Laser Electronics

Lecture Notes

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Topics

1. Review of Electromagnetic Waves
2. Matrix Formulation of Ray Tracing
3. Gaussian Beams
4. Resonant Cavities
5. Einstein A + B Coefficients
6. Lineshape
7. Light Amplification
8. Q-Switching
9. Mode Locking
10. Examples of Lasers
11. Dye Laser
12. Diode Laser
13. Nonlinear Optics
14. Applications

Textbooks

Electromagnetism

* 1. Introduction to Electrodynamics
by. D. Griffiths, Prentice Hall, Englewood Cliffs NJ, 1981

* 2. Classical Electromagnetic Radiation
by. J.B. Marion & M.A. Heald, Academic Press, 1980.

Optics

* 1. Optics
by E. Hecht & A. Zajac, Addison Wesley, 1974.

2. Principles of Optics
M. Born & E. Wolf, Pergamon Press, Oxford 1970.

3. Optics
M.V. Klein, John Wiley & Sons, 1970.

4. The Quantum Theory of Light
R. Loudon, Clarendon Press, Oxford, 1983.

5. The Principles of Nonlinear Optics
Y.R. Shen, John Wiley & Sons, 1984.

* 6. Optical Resonance and Two Level Atoms
L. Allen & J.H. Eberly, Dover Publications, 1987

Lasers

** 1. Laser Electronics
J. T. Verdeyen, Prentice Hall, Englewood Cliffs NJ, 1981.

2. Quantum Electronics
A. Yariv, John Wiley & Sons, 1975.

** 3. Laser Spectroscopy
W. Demtroder, Springer Verlag, 1988.

* 4. Lasers
A. E. Siegman, University Science Books, Mill Valley CA, 1986.

Electromagnetism Review

Maxwell's equations describe how electric and magnetic fields $\vec{E}$ + $\vec{B}$ are produced by charge and current densities ρ_f + $\vec{J}_f$. In the mks system, they are written as follows.

$$\nabla \cdot \vec{D} = \rho_f \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

The electric displacement $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ and magnetic field $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$.

Linear Materials

Polarization $\vec{P} = \epsilon_0 \chi_e \vec{E}$ such that $\vec{D} = \epsilon \vec{E}$.
Magnetization $\vec{M} = \chi_m \vec{H}$ " " $\vec{B} = \mu \vec{H}$

χ_e (χ_m) is called the electric (magnetic) susceptibility. ϵ is the dielectric constant and μ is the magnetic permeability.

Vacuum

$\epsilon \equiv \epsilon_0 = 8.85 \times 10^{-12} \frac{\text{Coul}^2}{\text{Nt m}^2}$ Permittivity of Free Space

$\mu \equiv \mu_0 = 4\pi \times 10^{-7} \frac{\text{Nt}}{\text{amp}^2}$ Permeability of Free Space.

Wave Equation

Maxwells equations in nonconducting linear media in the absence of free charges and currents are the following.

$$\nabla \cdot \vec{E} = 0 \qquad (1) \qquad\qquad \nabla \cdot \vec{B} = 0 \qquad (3)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (2) \qquad\qquad \nabla \times \vec{B} = \epsilon\mu \frac{\partial \vec{E}}{\partial t} \qquad (4)$$

Taking the curl of (2) we get:

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\epsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{using (4)}$$

$$\boxed{\nabla^2 \vec{E} - \epsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0.} \qquad \text{using (1)}$$

Exercise: Show that $\boxed{\nabla^2 \vec{B} - \epsilon\mu \frac{\partial^2 \vec{B}}{\partial t^2} = 0.}$

These two equations represent a wave having speed

$$v = \frac{1}{\sqrt{\epsilon\mu}}$$

For vacuum $v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3 \times 10^8$ m/sec

= speed of light!

Plane Wave Solution of Wave Equation

Consider the following plane wave propagating in the $+\hat{z}$ direction.

$$\vec{E} = \vec{E}_0 \, e^{i(kz - \omega t)}$$

$$\vec{B} = \vec{B}_0 \, e^{i(kz - \omega t)}$$

$\vec{E}_0$ + $\vec{B}_0$ are constant vector amplitudes.

$\vec{k} = k\hat{z}$ is the wavevector where $k = \frac{2\pi}{\lambda}$ and λ

is the wavelength.
$\omega = 2\pi\nu$ where ν is the frequency.

Exercise: Show that the wave equations are satisfied if:

$$k^2 - \epsilon\mu\omega^2 = 0$$

$$\boxed{k = \sqrt{\epsilon\mu}\,\omega}$$

Maxwell's Equations

The plane wave must also satisfy Maxwells 4 eqns.

a) $\nabla \cdot \vec{E} = 0 \implies \boxed{i\vec{k} \cdot \vec{E}_0 = 0}$ where $\vec{k} = k\hat{z}$

$\therefore \vec{E}_0$ is perpendicular to the direction of propagation.
i.e. $\vec{E}_0 \perp \vec{k}$

b) $\nabla \cdot \vec{B} = 0 \implies \boxed{i\vec{k} \cdot \vec{B}_0 = 0}$

$\therefore \vec{B}_0$ is perpendicular to the direction of propagation.
i.e. $\vec{B}_0 \perp \vec{k}$

c) $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \implies i\vec{k} \times \vec{E}_0 = -(-i\omega)\vec{B}_0$

$$\hat{k} \times \vec{E}_0 = \frac{1}{\sqrt{\epsilon\mu}} \vec{B}_0$$

$$\boxed{|\vec{E}_0| = \frac{1}{\sqrt{\epsilon\mu}} |\vec{B}_0|}$$

$\therefore \vec{k}$, $\vec{E}_0$ + $\vec{B}_0$ are three mutually perpendicular vectors.

d) $\nabla \times \vec{B} = \epsilon\mu \dfrac{\partial \vec{E}}{\partial t}$ yields no additional information

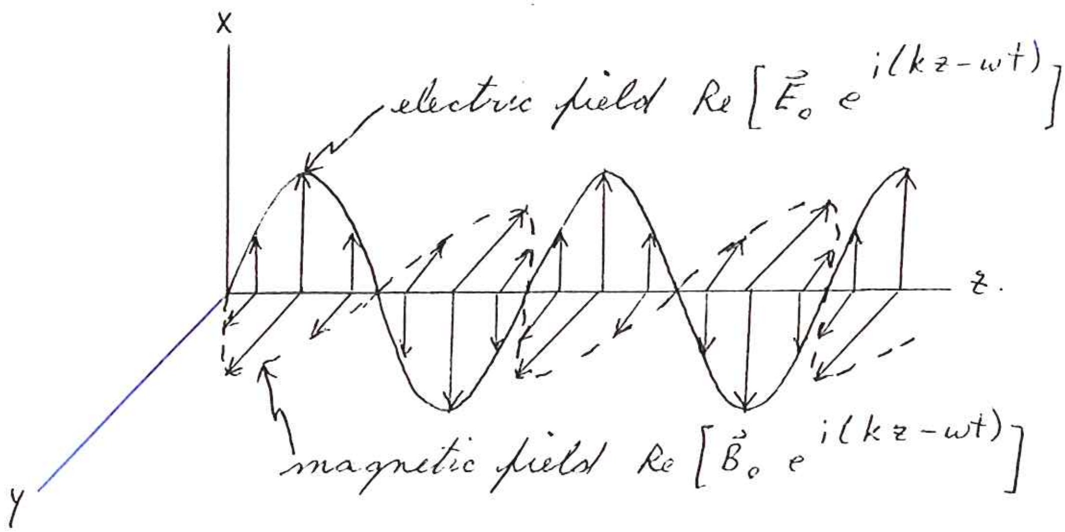
Exercise: Show this and explain why this is so.

Hence there are two linearly independent plane wave solutions propagating in the $\hat{k} = +\hat{z}$ direction.

1) $\vec{E}_0 \parallel \hat{x}$ & $\vec{B}_0 \parallel \hat{y}$

2) $\vec{E}_0 \parallel \hat{y}$ & $\vec{B}_0 \parallel \hat{x}$

Solution 1



Exercise: Sketch solution 2.

Colarization

The wave of solution 1 is said to be linearly polarized in the $\hat{x}$ direction since $\vec{E}_0 \parallel \hat{x}$. Similarly the wave of solution 2 is linearly polarized in the $\hat{y}$ direction since $\vec{E}_0 \parallel \hat{y}$.

Transverse Nature of Light Wave

A light wave is similar to a wave propagating down a string. A string oscillates in two possible modes: up-down or side to side. These two vibration directions are perpendicular or transverse to the wave propagation direction.

$\therefore$ light is called a transverse electromagnetic wave.

General Plane Wave Solution

$$\vec{E} = E_0 \, \hat{n} \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\vec{B} = \sqrt{\epsilon\mu}\, E_0 (\hat{k} \times \hat{n}) \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$k = \sqrt{\epsilon\mu}\;\omega$$

This is a plane wave propagating in direction $\vec{k}$ and linearly polarized in direction $\hat{n}$. For each $\vec{k}$, there are two independent polarization directions $\hat{n}$.

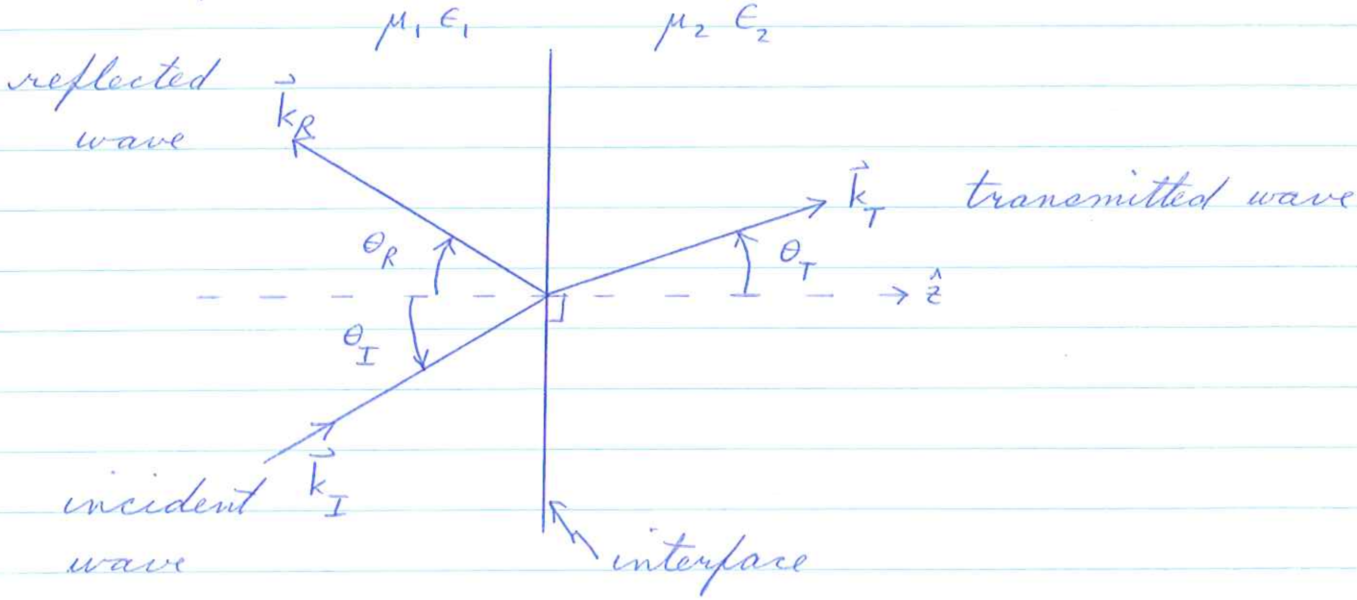
TABLE 8.1 The electromagnetic spectrum

Frequency (Hz)	Name of Radiation	Wavelength (m)
10^{22}		
10^{21}	Gamma rays	10^{-13}
10^{20}		10^{-12}
10^{19}		10^{-11}
10^{18}	*X* rays	10^{-10}
10^{17}		10^{-9}
10^{16}	Ultraviolet	10^{-8}
10^{15}		10^{-7}
10^{14}	Visible	10^{-6}
10^{13}	Infrared	10^{-5}
10^{12}		10^{-4}
10^{11}		10^{-3}
10^{10}	Microwave	10^{-2}
10^{9}		10^{-1}
10^{8}	TV, FM	1
10^{7}		10
10^{6}	Standard broadcast	10^{2}
10^{5}		10^{3}
10^{4}	Radiofrequency	10^{4}
10^{3}		10^{5}

Color	Wavelength (m)	Frequency (Hz)
Near ultraviolet	3.0×10^{-7}	10×10^{14}
Shortest visible blue	4.0×10^{-7}	7.5×10^{14}
Blue	4.6×10^{-7}	6.5×10^{14}
Green	5.4×10^{-7}	5.6×10^{14}
Yellow	5.9×10^{-7}	5.1×10^{14}
Orange	6.1×10^{-7}	4.9×10^{14}
Longest visible red	7.6×10^{-7}	3.9×10^{14}
Near infrared	10.0×10^{-7}	3.0×10^{14}

Reflection and Refraction

We shall study the reflection and transmission of EM waves propagating from one medium to another. We assume the two media are linear and nonconducting.



