| .08 | .0319 | .55 | .2088 | 1.02 | .3461 | 1.49 | .4319 |
| .09 | .0359 | .56 | .2123 | 1.03 | .3485 | 1.50 | .4332 |
| .10 | .0398 | .57 | .2157 | 1.04 | .3508 | 1.51 | .4345 |
| .11 | .0438 | .58 | .2190 | 1.05 | .3531 | 1.52 | .4357 |
| .12 | .0478 | .59 | .2224 | 1.06 | .3554 | 1.53 | .4370 |
| .13 | .0517 | .60 | .2258 | 1.07 | .3577 | 1.54 | .4382 |
| .14 | .0557 | .61 | .2291 | 1.08 | .3599 | 1.55 | .4394 |
| .15 | .0596 | .62 | .2324 | 1.09 | .3621 | 1.56 | .4406 |
| .16 | .0636 | .63 | .2357 | 1.10 | .3643 | 1.57 | .4418 |
| .17 | .0675 | .64 | .2389 | 1.11 | .3665 | 1.58 | .4430 |
| .18 | .0714 | .65 | .2422 | 1.12 | .3686 | 1.59 | .4441 |
| .19 | .0754 | .66 | .2454 | 1.13 | .3708 | 1.60 | .4452 |
| .20 | .0793 | .67 | .2486 | 1.14 | .3729 | 1.61 | .4463 |
| .21 | .0832 | .68 | .2518 | 1.15 | .3749 | 1.62 | .4474 |
| .22 | .0871 | .69 | .2549 | 1.16 | .3770 | 1.63 | .4485 |
| .23 | .0910 | .70 | .2580 | 1.17 | .3790 | 1.64 | .4495 |
| .24 | .0948 | .71 | .2612 | 1.18 | .3810 | 1.65 | .4505 |
| .25 | .0987 | .72 | .2642 | 1.19 | .3830 | 1.66 | .4515 |
| .26 | .1026 | .73 | .2673 | 1.20 | .3849 | 1.67 | .4525 |
| .27 | .1064 | .74 | .2704 | 1.21 | .3869 | 1.68 | .4535 |
| .28 | .1103 | .75 | .2734 | 1.22 | .3888 | 1.69 | .4545 |
| .29 | .1141 | .76 | .2764 | 1.23 | .3907 | 1.70 | .4554 |
| 30 | .1179 | .77 | .2794 | 1.24 | .3925 | 1.71 | .4564 |
| .31 | .1217 | .78 | 2823 | 1.25 | .3944 | 1.72 | .4573 |
| .32 | .1255 | .79 | .2852 | 1.26 | .3962 | 1.73 | .4582 |
| .33 | .1293 | .80 | .2881 | 1.27 | .3980 | 1.74 | .4591 |
| .34 | .1331 | .81 | .2910 | 1.28 | .3997 | 1.75 | .4599 |
| .35 | .1368 | .82 | .2939 | 1.29 | .4015 | 1.76 | 4608 |
| .36 | .1406 | .83 | .2967 | 1.30 | .4032 | 1.77 | .4616 |
| .37 | .1443 | .84 | .2996 | 1.31 | .4049 | 1.78 | .4625 |
| .38 | .1480 | .85 | .3023 | 1.32 | .4066 | 1.79 | .4633 |
| .39 | .1517 | .86 | .3051 | 1.33 | .4082 | 1.80 | .4641 |
| .40 | .1554 | .87 | .3079 | 1.34 | .4099 | 1.81 | .4649 |
| .41 | .1591 | .88 | .3106 | 1.35 | .4115 | 1.82 | .4656 |
| .42 | .1628 | .89 | .3133 | 1.36 | .4131 | 1.83 | .4664 |
| .43 | .1664 | .90 | .3159 | 1.37 | .4147 | 1.84 | .4671 |
| .44 | .1700 | .91 | .3186 | 1.38 | .4162 | 1.85 | .4678 |
| .45 | .1736 | .92 | .3212 | 1.39 | .4177 | 1.86 | .4686 |
| .46 | .1772 | .93 | .3238 | 1.40 | .4192 | 1.87 | .4693 |