Boundary Conditions

1) Tangential component of $\vec{E}$ is continuous

2) normal " " $\vec{D}$ " "

3) Tangential " " $\vec{H}$ " "

4) normal " " $\vec{B}$ " "

Application of Boundary Conditions

When applying the boundary conditions for the fields, we get expressions of the following form.

$$(\quad)\, e^{i(\vec{k}_I\cdot\vec{r}-\omega_I t)} + (\quad)\, e^{i(\vec{k}_R\cdot\vec{r}-\omega_R t)} = (\quad)\, e^{i(\vec{k}_T\cdot\vec{r}-\omega_T t)}$$

These equations must be satisfied at all points on the $z=0$ plane at all times. This implies the following.

1) $\omega_I t = \omega_R t = \omega_T t \quad \forall t$

or $\omega_I = \omega_R = \omega_T$

> The incident, reflected and transmitted waves all have the same color.

2) $\vec{k}_I\cdot\vec{r} = \vec{k}_R\cdot\vec{r} = \vec{k}_T\cdot\vec{r}$ where $\vec{r}=(x,y,0)\ \forall\ x,y.$

or $k_{Ix}x + k_{Iy}y = k_{Rx}x + k_{Ry}y = k_{Tx}x + k_{Ty}y$

$$\Rightarrow \quad k_{Ix} = k_{Rx} = k_{Tx} \qquad (0)$$

$$k_{Iy} = k_{Ry} = k_{Ty}$$

Defining the x axis such that $\vec{k}_I$ lies in the xz plane, i.e. $\vec{k}_I = (k_{Ix}, 0, k_{Iz})$ we get:

$$k_{Ry} = k_{Ty} = 0.$$

$\therefore$ wavevectors of incident, reflected and transmitted waves all lie in the same plane called the incident plane.

Next, we examine the consequences of (o).

$$k_{Ix} = k_{Rx}$$

$$k_I \sin\theta_I = k_R \sin\theta_R$$

But $k_I = k_R \Longrightarrow \sin\theta_I = \sin\theta_R$

$$\theta_I = \theta_R$$

Law of Reflection: Angle of incidence equals angle of reflection.

Also (o) $\Rightarrow$ $k_{Ix} = k_{Tx}$

$$k_I \sin\theta_I = k_T \sin\theta_T$$

$$n_1 \frac{\omega}{c} \sin\theta_I = n_2 \frac{\omega}{c} \sin\theta_T$$

$$\boxed{n_1 \sin\theta_I = n_2 \sin\theta_T}$$

This is <u>Snell's Law</u> of refraction which was discovered empirically in 1621.

Incident Plane Wave

$$\vec{E}_I = \vec{E}_{I0}\, e^{i(\vec{k}_I \cdot \vec{r} - \omega_I t)}$$

$$\vec{B}_I = \frac{1}{v_1}\, \hat{k}_I \times \vec{E}_I$$

where $v_1 = \dfrac{c}{n_1}$ and $n_1 \equiv \sqrt{\dfrac{\epsilon_1 \mu_1}{\epsilon_0 \mu_0}}$ is the index of refraction.

$$k_I = \sqrt{\epsilon_1 \mu_1}\, \omega_I$$

$$= n_1 \frac{\omega_I}{c}$$

Reflected Plane Wave

$$\vec{E}_R = \vec{E}_{R0}\, e^{i(\vec{k}_R \cdot \vec{r} - \omega_R t)}$$

$$\vec{B}_R = \frac{1}{v_1}\, \hat{k}_R \times \vec{E}_R$$

$$k_R = n_1 \frac{\omega_R}{c}$$

Transmitted Plane Wave

$$\vec{E}_T = \vec{E}_{T0}\, e^{i(\vec{k}_T \cdot \vec{r} - \omega_T t)}$$

$$\vec{B}_T = \frac{1}{v_2}\, \hat{k}_T \times \vec{E}_T .$$

$k_T = n_2 \dfrac{\omega_T}{c}$ where $n_2 \equiv \sqrt{\dfrac{\epsilon_2 \mu_2}{\epsilon_0 \mu_0}}$.

Detailed Application of Boundary Conditions

1) Continuity of $\vec{D}_\perp = \epsilon \vec{E}_\perp$

$$\epsilon_1 \left[\vec{E}_{I0} + \vec{E}_{R0} \right]_z = \epsilon_2 \vec{E}_{T0\,z} \qquad (1)$$

2) Continuity of $\vec{E}_{\parallel}$

$$\left[\vec{E}_{I0} + \vec{E}_{R0} \right]_{xy} = \vec{E}_{T0\;xy} \qquad (2)$$

3) Continuity of $\vec{B}_\perp$

$$n_1 \left[\hat{k}_I \times \vec{E}_{I0} \right]_z + n_1 \left[\hat{k}_R \times \vec{E}_{R0} \right]_z = n_2 \left[\hat{k}_T \times \vec{E}_{T0} \right]_z \qquad (3)$$

4) Continuity of $\vec{H}_{\parallel}$

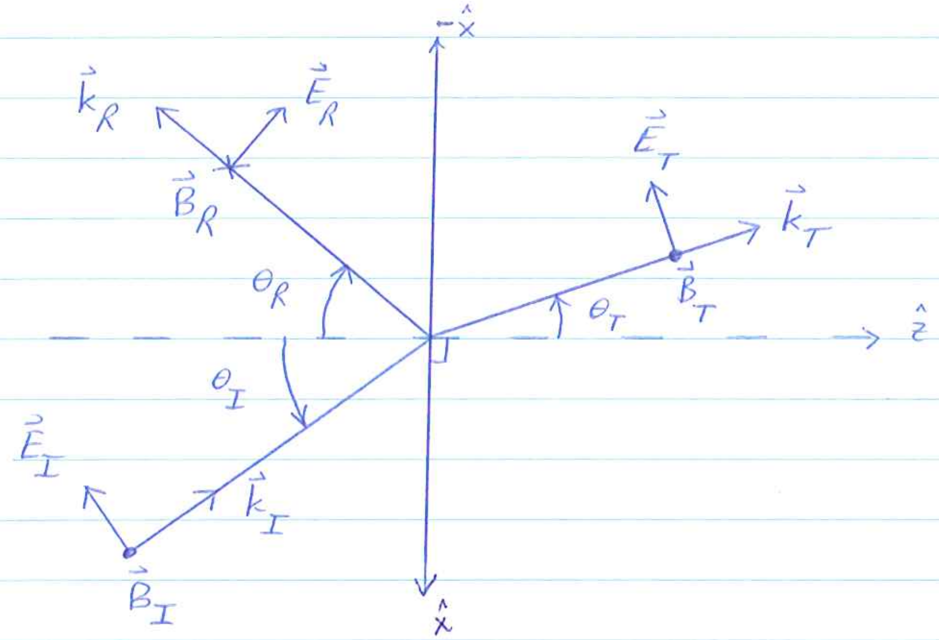
$$\frac{n_1}{\mu_1} \left[\hat{k}_I \times \vec{E}_{I0} \right]_{x,y} + \frac{n_1}{\mu_1} \left[\hat{k}_R \times \vec{E}_{R0} \right]_{x,y} = \frac{n_2}{\mu_2} \left[\hat{k}_T \times \vec{E}_{T0} \right]_{x,y} \qquad (4)$$

Case 1

The incident wave is polarized perpendicular to the plane of incidence, i.e. $\vec{E}_{I0} = (0, E_{I0}, 0)$ Such a wave is said to be s polarized. This case will be treated as a HW problem.

Case 2

The electric field of the incident wave lies in the plane of incidence. i.e. $\vec{E}_{I0} = E_{I0}(\cos\theta_I, 0, \sin\theta_I)$ Such a wave is said to be p polarized.



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The wavevectors may be written as follows. (exercise)

$$\vec{k}_I = k_I(-\sin\theta_I, 0, \cos\theta_I)$$

$$\vec{k}_R = k_I(-\sin\theta_I, 0, -\cos\theta_I)$$

$$\vec{k}_T = k_T(-\sin\theta_T, 0, \cos\theta_T)$$

Equation (1) then yields:

$$\epsilon_1\left[E_{I0}\sin\theta_I - E_{R0}\sin\theta_I\right] = \epsilon_2 E_{T0}\sin\theta_T$$

$$\frac{\mu_1\epsilon_1}{\mu_1}\sin\theta_I\left(E_{I0} - E_{R0}\right) = \frac{\mu_2\epsilon_2}{\mu_2}E_{T0}\sin\theta_T$$

$$\frac{n_1^2}{\mu_1}\sin\theta_I\left(E_{I0} - E_{R0}\right) = \frac{n_2^2}{\mu_2}\sin\theta_T\, E_{T0}$$

Using Snell's Law and defining $\beta \equiv \frac{\mu_1}{\mu_2} \frac{n_2}{n_1}$ we obtain

$$E_{I0} - E_{R0} = \beta E_{T0} \qquad (5)$$

Equation (2) yields:

$$E_{I0} \cos\theta_I + E_{R0} \cos\theta_I = E_{T0} \cos\theta_T$$

Defining $\alpha \equiv \dfrac{\cos\theta_T}{\cos\theta_I}$ we obtain:

$$E_{I0} + E_{R0} = \alpha E_{T0} \qquad (6)$$

Exercise: Show $\alpha = \dfrac{\sqrt{1 - \left(\frac{n_1}{n_2} \sin\theta_I\right)^2}}{\cos\theta_I}$

One can show that (5) + (6) are consistent with the results of (3) + (4). Eqns. (5) + (6) yield the so called Fresnel equations.

$$\boxed{\begin{aligned} E_{R0} &= \frac{\alpha - \beta}{\alpha + \beta} E_{I0} \\ E_{T0} &= \frac{2}{\alpha + \beta} E_{I0} \end{aligned}}$$

Reflection & Transmission Coefficients

The time averaged Poynting vector gives the average energy passing through a unit area per unit time.

$$\langle \vec{S} \rangle = \frac{1}{2} \mathrm{Re} \left(\vec{E} \times \vec{H}^* \right)$$

Exercise: For the incident wave, show that

$$\langle \vec{S}_I \rangle = \frac{\epsilon_1}{2} v_1 E_{I0}^2 \; \hat{k}_I$$

Hence the intensity (Power per unit area) incident on the $z=0$ boundary is:

$$I_I = \langle \vec{S}_I \rangle \cdot \hat{z}$$

$$= \frac{\epsilon_1}{2} v_1 E_{I0}^2 \cos\theta_I$$

Similarly one can show (exercise) that:

$$I_R = \frac{\epsilon_1}{2} v_1 E_{R0}^2 \cos\theta_R \quad \text{reflected intensity}$$

$$I_T = \frac{\epsilon_2}{2} v_2 E_{T0}^2 \cos\theta_T \quad \text{transmitted intensity}$$

Reflection Coefficient

$$R \equiv I_R / I_I$$

$$= E_{R0}^2 / E_{I0}^2$$

$$R = \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^2$$

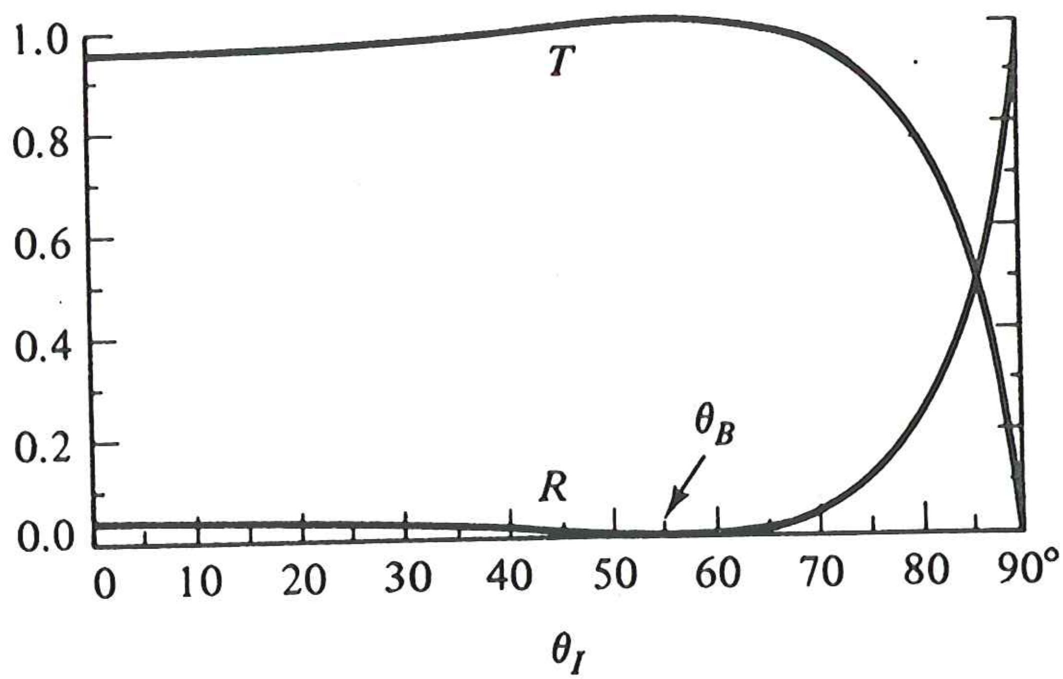
Transmission Coefficient

$$T \equiv I_T / I_I$$

$$= \frac{\epsilon_2 \, v_2}{\epsilon_1 \, v_1} \left(\frac{E_{T0}}{E_{I0}} \right)^2 \frac{\cos\theta_T}{\cos\theta_I}$$

$$T = \alpha\beta \left(\frac{2}{\alpha + \beta} \right)^2$$

Exercise: Check that $R + T = 1$.



Example

Find the reflectivity & transmission coeffs. for p polarized light normally incident on a glass surface.

$n_1 = n_{air} = 1 \qquad \mu_{air} = \mu_{glass} \simeq 1$

$n_2 = n_{glass} \simeq 1.5$

$\theta_I = 0$

$$\alpha = \frac{\sqrt{1 - \left(\frac{n_1}{n_2} \sin\theta_I\right)^2}}{\cos\theta_I} = 1$$

$$\beta = \frac{\mu_1}{\mu_2}\frac{n_2}{n_1} = 1.5$$

$$\therefore \; R = \left(\frac{1 - 1.5}{1 + 1.5}\right)^2 = .04$$

$$T = 1.5\left(\frac{2}{1.5 + 1}\right)^2 = .96$$

Brewsters Angle

Note that the reflectivity R is zero when $\alpha = \beta$. The corresponding angle of incidence is called Brewsters angle.

Exercise: Show $\tan\theta_B = n_2/n_1$

When designing lasers, reflection losses are minimized by tilting optical elements by θ_B.

eg. air-glass interface $\tan\theta_B = \frac{n_{glass}}{n_{air}} = 1.5 \Rightarrow \theta_B = 56.3^\circ$

Propagation of EM Waves in Anisotropic Crystals

Maxwell's equations in a nonmagnetic crystal having no free charges or currents are:

$$\nabla \cdot \vec{D} = 0 \qquad\qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad\qquad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

where $\vec{B} = \mu_0 \vec{H}$. We consider an anisotropic crystal such that $D_j = \epsilon_{jl} E_l$ where ϵ_{jl} is the dielectric tensor. Hence, in general $\vec{D} \nparallel \vec{E}$. The preceding expression simplifies to $D_j = \epsilon_j E_j$ $\quad j = x, y, z$ where x, y, z are the crystal principal axes.

We shall examine how a plane wave having phase factor $\exp i(\omega t - \vec{k} \cdot \vec{r})$ propagates in the crystal. The phase velocity $\vec{v}_p \equiv \frac{\omega}{k} \hat{k} \equiv \frac{c}{n} \hat{k}$ where n is the index of refraction. We shall find the relationship between n and the dielectric tensor.

Exercise: For the plane wave, show Maxwells equations yield:

$$\hat{k} \cdot \vec{D} = 0 \implies \hat{k} \perp \vec{D}$$

$$\hat{k} \cdot \vec{B} = 0 \implies \hat{k} \perp \vec{B}$$

$$\vec{H} = \frac{n}{\mu_0 c} \hat{k} \times \vec{E} \qquad (1)$$

$$\vec{D} = -\frac{n}{c} \hat{k} \times \vec{H} \qquad (2)$$

Substituting (1) into (2) we get:

$$\vec{D} = \frac{-n^2}{\mu_0 c^2} \hat{k} \times (\hat{k} \times \vec{E})$$

$$= \frac{n^2}{\mu_0 c^2} \left[\vec{E} - \hat{k}(\hat{k}.\vec{E}) \right]$$

$$D_j = \frac{n^2}{\mu_0 c^2} \left[E_j - \hat{k}_j (\hat{k}.\vec{E}) \right] .$$

$$\epsilon_j E_j = \frac{n^2}{\mu_0 c^2} \left[E_j - \hat{k}_j (\hat{k}.\vec{E}) \right]$$

$$E_j \left[\epsilon_j \mu_0 c^2 - n^2 \right] = -n^2 \hat{k}_j (\hat{k}.\vec{E})$$

$$E_j = n^2 \frac{\hat{k}_j \quad (\hat{k}.\vec{E})}{n^2 - \epsilon_j \mu_0 c^2}$$

$$\sum_j \hat{k}_j E_j = n^2 \sum_j \frac{\hat{k}_j^2}{n^2 - \epsilon_j'} (\hat{k}.\vec{E}) \quad \text{where } \epsilon_j' \equiv \frac{\epsilon_j}{\epsilon_0}$$

$$\therefore \boxed{\frac{1}{n^2} = \frac{\hat{k}_x^2}{n^2 - \epsilon_x'} + \frac{\hat{k}_y^2}{n^2 - \epsilon_y'} + \frac{\hat{k}_z^2}{n^2 - \epsilon_z'}} \qquad (3)$$

Exercise: Show this equation is quadratic in n^2 not cubic as it appears.

$\therefore$ (3) yields two values for n^2 and hence n. The crystal may have two indices of refraction and is said to be birefringent. Rather than solving (3), we use the following easier approach.

The Index Ellipsoid

The electric energy density is given by:

$$\mathcal{U}_E = \frac{\vec{E}\cdot\vec{D}}{2}$$

$$= \frac{1}{2}\left[E_x D_x + E_y D_y + E_z D_z \right]$$

$$= \frac{1}{2}\left[\frac{D_x^2}{\epsilon_x' \epsilon_o} + \frac{D_y^2}{\epsilon_y' \epsilon_o} + \frac{D_z^2}{\epsilon_z' \epsilon_o} \right]$$

$$1 = \frac{1}{2\epsilon_o \mathcal{U}_E}\left[\frac{D_x^2}{n_x^2} + \frac{D_y^2}{n_y^2} + \frac{D_z^2}{n_z^2} \right] \quad \text{where } n_\ell \equiv \sqrt{\epsilon_\ell'}$$

Defining $\vec{F} \equiv \dfrac{\vec{D}}{\sqrt{2\epsilon_o \mathcal{U}_E}}$ we obtain the so called index

ellipsoid $\boxed{1 = \dfrac{F_x^2}{n_x^2} + \dfrac{F_y^2}{n_y^2} + \dfrac{F_z^2}{n_z^2}}$

Exercise: Show 1) $\vec{F}\cdot\vec{F} = n^2$

2) $\hat{k}\cdot\vec{F} = 0$.

Uniaxial Crystals

We shall consider uniaxial crystals whose index of refraction is unchanged for rotations about the z principal axis. The z axis is called the optical axis.

i.e. $n_x = n_y \equiv n_o$

$n_z \equiv n_e$

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*To check box, double click and set default value to 'checked'

Exercise

Show $\vec{F}.\vec{F} = n^2$

pg. 19 $$D_j = \frac{n^2}{\mu_o c^2}\left[E_j - \hat{k}_j(\hat{k}.\vec{E})\right]$$

$$\sum_j D_j D_j = \frac{n^2}{\mu_o c^2}\left[\sum_j E_j D_j - \underbrace{\sum_j \hat{k}_j D_j}_{=0 \text{ since } \hat{k} \perp \vec{D}} (\hat{k}.\vec{E})\right]$$

$$\vec{D}.\vec{D} = \frac{n^2}{\mu_o c^2} 2U_E$$

$$n^2 = \frac{\vec{D}.\vec{D}}{2\epsilon_o U_E}$$

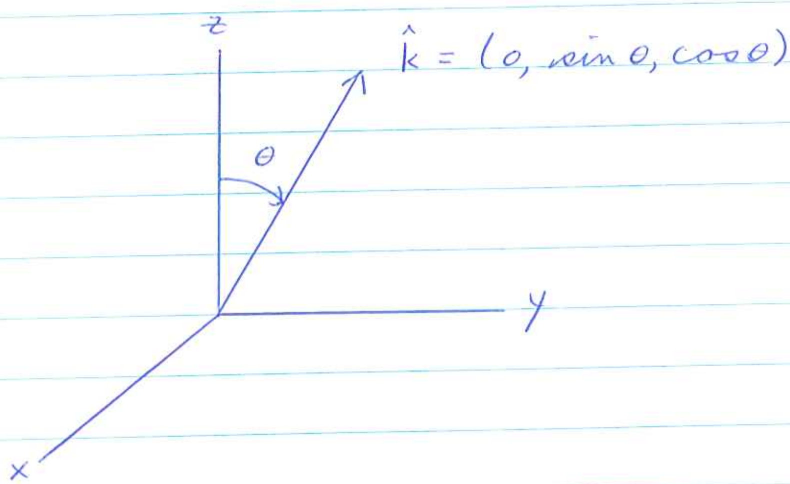
$$\therefore n^2 = \vec{F}.\vec{F} = F^2$$

The index ellipsoid then becomes $1 = \dfrac{F_x^2 + F_y^2}{n_o^2} + \dfrac{F_z^2}{n_e^2}$

If $n_e < n_o$ the crystal is said to be negative.
" $n_e > n_o$ " " positive.

Field Directions

We define the y axis such that $\hat{k}$ lies in the yz plane.



There are two linearly independent choices for $\vec{D}$ since $\vec{D} \perp \hat{k}$.

Case 1: $\vec{D} = D\,\hat{x}$

Exercise: Show index of refraction $n = n_o$.

Electric Field $\vec{E} = \left(\dfrac{D}{\epsilon_o\, n_o^2}, 0, 0 \right) = \dfrac{\vec{D}}{\epsilon_o\, n_o^2} \Rightarrow \vec{E} \perp \hat{k}$

Direction of Energy Flow or Poynting Vector

$$\vec{S} = \vec{E} \times \vec{H}$$
$$= \vec{E} \times \frac{n}{\mu_o c} (\hat{k} \times \vec{E}) \quad \text{using (1)}$$

$$\vec{S} = \frac{n}{\mu_o c}\left[\hat{k}\,(\vec{E}\cdot\vec{E}) - \vec{E}\,(\hat{k}\cdot\vec{E})\right]$$

$$= \frac{nE^2}{\mu_o c}\,\hat{k}$$

$\therefore$ the light ray travels in the same direction as the wavefront. This ray is called the ordinary ray.

Case 2: $\vec{D} = D(0, -\cos\theta, \sin\theta)$

$$\therefore F_x = 0, \quad F_y = \frac{-D\cos\theta}{\sqrt{2\epsilon_o U_E}}, \quad F_z = \frac{D\sin\theta}{\sqrt{2\epsilon_o U_E}}$$

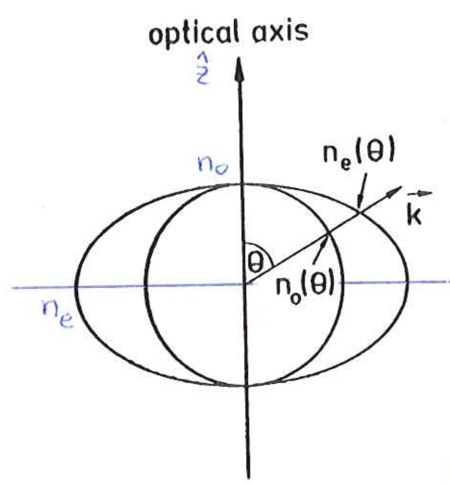
Index Ellipsoid $\Rightarrow$ $1 = \dfrac{1}{n_o^2}\,\dfrac{D^2\cos^2\theta}{2\epsilon_o U_E} + \dfrac{1}{n_e^2}\,\dfrac{D^2\sin^2\theta}{2\epsilon_o U_E}$

$$\frac{2\epsilon_o U_E}{D^2} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

Index of Refraction $n^2 = \vec{F}\cdot\vec{F} = \dfrac{D^2}{2\epsilon_o U_E}$

$$\therefore \frac{1}{n^2} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

$\therefore$ the index of refraction and hence the phase velocity depend on the angle between $\hat{k}$ + $\hat{z}$.



Electric Field $\vec{E} = \frac{D}{\epsilon_0}\left(0, -\frac{\cos\theta}{n_o^2}, \frac{\sin\theta}{n_e^2}\right) \not\parallel \vec{D}$

Exercise: Show 1) $\vec{E}$ is not perpendicular to $\hat{k}$
2) direction of energy flow $\vec{S} \not\parallel \hat{k}$

$\therefore$ the light ray does not propagate in the same direction as the wavefront.

How is this possible?

Huygens principle states that each point passed by the wavefront, acts as a source of waves. For the ordinary ray, these waves have velocity independent of θ and therefore retain their spherical shape. For the so called extraordinary ray, the wavefronts are distorted since $v_\perp \neq v_\parallel$ ($\parallel$ to optical axis) as is shown below.