## Appendix Table 2 (continued)

| z | A | z | A | z | A | z | A |
|---|---|---|---|---|---|---|---|
| 1.88 | 0.4700 | 2.41 | 0.4920 | 2.94 | 0.4984 | 3.47 | 0.4997 |
| 1.89 | .4706 | 2.42 | .4922 | 2.95 | .4984 | 3.48 | .4998 |
| 1.90 | .4713 | 2.43 | .4925 | 2.96 | .4985 | 3.49 | .4998 |
| 1.91 | .4719 | 2.44 | .4927 | 2.97 | .4985 | 3.50 | .4998 |
| 1.92 | .4726 | 2.45 | .4929 | 2.98 | .4986 | 3.51 | .4998 |
| 1.93 | .4732 | 2.46 | .4931 | 2.99 | .4986 | 3.52 | .4998 |
| 1.94 | .4738 | 2.47 | .4932 | 3.00 | .4987 | 3.53 | .4998 |
| 1.95 | .4744 | 2.48 | .4934 | 3.01 | .4987 | 3.54 | .4998 |
| 1.96 | .4750 | 2.49 | .4936 | 3.02 | .4987 | 3.55 | .4998 |
| 1.97 | .4756 | 2.50 | .4938 | 3.03 | .4988 | 3.56 | .4998 |
| 1.98 | .4762 | 2.51 | .4940 | 3.04 | .4988 | 3.57 | .4998 |
| 1.99 | .4767 | 2.52 | .4941 | 3.05 | .4989 | 3.58 | .4998 |
| 2.00 | .4773 | 2.53 | .4943 | 3.06 | .4989 | 3.59 | .4998 |
| 2.01 | .4778 | 2.54 | .4945 | 3.07 | .4989 | 3.60 | .4998 |
| 2.02 | .4783 | 2.55 | .4946 | 3.08 | .4990 | 3.61 | .4999 |
| 2.03 | .4788 | 2.56 | .4948 | 3.09 | .4990 | 3.62 | .4999 |
| 2.04 | .4793 | 2.57 | .4949 | 3.10 | .4990 | 3.63 | .4999 |
| 2.05 | .4798 | 2.58 | .4951 | 3.11 | .4991 | 3.64 | .4999 |
| 2.06 | .4803 | 2.59 | .4952 | 3.12 | .4991 | 3.65 | .4999 |
| 2.07 | .4808 | 2.60 | .4953 | 3.13 | .4991 | 3.66 | .4999 |
| 2.08 | .4812 | 2.61 | .4955 | 3.14 | .4992 | 3.67 | .4999 |
| 2.09 | .4817 | 2.62 | .4956 | 3.15 | .4992 | 3.68 | .4999 |
| 2.10 | .4821 | 2.63 | .4957 | 3.16 | .4992 | 3.69 | .4999 |
| 2.11 | .4826 | 2.64 | .4959 | 3.17 | .4992 | 3.70 | .4999 |
| 2.12 | .4830 | 2.65 | .4960 | 3.18 | .4993 | 3.71 | .4999 |
| 2.13 | .4834 | 2.66 | .4961 | 3.19 | .4993 | 3.72 | .4999 |
| 2.14 | .4838 | 2.67 | .4962 | 3.20 | .4993 | 3.73 | .4999 |
| 2.15 | .4842 | 2.68 | .4963 | 3.21 | .4993 | 3.74 | .4999 |
| 2.16 | .4846 | 2.69 | .4964 | 3.22 | .4994 | 3.75 | .4999 |
| 2.17 | .4850 | 2.70 | .4965 | 3.23 | .4994 | 3.76 | .4999 |
| 2.18 | .4854 | 2.71 | .4966 | 3.24 | .4994 | 3.77 | .4999 |
| 2.19 | .4857 | 2.72 | .4967 | 3.25 | .4994 | 3.78 | .4999 |
| 2.20 | .4861 | 2.73 | .4968 | 3.26 | .4994 | 3.79 | .4999 |
| 2.21 | .4865 | 2.74 | .4969 | 3.27 | .4995 | 3.80 | .4999 |
| 2.22 | .4868 | 2.75 | .4970 | 3.28 | .4995 | 3.81 | .4999 |
| 2.23 | .4871 | 2.76 | .4971 | 3.29 | .4995 | 3.82 | .4999 |
| 2.24 | .4875 | 2.77 | .4972 | 3.30 | .4995 | 3.83 | .4999 |
| 2.25 | .4878 | 2.78 | .4973 | 3.31 | .4995 | 3.84 | .4999 |
| 2.26 | .4881 | 2.79 | .4974 | 3.32 | .4996 | 3.85 | .4999 |
| 2.27 | .4884 | 2.80 | .4974 | 3.33 | .4996 | 3.86 | .4999 |
| 2.28 | .4887 | 2.81 | .4975 | 3.34 | .4996 | 3.87 | .5000 |
| 2.29 | .4890 | 2.82 | .4976 | 3.35 | .4996 | 3.88 | .5000 |
| 2.30 | .4893 | 2.83 | .4977 | 3.36 | .4996 | 3.89 | .5000 |
| 2.31 | .4896 | 2.84 | .4977 | 3.37 | .4996 | | |
| 2.32 | .4898 | 2.85 | .4978 | 3.38 | .4996 | | |
| 2.33 | .4901 | 2.86 | .4979 | 3.39 | .4997 | | |
| 2.34 | .4904 | 2.87 | .4980 | 3.40 | .4997 | | |
| 2.35 | .4906 | 2.88 | .4980 | 3.41 | .4997 | | |
| 2.36 | .4909 | 2.89 | .4981 | 3.42 | .4997 | | |
| 2.37 | .4911 | 2.90 | .4981 | 3.43 | .4997 | | |
| 2.38 | .4910 | 2.91 | .4982 | 3.44 | .4997 | | |
| 2.39 | .4916 | 2.92 | .4983 | 3.45 | .4997 | | |
| 2.40 | .4918 | 2.93 | .4983 | 3.46 | .4997 | | |