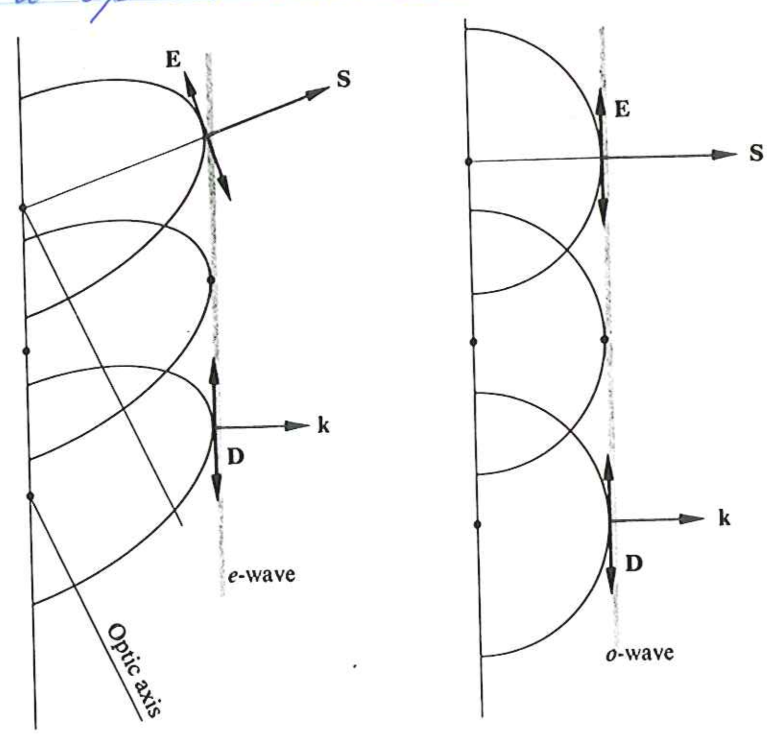
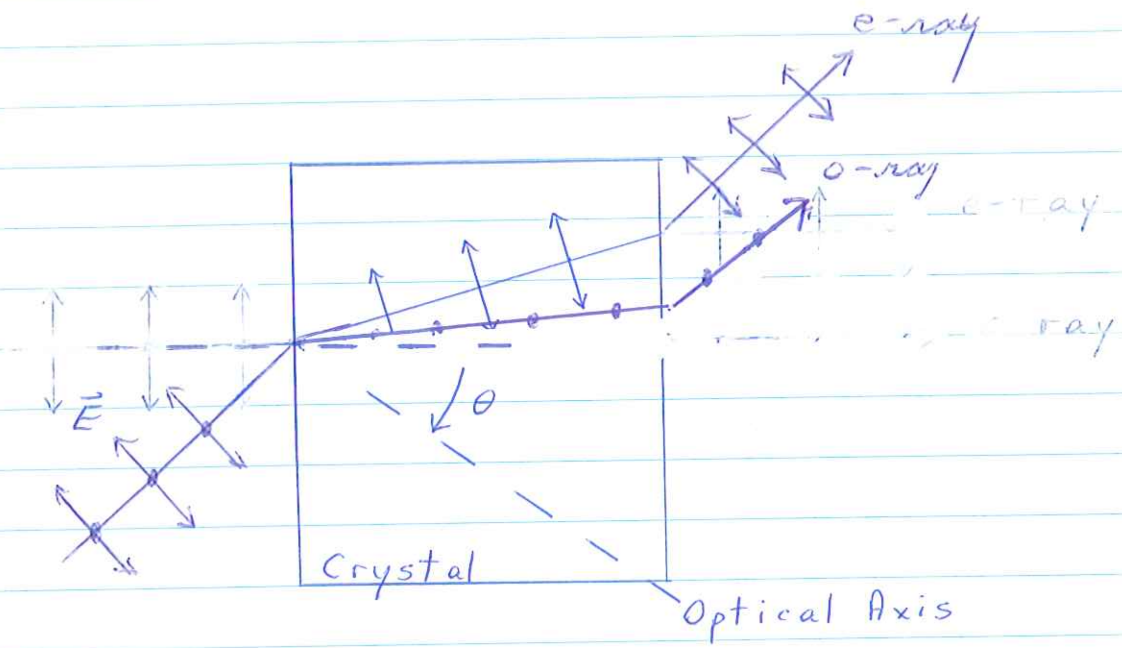


Fig. 8.26 Orientations of the E-, D-, S- and k-vectors.

If unpolarized light is incident on a uniaxial crystal, the light beam is split into ordinary and extraordinary rays which are orthogonally polarized.



Alternative Derivation of n^2 from (3)

$$\frac{1}{n^2} = \frac{k_x^2}{n^2-\epsilon_x} + \frac{k_y^2}{n^2-\epsilon_y} + \frac{k_z^2}{n^2-\epsilon_z}$$

For writing ease, we dropped ^ and primes.

$$(n^2-\epsilon_x)(n^2-\epsilon_y)(n^2-\epsilon_z) = n^2\Big\{ k_x^2(n^2-\epsilon_y)(n^2-\epsilon_z) + k_y^2(n^2-\epsilon_x)(n^2-\epsilon_z) + k_z^2(n^2-\epsilon_x)(n^2-\epsilon_y)\Big\}$$

$$n^6 - n^4(\epsilon_x+\epsilon_y+\epsilon_z) + n^2(\epsilon_x\epsilon_y + \epsilon_x\epsilon_z + \epsilon_y\epsilon_z) - \epsilon_x\epsilon_y\epsilon_z$$

$$= n^2\Big\{ n^4(k_x^2+k_y^2+k_z^2) + n^2k_x^2(-\epsilon_y-\epsilon_z) + n^2k_y^2(-\epsilon_x-\epsilon_z) + n^2k_z^2(-\epsilon_x-\epsilon_y) + k_x^2\epsilon_y\epsilon_z + k_y^2\epsilon_x\epsilon_z + k_z^2\epsilon_x\epsilon_y\Big\}$$

Now $\hat{k}^2 = k_x^2+k_y^2+k_z^2 = 1$. Also let $x \equiv n^2$

$$x^2\left[\epsilon_x+\epsilon_y+\epsilon_z - k_x^2(\epsilon_y+\epsilon_z) - k_y^2(\epsilon_x+\epsilon_z) - k_z^2(\epsilon_x+\epsilon_y)\right]$$

$$+ x\left[-\epsilon_x\epsilon_y - \epsilon_x\epsilon_z - \epsilon_y\epsilon_z + k_x^2\epsilon_y\epsilon_z + k_y^2\epsilon_x\epsilon_z + k_z^2\epsilon_x\epsilon_y\right]$$

$$+ \epsilon_x\epsilon_y\epsilon_z = 0.$$

This has the form $ax^2 + bx + c = 0$

which has solution $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Hence there are 2 solutions for n which depend on the crystal parameters ϵ_x, ϵ_y & ϵ_z.

<u>Special Uniaxial Case.</u>

Let $\epsilon_x = \epsilon_y \equiv \epsilon$

$$x^2\left[2\epsilon + \epsilon_z - k_x^2(\epsilon+\epsilon_z) - k_y^2(\epsilon+\epsilon_z) - 2\epsilon k_z^2\right]$$
$$+x\left[-\epsilon^2 - \epsilon\epsilon_z - \epsilon\epsilon_z + k_x^2\epsilon\epsilon_z + k_y^2\epsilon\epsilon_z + k_z^2\epsilon^2\right]$$
$$+\epsilon^2\epsilon_z = 0.$$

Using $k_x^2 + k_y^2 + k_z^2 = 1$ we get:

$$x^2\left[2\epsilon + \epsilon_z - (1-k_z^2)(\epsilon+\epsilon_z) - 2\epsilon k_z^2\right]$$
$$+x\left[-\epsilon^2 - 2\epsilon\epsilon_z + (1-k_z^2)\epsilon\epsilon_z + k_z^2\epsilon^2\right]$$
$$+\epsilon^2\epsilon_z = 0.$$

$$0 = x^2\left[\epsilon + k_z^2\epsilon_z - k_z^2\epsilon\right]$$

$$+ x\left[-\epsilon^2 - \epsilon\epsilon_z - k_z^2\epsilon\epsilon_z + k_z^2\epsilon^2\right] + \epsilon^2\epsilon_z$$

$$= x^2\left[\epsilon + k_z^2(\epsilon_z - \epsilon)\right]$$

$$-\epsilon x\left[\epsilon + k_z^2(\epsilon_z - \epsilon) + \epsilon_z\right]$$

$$+ \epsilon^2\epsilon_z$$

$$= \left[x - \epsilon\right]\left[x\left(\epsilon + k_z^2(\epsilon_z - \epsilon)\right) - \epsilon_z\epsilon\right] = 0 .$$

$$\therefore x = \epsilon, \quad \frac{\epsilon_z\epsilon}{\epsilon + k_z^2(\epsilon_z - \epsilon)}$$

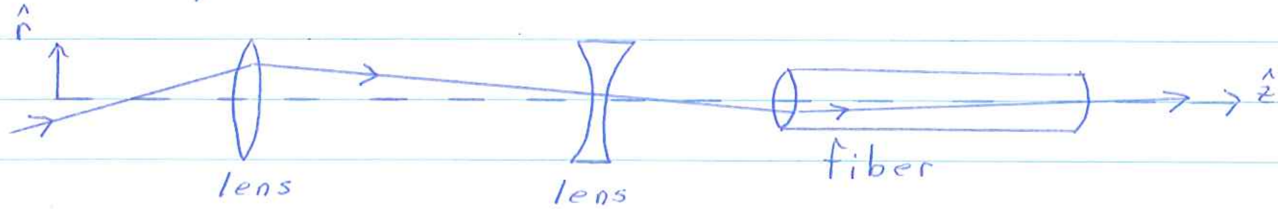
Exercise: Show $x = \dfrac{\epsilon_z\,\epsilon}{\epsilon + k_z^2(\epsilon_z - \epsilon)}$

reduces to $\dfrac{1}{n^2} = \dfrac{\cos^2\theta}{n_o^2} + \dfrac{\sin^2\theta}{n_e^2}$

where $n_o \equiv \sqrt{\epsilon}$ and $n_e \equiv \sqrt{\epsilon_z}$

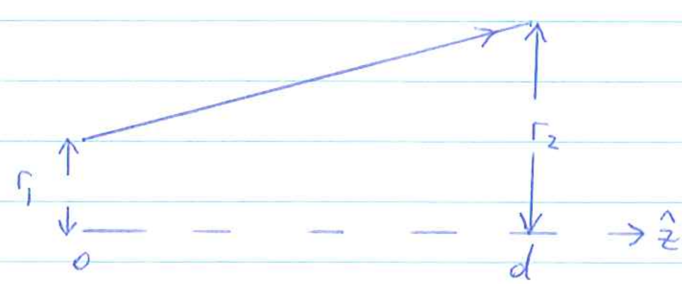
Ray Tracing

From every day experience, we "know" light travels in straight lines. The wavelike properties of light are not apparent since an optical wavelength $\sim 5 \times 10^{-7}$ m. is so small. Consider a light ray traversing an optical system.



A light ray is specified by its radial distance r from the central z axis and its direction or slope $r' \equiv \dfrac{dr}{dz}$.

Consider a ray traversing a homogeneous (or uniform) material of length d.



$$r_2 = r_1 + r_1' d$$

$$r_2' = r_1'$$

This can be conveniently written in matrix form.

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

Suppose after having traversed an initial distance d_1, an additional distance d_2 is traversed. The ray is then described by:

$$\begin{pmatrix} r_3 \\ r_3' \end{pmatrix} = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r_2 \\ r_2' \end{pmatrix}$$

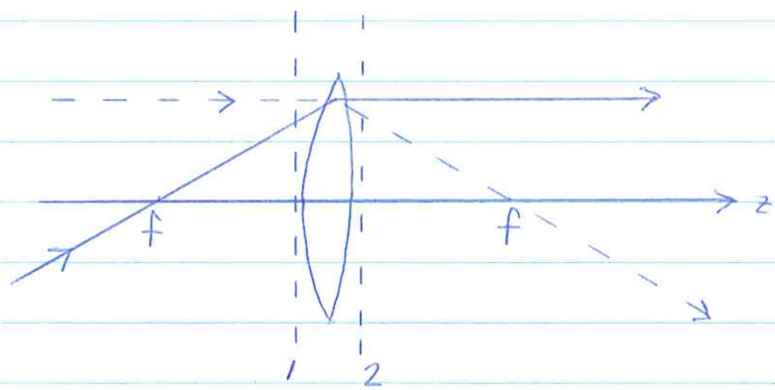
$$= \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & d_1 + d_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

The above illustrates the utility of the matrix approach. The effect of an optical system on a light ray is described by a matrix which is the product of matrices representing the individual optical elements.

Thin Lens Matrix

Consider a lens having focal length f.



$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

By thin, we mean that the lens thickness is negligible such that $r_2 = r_1$. $\Rightarrow A = 1,\ B = 0$.

To find C & D, we first consider an incident ray that is parallel to the z axis. The incoming ray has slope $r_1' = 0$ while the outgoing ray has slope $r_2' = \frac{-r_1}{f}$.

$$r_2' = C r_1 + D r_1'$$

$$\frac{-r_1}{f} = C r_1 + D.0$$

$$C = \frac{-1}{f}$$

Next consider a ray emanating from the focus.
$r_1' = \frac{r_1}{f}$ & $r_2' = 0$.

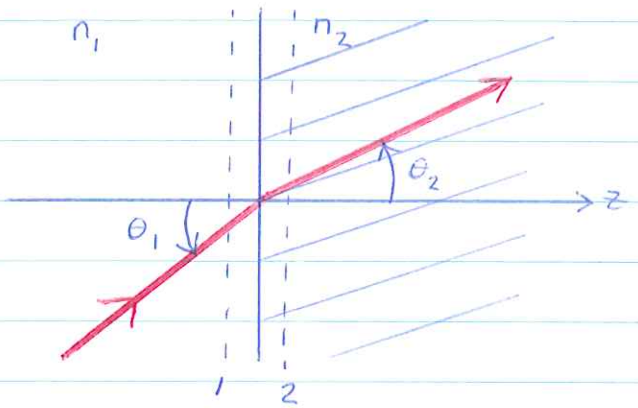
$$r_2' = C r_1 + D r_1'$$

$$0 = \frac{-1}{f} r_1 + D \frac{r_1}{f}$$

$$D = 1$$

$\therefore$ matrix for thin lens is $\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}$.

Matrix For Dielectric Interface



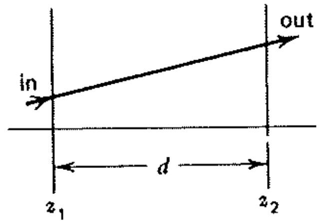
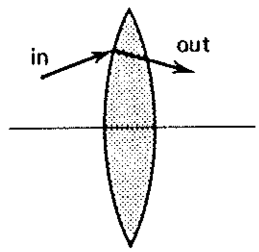
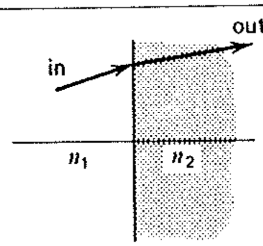
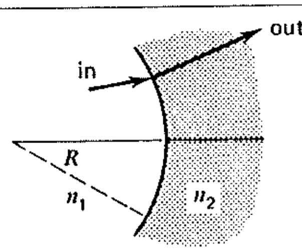
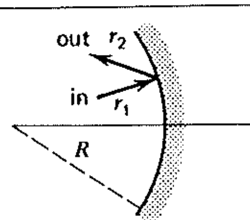
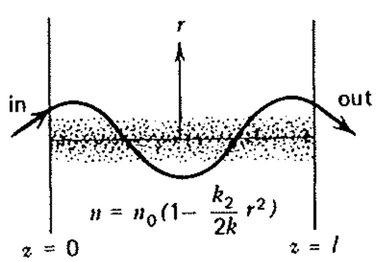
Obviously $\quad r_2 = r_1 \qquad (1)$

Snell's Law $\quad n_2 \sin\theta_2 = n_1 \sin\theta_1$

For rays nearly parallel to the z axis, also known as paraxial rays $\sin\theta \simeq \tan\theta = \dfrac{dr}{dz}$ and the above equation becomes: $\quad n_2 r_2' = n_1 r_1' \qquad (2)$

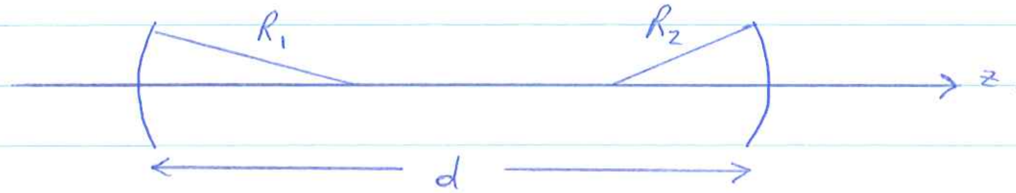
$$(1) + (2) \Rightarrow \begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{n_1}{n_2} \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

Table 6.1. Ray Matrices for Some Common Optical Elements and Media

(1) Straight Section Length d		$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$
(2) Thin Lens: Focal length f ($f>0$, converging; $f<0$, diverging)		$\begin{bmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{bmatrix}$
(3) Dielectric Interface: Refractive indices n_1, n_2		$\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$
(4) Spherical Dielectric Interface: Radius R		$\begin{bmatrix} 1 & 0 \\ \frac{n_2-n_1}{n_2}\frac{1}{R} & \frac{n_1}{n_2} \end{bmatrix}$
(5) Spherical Mirror: Radius of curvature R		$\begin{bmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{bmatrix}$
(6) A medium with a quadratic index profile		$\begin{bmatrix} \cos\left(\sqrt{\frac{k_2}{k}}\,l\right) & \sqrt{\frac{k}{k_2}}\sin\left(\sqrt{\frac{k_2}{k}}\,l\right) \\ -\sqrt{\frac{k_2}{k}}\sin\left(\sqrt{\frac{k_2}{k}}\,l\right) & \cos\left(\sqrt{\frac{k_2}{k}}\,l\right) \end{bmatrix}$

Application of Ray Tracing - Optical Cavity

An optical cavity consists of two mirrors having radii of curvature R_1 and R_2 as shown below.

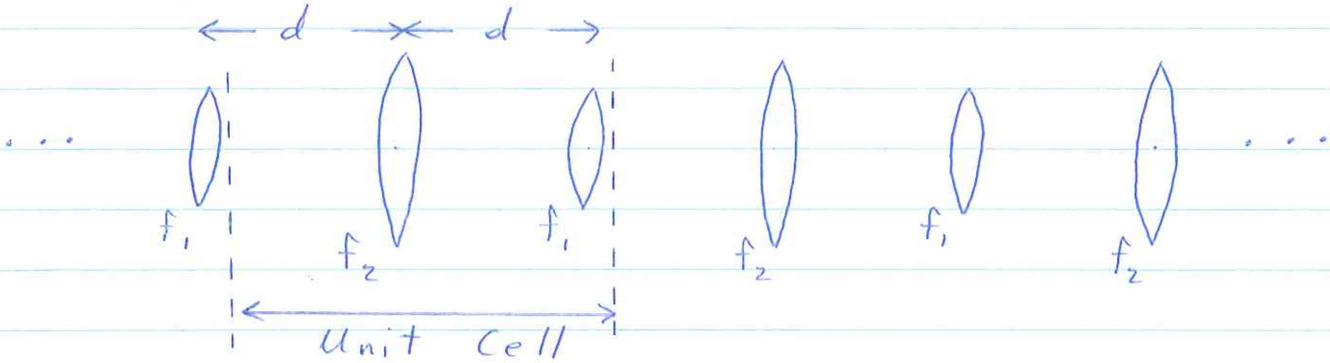


Laser designers commonly insert the "light amplifier" inside a cavity. The overall amplification factor then equals the single pass amplification times the number of passes.

A cavity is said to be stable if a light ray remains inside it after many bounces. If the ray escapes past a mirror, the cavity is called unstable.

Lens Model of Cavity

The above cavity can be modelled by the following lens system.



$f_1 = \frac{R_1}{2}$, $f_2 = \frac{R_2}{2}$ so that the lens matrix is the same as the mirror matrix of the cavity.

A cavity round trip corresponds to traversing the unit cell, which is described by matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f_1} & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{d}{f_2} & d + d\left(1 - \frac{d}{f_2}\right) \\ \frac{-1}{f_1} - \frac{1}{f_2}\left(1 - \frac{d}{f_1}\right) & \left(1 - \frac{d}{f_1}\right)\left(1 - \frac{d}{f_2}\right) - \frac{d}{f_1} \end{pmatrix}$$

Exercise: Show $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1$

After the $(s+1)$th unit cell, a ray is described by:

$$\begin{pmatrix} r_{s+1} \\ r'_{s+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} r_s \\ r'_s \end{pmatrix}$$

or $\quad r_{s+1} = A r_s + B r'_s \qquad (1)$

$\quad r'_{s+1} = C r_s + D r'_s \qquad (2)$

We shall find a second order difference equation for the ray.

$(1) \Rightarrow \quad r'_s = \frac{1}{B}\left(r_{s+1} - A r_s\right) \qquad (3)$

or $\quad r'_{s+1} = \frac{1}{B}\left(r_{s+2} - A r_{s+1}\right) \qquad (4)$

Substituting (3) & (4) into (2) gives:

$$\frac{1}{B}\left(r_{s+2} - A r_{s+1}\right) = C r_s + \frac{D}{B}\left(r_{s+1} - A r_s\right)$$

$$r_{s+2} - A r_{s+1} = BC r_s + D r_{s+1} - AD r_s$$

$$r_{s+2} - (A+D) r_{s+1} + (AD - BC) r_s = 0$$

$$r_{s+2} - (A+D) r_{s+1} + r_s = 0 \quad \text{using exercise result.}$$

We now consider a solution of the form $r_s = r_0 e^{is\theta}$.

$$r_0 e^{i(s+2)\theta} - (A+D) r_0 e^{i(s+1)\theta} + r_0 e^{is\theta} = 0.$$

$$e^{2i\theta} - (A+D) e^{i\theta} + 1 = 0$$

$$e^{i\theta} = \frac{A+D \pm \sqrt{(A+D)^2 - 4}}{2}$$

$$= \frac{A+D}{2} \pm i\sqrt{1 - \left(\frac{A+D}{2}\right)^2}$$

Note that $e^{i\theta}$ is real if $\left(\frac{A+D}{2}\right)^2 > 1$. The ray position $r_s \sim (\text{real}\#)^s$ grows without limit. Hence the condition for a stable cavity is $\left(\frac{A+D}{2}\right)^2 \leq 1$.

The solution then is $r_s = r_1 e^{is\theta} + r_2 e^{-is\theta}$

or $r_s = r_{max} \sin(s\theta + \alpha)$

Condition For Stable Cavity

$$\left(\frac{A+D}{2}\right)^2 \le 1$$

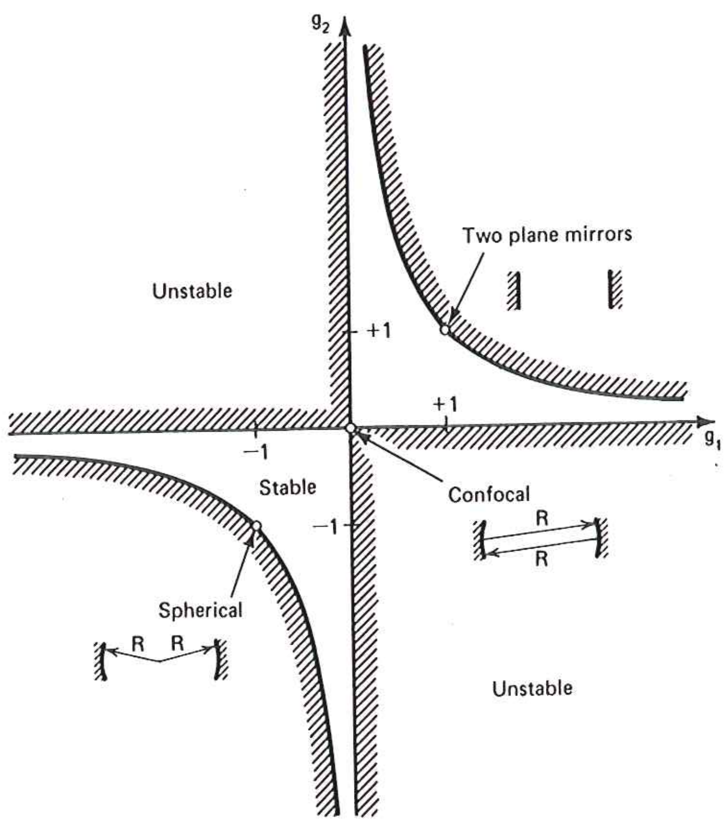
$$-1 \le \frac{A+D}{2} \le 1$$

$$0 \le \frac{A+D}{2} + 1 \le 2$$

$$0 \le \frac{A+D+2}{4} \le 1$$

Exercise: Show $\frac{A+D+2}{4} = \left(1-\frac{d}{R_1}\right)\left(1-\frac{d}{R_2}\right)$

Defining $g \equiv 1 - \frac{d}{R}$, the stability condition becomes: $\boxed{0 \le g_1 g_2 \le 1}$



Unstable Cavity

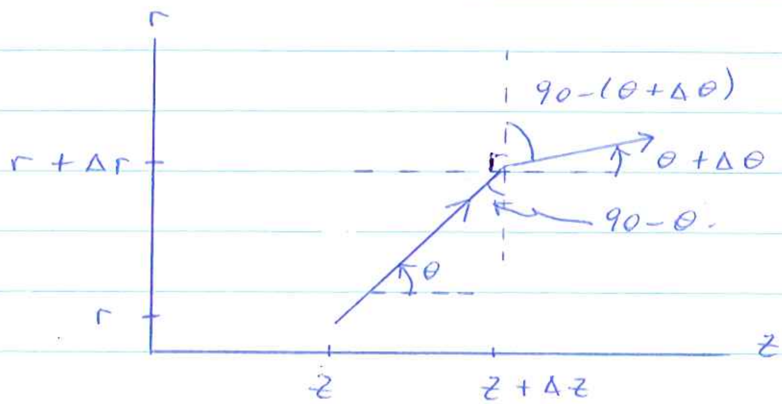
Sometimes lasers are designed using unstable cavities.

eg. Consider a beam that undergoes 10 reflections before it misses a cavity mirror and is amplified by a factor 5 every cavity traversal. If the initial beam has power 1 mwatt, then output beam power is $5^{10} \times 1 \simeq 10$ kwatt.