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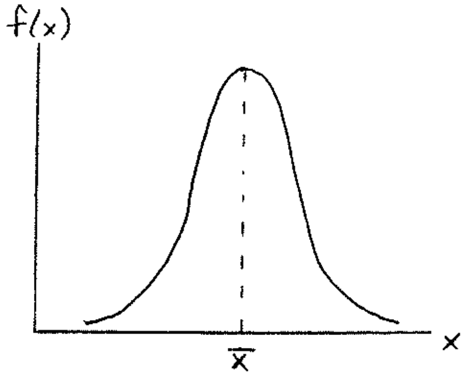
# Gaussian or Normal Probability Distribution

Frequently a probability distribution can be approximated by the Gaussian or Normal distribution given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2}$$



$f(x)$ is $\overbrace{\text{centered}}^{\text{bell shaped and}}$ about the mean $\bar{x}$

Note that $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1.$

Proof: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \, dz. \qquad z = \frac{x-\bar{x}}{\sigma} \qquad dz = \frac{dx}{\sigma.}$$

$= 1$

I ask that you accept this last step!

Also note that by symmetry $\displaystyle\int_{0}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = \int_{-\infty}^{0} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = \frac{1}{2}.$

## Meaning of $\sigma$

Let's compute the probability that $x \in [\bar{x}-\sigma, \bar{x}+\sigma]$.

$$P(x \in [\bar{x}-\sigma, \bar{x}+\sigma]) = \int_{\bar{x}-\sigma}^{\bar{x}+\sigma} f(x)\, dx.$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\bar{x}-\sigma}^{\bar{x}+\sigma} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-z^2/2} dz \quad \text{where } z = \frac{x-\bar{x}}{\sigma}$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} \int_{0}^{1} e^{-z^2/2} dz.$$

$$= 2 \times .3413 \quad \text{using table}$$

$$= .6826.$$

$\therefore$ there is a 68% probability that $x \in [\bar{x}-\sigma, \bar{x}+\sigma]$.
Similarly " " 95.5% " " $[\bar{x}-2\sigma, \bar{x}+2\sigma]$.