$\therefore$ unstable cavities are useful for pulsed lasers.

Ray Propagation In Inhomogeneous Media

Consider a material having refraction index $n(r)$.



Snell's Law $\quad n(r)\sin(90-\theta) = n(r+\Delta r)\sin(90-(\theta+\Delta\theta))$

$$n(r)\cos\theta = n(r+\Delta r)\cos(\theta+\Delta\theta)$$

$$\simeq \left[n(r) + \frac{dn}{dr}\Delta r\right]\left(\cos\theta\cos\Delta\theta - \sin\theta\sin\Delta\theta\right)$$

$$\simeq \left(n(r) + \frac{dn}{dr}\Delta r\right)\left(\cos\theta - \Delta\theta\sin\theta\right)$$

$$\simeq n(r)\cos\theta - \Delta\theta\, n(r)\sin\theta + \Delta r\frac{dn}{dr}\cos\theta$$

$$\frac{dn}{dr}\Delta r\cos\theta = n(r)\,\Delta\theta\sin\theta.$$

$$\frac{dn}{dr} = n(r)\frac{\Delta\theta}{\Delta r}\tan\theta$$

$$= n(r)\frac{\Delta\theta}{\Delta r}\frac{\Delta r}{\Delta z}$$

$$= n(r)\frac{\Delta\theta}{\Delta z}.$$

But $\theta \simeq \dfrac{\Delta r}{\Delta z}$ and taking $\lim\limits_{\Delta z \to 0}$ we get:

$$\frac{dn}{dr} = n(r)\,\frac{d^2 r}{dz^2}.$$

or $$\boxed{\frac{d^2 r}{dz^2} = \frac{1}{n(r)}\,\frac{dn}{dr}}$$

Example.

Consider an optical fiber whose refraction index is given by:

$$n(r) = n_0\left(1 - \frac{k_2}{2k}\,r^2\right) \quad \text{where } \frac{k_2 r^2}{2k} \ll 1.$$

$$\frac{d^2 r}{dz^2} = \frac{1}{n_0}\, n_0 \left(-\frac{k_2 r}{k}\right)$$

$$\frac{d^2 r}{dz^2} = -\frac{k_2}{k}\, r$$

$$r(z) = r_1 \cos \sqrt{\frac{k_2}{k}}\, z + r_1' \sqrt{\frac{k}{k_2}} \sin \sqrt{\frac{k_2}{k}}\, z.$$

Differentiating we get:

$$r'(z) = -r_1 \sqrt{\frac{k_2}{k}} \sin \sqrt{\frac{k_2}{k}}\, z + r_1' \cos \sqrt{\frac{k_2}{k}}\, z.$$

or $$\begin{pmatrix} r(z) \\ r'(z) \end{pmatrix} = \begin{pmatrix} \cos \sqrt{\frac{k_2}{k}}\, z & \sqrt{\frac{k_1}{k_2}} \sin \sqrt{\frac{k_2}{k_1}}\, z \\ -\sqrt{\frac{k_2}{k}} \sin \sqrt{\frac{k_2}{k}}\, z & \cos \sqrt{\frac{k_2}{k}}\, z \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

Gaussian Beams

We shall solve the equation for EM waves in a cavity.

$$\nabla^2 \vec{E} - \frac{n^2}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

We consider a linearly polarized cavity wave having electric field

$$E(x,y,z) = E_0\,\psi(x,y,z)\,e^{i(-kz+\omega t)} \qquad \text{where } k = \frac{\omega}{c}n. \qquad (0)$$

The wave equation then becomes:

$$\nabla_t^2 E + \frac{\partial^2 E}{\partial z^2} + \frac{\omega^2 n^2}{c^2}E = 0$$

where the transverse Laplacian $\nabla_t^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

$$\frac{\partial E}{\partial z} = E_0\left(-ik\psi + \frac{\partial \psi}{\partial z}\right)e^{i(-kz+\omega t)}$$

$$\frac{\partial^2 E}{\partial z^2} = E_0\left(-k^2\psi - 2ik\frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2}\right)e^{i(-kz+\omega t)}$$

We then obtain:

$$\nabla_t^2 \psi - 2ik\frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

For a laser beam propagating in the z direction, ψ changes less in the z direction than along the transverse directions. Hence $\frac{\partial^2 \psi}{\partial z^2}$ is neglected.

The middle term is kept since $\frac{\partial \psi}{\partial z}$ is multiplied by the large number k.

$$\therefore \quad \boxed{\nabla_t^2 \psi - 2ik\frac{\partial \psi}{\partial z} = 0} \qquad (1)$$

Lowest Order TEM_{00} Mode

To minimize arithmetic, we consider the cylindrically symmetric case. The above equation then becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) - 2ik\frac{\partial \psi}{\partial z} = 0. \qquad (2)$$

We now consider the solution $\psi_0 = \exp\left[-i\left(P(z) + \frac{kr^2}{2q(z)}\right)\right]$

Exercise: Substituting ψ_0 into (2), show one obtains:

$$\left\{\left[\frac{k^2}{q^2(z)}\left(q'(z)-1\right)\right]r^2 - 2k\left[P'(z) + \frac{i}{q(z)}\right]\right\}\psi_0 = 0.$$

Equating the coefficients of r^2 and r^0 to zero yields:

$$P'(z) = \frac{-i}{q(z)} \qquad (3)$$

$$q'(z) = 1 \qquad (4)$$

Integrating (4) we obtain: $q(z) = z + q_0$

Determination of q_0

$$\psi_0(z) = \exp\left(\frac{-ikr^2}{2q(z)}\right) \exp\left(-iP(z)\right)$$

If q is real then the first term oscillates more and more rapidly as r increases. This differs from our expectation that $\psi_0 \to 0$ as $r \to \infty$. Hence we conclude that q is complex. Incorporating the real portion of q_0 into z, we write:

$$q(z) = z + iz_0. \qquad z, z_0 \in R.$$

Note: $z=0$ is then defined to be the point where $q(z)$ is strictly imaginary.

Exercise: Show that $\dfrac{1}{q(z)} = \dfrac{z}{z^2+z_0^2} - \dfrac{iz_0}{z^2+z_0^2}$

$$\therefore \psi_0(z) = \exp\left(\frac{-kz_0 r^2}{2(z^2+z_0^2)}\right) \exp\left(\frac{-ikzr^2}{2(z^2+z_0^2)}\right) \exp\left(-iP(z)\right)$$

$$= \exp\left(\frac{-r^2}{\omega^2(z)}\right) \exp\left(\frac{-ikr^2}{2R(z)}\right) \exp\left(-iP(z)\right) \qquad (5)$$

where $\omega^2(z) \equiv \dfrac{2z_0}{k}\left(1 + \left(\dfrac{z}{z_0}\right)^2\right)$

$= \omega_0^2\left(1 + \left(\dfrac{z}{z_0}\right)^2\right)$ where $\omega_0^2 \equiv \dfrac{2z_0}{k}$

and $R(z) \equiv \dfrac{z^2 + z_0^2}{z}$

Determination of $P(z)$

$$P'(z) = \frac{-i}{q(z)} \qquad (3)$$

$$\frac{dP}{dz} = \frac{-i}{z + iz_0}$$

$$P(z) - P(0) = -i\Big[\ln(z + iz_0)\Big]_{z=0}^{z}$$

$$= -i \ln\left(\frac{z + iz_0}{iz_0}\right)$$

We neglect $P(0)$ since this can be incorporated into E_0.

$$\therefore\ iP(z) = \ln\left(1 - i\frac{z}{z_0}\right)$$

Now $1 - \dfrac{iz}{z_0} = \left[1 + \left(\dfrac{z}{z_0}\right)^2\right]^{1/2} \exp\left[-i \,\mathrm{Arctan}\left(\dfrac{z}{z_0}\right)\right]$

Exercise: Show that $\exp[-iP(z)] = \dfrac{1}{\sqrt{1 + \left(\frac{z}{z_0}\right)^2}} \exp\left[i \,\mathrm{Arctan}\left(\dfrac{z}{z_0}\right)\right]$. (6)

Solution

Using equations (0), (5) & (6) we get the result:

$$\boxed{\begin{aligned} \frac{E(r,z)}{E_o} = \; & \frac{w_o}{w(z)} \exp\left(\frac{-r^2}{w^2(z)}\right) && \text{amplitude factor} \\ & \cdot \exp\left\{-i\left[kz - \operatorname{Arctan}\left(\frac{z}{z_o}\right)\right]\right\} && \text{longitudinal phase factor} \\ & \cdot \exp\left[-i\frac{kr^2}{R(z)}\right] && \text{radial phase factor} \\ & \cdot \exp(i\omega t) \end{aligned}}$$

where $w^2(z) = w_o^2\left[1 + \left(\frac{\lambda z}{\pi w_o^2}\right)^2\right]$

$$R(z) = z\left[1 + \left(\frac{\pi w_o^2}{\lambda z}\right)^2\right]$$

$$z_o = \frac{\pi w_o^2}{\lambda}.$$

Exercise: Derive these expressions for $w(z)$, $R(z)$ & z_o.

Interpretation of Solution

Field Amplitude

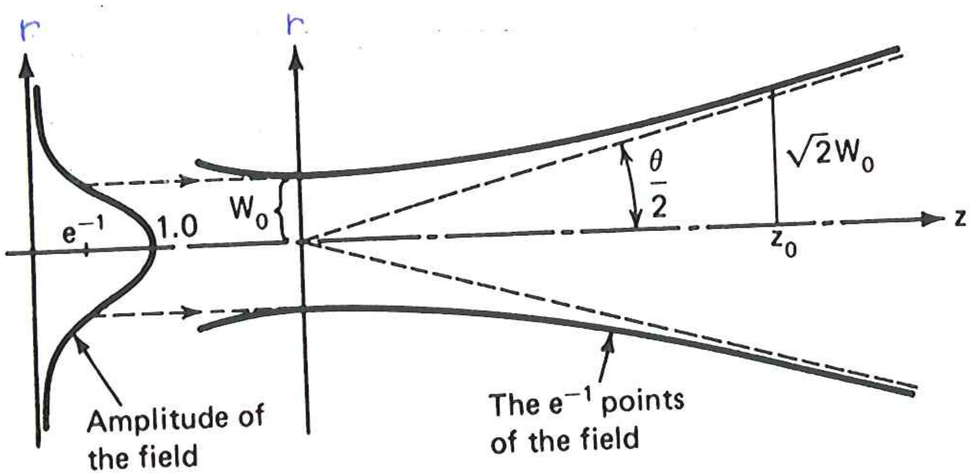
$$\left|\frac{E(r,z)}{E_0}\right| = \frac{w_0}{w(z)} \exp\left(\frac{-r^2}{w^2(z)}\right)$$

$w(z)$ is the radial position where the electric field is reduced by a factor $1/e$ from its peak value $E(r=0)$. Most of the beam energy is contained inside the region $r < w(z)$ since the laser power is proportional to the square of the electric field. Therefore w is called the spot size.

The spot size changes as the beam propagates since:

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$$

At $z = 0$ the beam has minimum width w_0.
At $z = z_0$ the beam width is $\sqrt{2}\, w_0$.



Beam Divergence

When $z \gg z_0$ $\quad w(z) \simeq w_0 \dfrac{z}{z_0}$

The beam divergence angle θ, shown in the preceding figure is given by:

$$\frac{\theta}{2} \simeq \frac{w(z)}{z}$$

Exercise: Show $\theta = \dfrac{2\lambda}{\pi w_0}$

Note that the smaller the minimum beamwidth w_0, the greater is the divergence angle θ.

Beam Power

The power in the beam is given by:

$$P = \frac{v\epsilon}{2} \int_{\substack{\text{beam} \\ \text{cross section}}} E^2 \, da.$$

Exercise: Show $P = \dfrac{1}{2} \dfrac{E_0^2}{\eta} \dfrac{\pi w_0^2}{2}$ where $\eta \equiv \sqrt{\dfrac{\mu}{\epsilon}}$

Hence the power P is independent of z as expected due to energy conservation.

Longitudinal Phase Factor

The so called longitudinal phase

$$\phi = kz - \arctan\left(\frac{z}{z_0}\right)$$

$$= \left[k - \frac{1}{z}\arctan\left(\frac{z}{z_0}\right)\right] z.$$

This gives rise to a wave having phase velocity

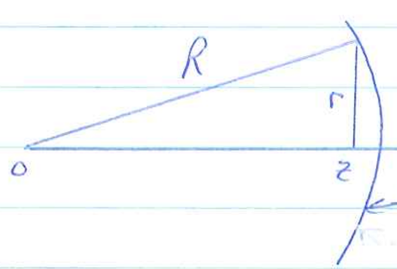
$$v_p = \frac{\omega}{k - \frac{1}{z}\arctan\left(\frac{z}{z_0}\right)}$$

$$= \frac{c/n}{1 - \frac{1}{kz}\arctan\left(\frac{z}{z_0}\right)} \qquad \text{using } k = \frac{\omega}{c} n$$

Hence the phase velocity of the wave is slightly greater than that of a plane wave in a uniform medium.

Radial Phase Factor

The radial phase factor $\exp\left[\frac{-ikr^2}{2R(z)}\right]$ arises because the wavefront is spherical. To see this, consider a spherical wave emanating from the origin with curvature R.



wavefront $E \sim \frac{1}{R} e^{-ikR}$

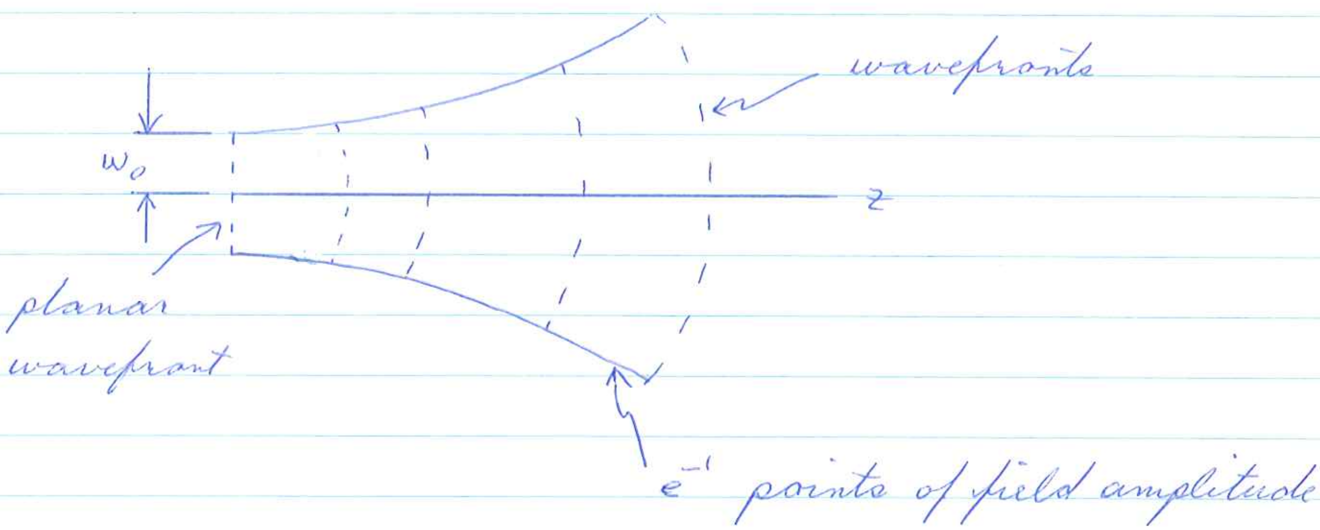
Exercise: Show for $z >> r$ that $R \simeq z + \frac{r^2}{2R}$

Then $E \sim \frac{1}{R} \exp(-ikz) \exp\left(\frac{-ikr^2}{2R}\right)$.

For the gaussian wave, the curvature is not a constant but is given by:

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

When $z >> z_0$, $R(z) \simeq z$ and the wave appears to emanate from the origin.
When $z = 0$, $R(z) = \infty$ and the wavefront is planar.



Higher Order Modes

The general solution of the equation

$$\nabla_t^2 \psi - 2ik \frac{\partial \psi}{\partial z} = 0$$

is:

$$\frac{E_{mn}(x,y,z)}{E_o} = H_m\left(\frac{\sqrt{2}\, x}{w(z)}\right) H_n\left(\frac{\sqrt{2}\, y}{w(z)}\right)$$

$$\cdot \frac{w_o}{w(z)} \exp -\left(\frac{x^2+y^2}{w^2(z)}\right)$$

$$\cdot \exp\left\{-i\left[kz - (1+m+n)\,\text{Arctan}\left(\frac{z}{z_o}\right)\right]\right\}$$

$$\cdot \exp\left(\frac{-ikr^2}{2R(z)}\right) \cdot \exp(i\omega t)$$

where $w(z)$, w_o, z_o, $R(z)$ are defined as previously. E_{mn} is also called the TEM_{mn} mode.

Hermite Polynomials

$H_n(u)$ is the Hermite polynomial of degree n and is defined by:

$$H_n(u) \equiv (-1)^n e^{u^2} \frac{d^n e^{-u^2}}{du^n}$$

$H_o(u) = 1$

$H_1(u) = 2u$

$H_2(u) = 2(2u^2 - 1)$

The Hermite polynomials strongly affect the field in the transverse plane as shown in the following figure.

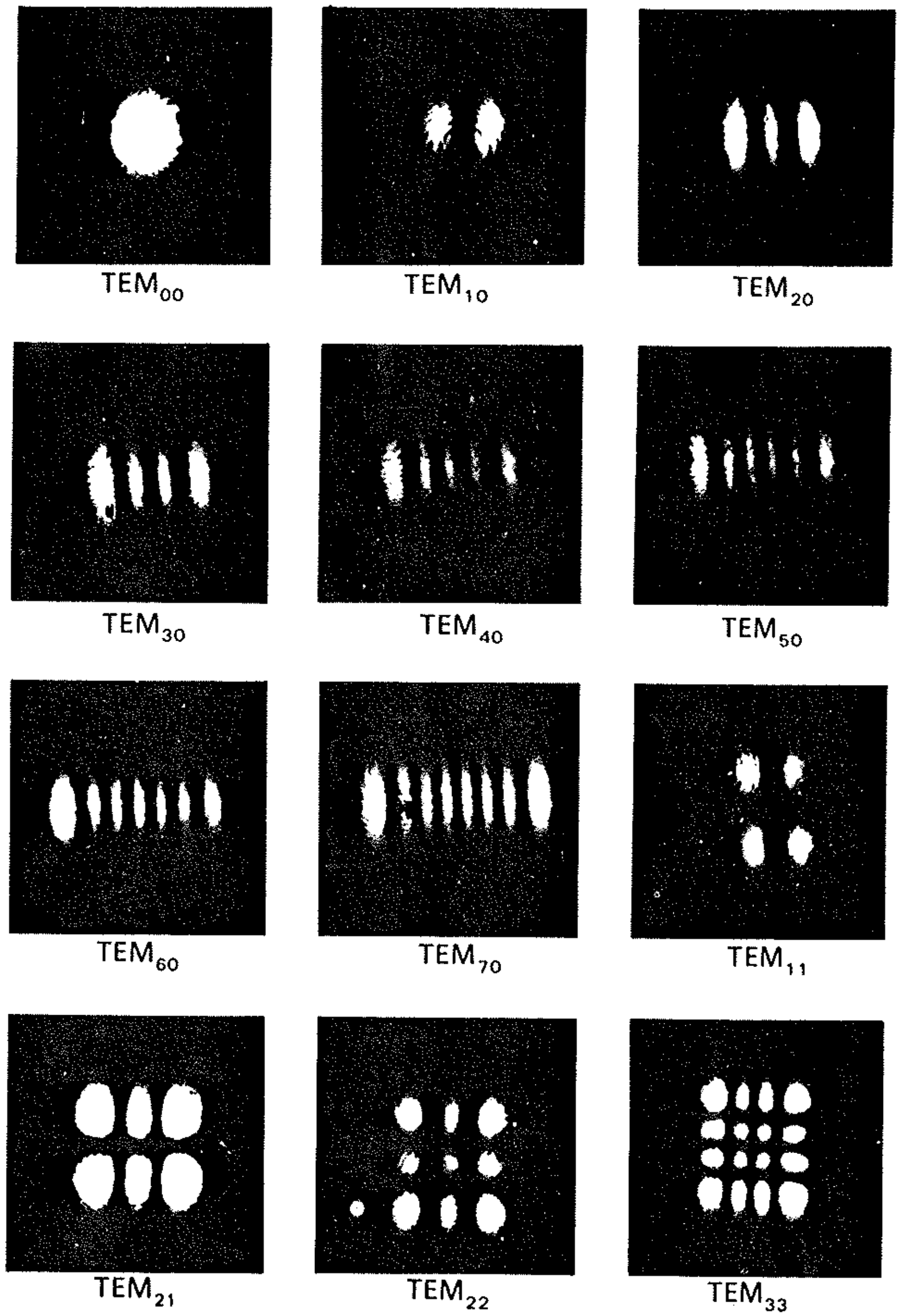


Figure 6.7 Some low-order optical-beam modes. (After Ref. 9.)

Propagation of Gaussian Beams.

Question: How do Hermite - Gaussian beams propagate in an optical system?

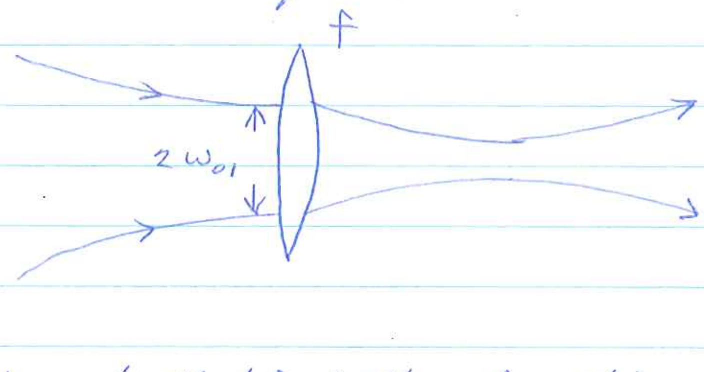
Answer: The beam parameter $\frac{1}{q} \equiv \frac{1}{R} - \frac{i\lambda}{\pi w^2}$ changes according to:

$$q_2 = \frac{A q_1 + B}{C q_1 + D} \quad OR \quad \frac{1}{q_2} = \frac{C + D/q_1}{A + B/q_1}$$

where $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the same matrix as is used to study ray propagation through an optical system. The proof of the above statement is omitted.

Example: Beam Focusing

Consider a gaussian beam incident on a lens. At the lens, the incoming wave has waist w_{01} and a planar wavefront.



We shall find the location of the focus and the focal beam size.

Solution

at the lens, the incoming beam parameter $\frac{1}{q_1} = \frac{1}{\infty} - \frac{i\lambda}{\pi \omega_{01}^2}$

At position z after the lens, the beam parameter $\frac{1}{q_2} = \frac{1}{R(z)} - \frac{i\lambda}{\pi \omega^2(z)}$

The matrix for the lens plus a length of free space z is

$$T = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1 - z/f & z \\ -1/f & 1 \end{pmatrix}$$

$$\therefore \frac{1}{q_2} = \frac{C + D/q_1}{A + B/q_1}$$

$$= \frac{-\frac{1}{f} + \frac{1}{q_1}}{1 - \frac{z}{f} + \frac{z}{q_1}}$$

Exercise: Show that $$\frac{1}{R(z)} = \frac{-\frac{1}{f} + z\left(\frac{1}{f^2} + \frac{1}{z_{01}^2}\right)}{\left(1 - \frac{z}{f}\right)^2 + \left(\frac{z}{z_{01}}\right)^2}$$

$$\frac{\lambda}{\pi \omega^2(z)} = \frac{1/z_{01}}{\left(1 - \frac{z}{f}\right)^2 + \left(\frac{z}{z_{01}}\right)^2}$$

where $z_{01} \equiv \frac{\pi \omega_{01}^2}{\lambda}$.

The focus occurs when the beam waist is smallest.

i.e. $\left.\dfrac{\partial w}{\partial z}\right|_{z=z_m} = 0$

<u>Exercise</u>: Show $z_m = \dfrac{f}{1+(f/z_{01})^2}$

Note that in the limit $\left(\dfrac{f}{z_{01}}\right)^2 \ll 1$, the focus is at the expected position $z_m \simeq f$.

<u>Exercise</u>: In the limit $\left(\dfrac{f}{z_{01}}\right)^2 \ll 1$, show that the focal waist $w_0(z_m) \simeq \dfrac{\lambda f}{\pi w_{01}}$.