$\therefore$ $\sigma$ is a measure of the spread of data from the mean
$\equiv$ standard deviation of the normal distribution

## Examples

1) Heights of male students are normally distributed with a mean 68.5 inches and standard deviation 2.5 inches.

i.e. $\bar{x} = 68.5 \qquad \sigma = 2.5$

a) What is probability a student is over 6 ft?

b) What percentage of students is between 70 + 72 inches?

## Solution

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2}$$

a) $$P(x > 6\,ft) = \int_{72}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2} dx. \qquad z = \frac{x-\bar{x}}{\sigma}$$

$$x = 72 \Rightarrow z = \frac{72 - 68.5}{2.5} = 1.4$$

$$= \frac{1}{\sqrt{2\pi}} \int_{1.4}^{\infty} e^{-z^2/2}\, dz.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2}\, dz - \frac{1}{\sqrt{2\pi}} \int_{0}^{1.4} e^{-z^2/2}\, dz.$$

$$= .500 - .419$$

$$= .081$$

$\therefore$ probability student is over 6 ft. is 8%.

b) $P(70 < x < 72) = \int_{70}^{72} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2} dx.$ $\quad z = \frac{x-\bar{x}}{\sigma}$

$$= \frac{1}{\sqrt{2\pi}} \int_{.60}^{1.4} e^{-z^2/2} dz \qquad x = 70 \Rightarrow z = \frac{70-68.5}{2.5} = .60$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{1.4} e^{-z^2/2} dz - \frac{1}{\sqrt{2\pi}}\int_{0}^{.60} e^{-z^2/2} dz.$$

$$= .419 - .226$$

$$= .193$$

∴ 19.3% of students are between 70 + 72 inches tall.

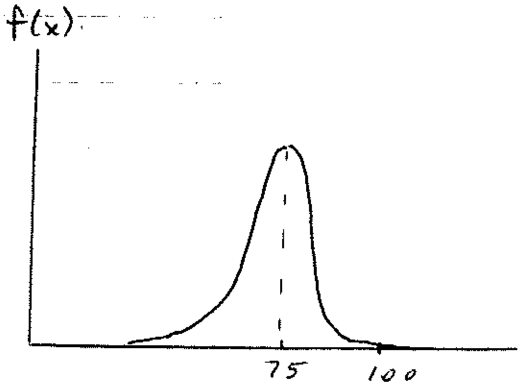
2) The mean grade of 500 students is 75% and standard deviation is 8%.
Assuming a normal distribution find:
a) number of grades above 85%

b) grade limits of middle 400.

<u>Solution</u>

$\bar{x} = 75 \qquad \sigma = 8.$

a) $P(x > 85) = \int_{85}^{100} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\bar{x}}{\sigma}\right)^2} dx.$



~~Part of~~ f(x) is negligible when $x > 100$. ~~is negligible~~. $\therefore$ 100 can be replaced by $\infty$.

$$\therefore \; P(x > 85) = \int_{85}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{1.25}^{\infty} e^{-z^2/2} dz \qquad x = 85, \; z = \frac{85-75}{8} = 1.25$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_{0}^{1.25} e^{-z^2/2} dz$$

$$= \frac{1}{2} - .3944.$$

$$= .105$$

$\therefore$ 10.5% of class got over 85% or $10.5\% \times 500 \simeq 53$ studen

b) Let middle 400 have grades from $\bar{x} - w$ to $\bar{x} + w$.

$$P(x \in [\bar{x} - w, \bar{x} + w]) = \frac{400}{500} = \int_{\bar{x}-w}^{\bar{x}+w} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2} dx.$$

$$.80 = \int_{-w/\sigma}^{w/\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz.$$

$$.40 = \frac{1}{\sqrt{2\pi}} \int_0^{w/\sigma} e^{-z^2/2}\, dz.$$

$$\therefore \frac{w}{\sigma} = 1.28.$$

$$w = 1.28 \times 8$$
$$= 10.2.$$

$\therefore$ middle 400 grades are from 65% to 85%.