<u>Numerical Example</u>

$w_{01} = 1$ mm
$\lambda = 5900$ Å (yellow light)
$f = 1$ cm.

$$z_{01} = \frac{\pi w_{01}^2}{\lambda} = \frac{\pi (10^{-3})^2}{5900 \times 10^{-10}} \simeq 5 \text{ m}$$

$$\left(\frac{f}{z_{01}}\right)^2 \simeq \frac{1}{25{,}000} \ll 1$$

focal spot size $w_0(z_m) = \dfrac{\lambda f}{\pi w_{01}} = \dfrac{5900 \times 10^{-10} \times 10^{-2}}{\pi \times 10^{-3}}$

$= 2 \times 10^{-6}$ m

Stable Cavities

Cavities only allow waves having certain beam profiles to be reflected back and forth without loss. For the stable cavities that we consider, we assume that these so called eigenmodes are the Hermite-Gaussian beams. The complex beam parameter must then satisfy the following

$$q(z + \text{round trip}) = q(z) \quad \text{where } z \in \text{cavity}$$

Now $q(z + \text{round trip}) = \dfrac{A\,q(z) + B}{C\,q(z) + D}$ where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

is the matrix for a cavity round trip.

$$\therefore \quad q = \frac{Aq + B}{Cq + D}$$

$$Cq^2 + Dq = Aq + B$$

$$\frac{B}{q^2} + \frac{(A-D)}{q} - C = 0$$

$$\frac{1}{q} = \frac{-(A-D) \pm \left[(A-D)^2 + 4BC\right]^{1/2}}{2B}$$

$$= \frac{(D-A)}{2B} \pm \frac{1}{B}\left[\left(\frac{A-D}{2}\right)^2 + BC\right]^{1/2}$$

Exercise: Show that $AD - BC = 1$ for lenses and propagation through empty space.

$$\therefore \frac{1}{q} = \frac{(D-A)}{2B} \pm \frac{1}{B}\left[\left(\frac{A+D}{2}\right)^2 - 1\right]^{1/2}$$

$$= \frac{(D-A)}{2B} - \frac{i}{B}\left[1 - \left(\frac{A+D}{2}\right)^2\right]^{1/2}$$

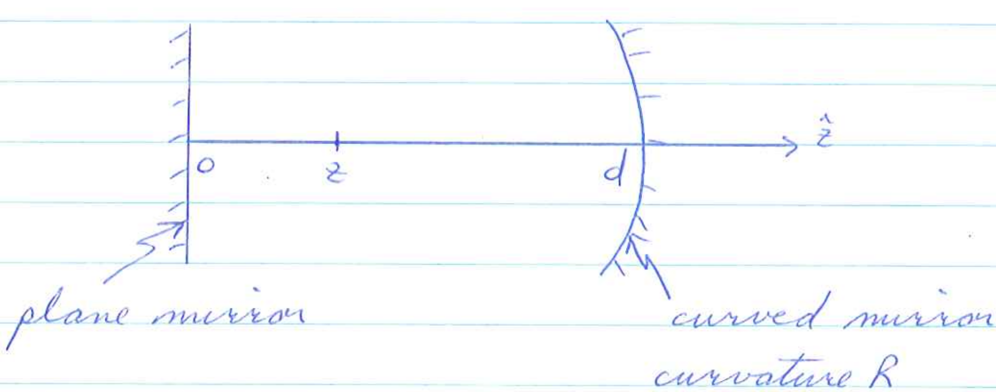
Comparing this to the definition $\frac{1}{q} = \frac{1}{R} - \frac{i\lambda}{\pi w^2}$

we obtain:

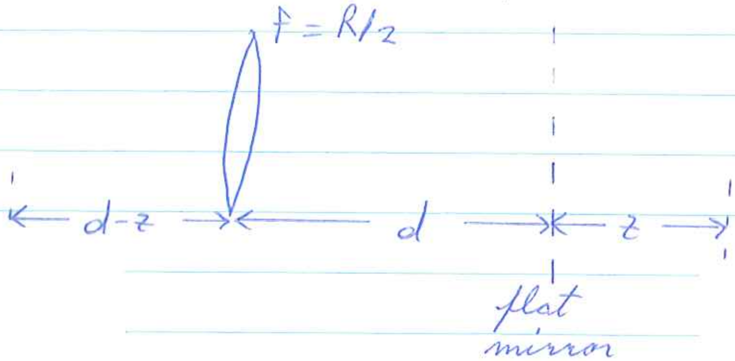
$$\boxed{\begin{aligned} R(z) &= \frac{2B}{D-A} && (1) \\ \frac{\pi w^2}{\lambda} &= \frac{B}{\left[1 - \left(\frac{A+D}{2}\right)^2\right]^{1/2}} && (2) \end{aligned}}$$

Example

Find the gaussian eigenmode for the following cavity.



Beginning at a point z in the cavity, the equivalent lens system for one cavity round trip is:



This unit cell has the matrix:

$$T = \begin{pmatrix} 1 & d+z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d-z \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{(d+z)}{f} & 2d - \frac{(d^2 - z^2)}{f} \\ \frac{-1}{f} & 1 - \frac{(d-z)}{f} \end{pmatrix}$$

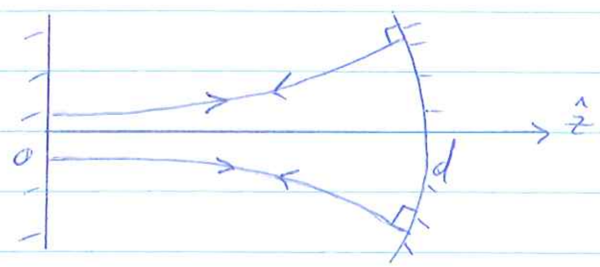
Now $$R(z) = \frac{2B}{D - A} = \frac{2\left[2d - \frac{(d^2 - z^2)}{f}\right]}{\frac{2z}{f}}$$

Exercise: Show $R(z) = \frac{d(R-d)}{z} + z$.

Note that $R(0) = \infty$ = left mirror curvature

" " $R(d) = R$ = right " "

Hence the Gaussian eigenmode "fits" into the cavity.



Also
$$\frac{\pi w^2}{\lambda} = \frac{B}{\left[1 - \left(\frac{A+D}{2}\right)^2\right]^{1/2}}$$
$$= \frac{2d - \frac{(d^2 - z^2)}{f}}{\left[1 - \left(1 - \frac{d}{f}\right)^2\right]^{1/2}}$$
$$= \frac{d - \frac{(d^2 - z^2)}{R}}{\sqrt{\frac{d}{R}\left(1 - \frac{d}{R}\right)}}$$

Exercise: Show that the minimum beam waist at $z = 0$, is given by
$$w_0 = \left(\frac{\lambda}{\pi}\right)^{1/2} \left[dR\,(1 - d/R)\right]^{1/4}.$$

Example

If $\lambda = 632.8$ nm (red light)
$d = 1$ m.
$R = 20$ m.
$\Rightarrow$ spot size $w_0 = \left(\frac{632.8 \times 10^{-9}}{\pi}\right)^{1/2} \left[1.20\,(1 - 1/20)\right]^{1/4}$
$= .94$ mm.

Exercise: Evaluate the spotsize on the curved mirror.
(Answer = .96 mm)

Mode Volume

Mode Volume $V \equiv \frac{1}{E_0^2} \int_0^d \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(xyz)\, E^*(xyz)\, dx\, dy\, dz$

where E_0 is the peak value of the electric field E inside the cavity of length d. V gives a measure of the volume occupied by a mode. This may be used to compute the # of atoms the beam interacts with, and hence its attenuation or amplification.

For the Hermite-Gaussian mode E_{mn} we get:

$$V_{mn} = \int_0^d \frac{w_0^2}{w^2(z)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_m^2\left(\frac{\sqrt{2}\,x}{w}\right) e^{-2x^2/w^2} H_n^2\left(\frac{\sqrt{2}\,y}{w}\right) e^{-2y^2/w^2}\, dx\, dy\, dz$$

$$= \int_0^d \frac{w_0^2}{2}\, dz \int_{-\infty}^{\infty} H_m^2(u)\, e^{-u^2}\, du \int_{-\infty}^{\infty} H_n^2(u)\, e^{-u^2}\, du$$

using $u = \frac{\sqrt{2}\,x}{w}, \frac{\sqrt{2}\,y}{w}$

Now $\int_{-\infty}^{\infty} H_m^2(u)\, e^{-u^2}\, du = 2^m\, m!\, \sqrt{\pi}$

$$\therefore \boxed{V_{mn} = \frac{\pi w_0^2}{2}\, d\; m!\, n!\, 2^{m+n}}$$

Example

For the TEM_{00} mode in the cavity considered previously

$w_o = .94\ mm$

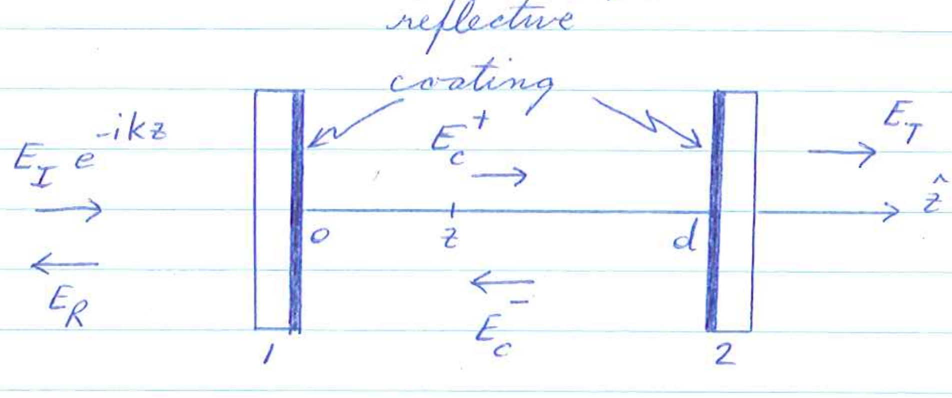
$d = 1\ m.$

$$\Longrightarrow V_{00} = \frac{\pi\,(.94\ mm)^2}{2}\ 1m.$$

$$= 1.39\ cm^3.$$

Resonant Cavities

Consider a cavity having planar mirrors.



E_I is incident electric field amplitude
E_R " reflected " "
E_T " transmitted " "

$E_c^{\pm}$ are fields in cavity propagating in $\pm\hat{z}$ directions. They are produced by transmission of the incident wave into the cavity and subsequent reflection from mirrors 1 & 2.

$$E_c^+(z) = t_1 E_I e^{-ikz}\Big[\underbrace{1}_{\text{from incident wave}} + \underbrace{r_1 r_2 e^{-i\phi}}_{\text{after 1 round trip}} + \underbrace{r_1^2 r_2^2 e^{-2i\phi}}_{\text{after 2 round trips}} + \cdots\Big]$$

r (t) is the electric field reflectance (transmittance). ϕ is the phase shift due to a round trip. For a plane wave $\phi = 2kd$ where $k = \frac{\omega}{c}$.

$$E_c^+(z) = \frac{t_1 E_I e^{-ikz}}{1 - r_1 r_2 e^{-i\phi}} \qquad (1)$$

The incident and reflected fields are related to the cavity fields by the following:

$$E_R = -r_1 E_I + t_1 E_c^-(0) \qquad (2)$$

- not + since light is reflected inside glass at air interface

$$E_c^+(0) = t_1 E_I + r_1 E_c^-(0) \qquad (3)$$

where $r_1^2 + t_1^2 = 1$.

Solving (3) for $E_c^-(0)$ and substituting in (2) gives:

$$E_R = -r_1 E_I + \frac{t_1}{r_1}\left[E_c^+(0) - t_1 E_I\right]$$

$$= \frac{-E_I + t_1 E_c^+(0)}{r_1}$$

Using (1) we get:

$$\frac{E_R}{E_I} = \frac{1}{r_1}\left[-1 + \frac{t_1^2}{1 - r_1 r_2 e^{-i\phi}}\right]$$

$$= \frac{-1 + r_1 r_2 e^{-i\phi} + t_1^2}{r_1(1 - r_1 r_2 e^{-i\phi})}$$

$$= \frac{-r_1 + r_2 e^{-i\phi}}{1 - r_1 r_2 e^{-i\phi}}$$

Intensity reflection coefficient

$$R \equiv \left| \frac{E_R}{E_I} \right|^2$$

$$= \frac{-r_1 + r_2 e^{-i\phi}}{1 - r_1 r_2 e^{-i\phi}} \cdot \frac{-r_1 + r_2 e^{i\phi}}{1 - r_1 r_2 e^{i\phi}}$$

$$= \frac{r_1^2 + r_2^2 - 2 r_1 r_2 \cos\phi}{1 - 2 r_1 r_2 \cos\phi + r_1^2 r_2^2}$$

$$= \frac{R_1 + R_2 - 2\sqrt{R_1 R_2}\cos\phi}{1 - 2\sqrt{R_1 R_2}\cos\phi + R_1 R_2} \qquad \text{where } R \equiv r^2$$

Intensity transmission coefficient

$$T = 1 - R$$

Exercise: Show that $\boxed{T = \dfrac{(1-R_1)(1-R_2)}{(1-\sqrt{R_1 R_2})^2 + 4\sqrt{R_1 R_2}\sin^2 \phi/2}}$

where $\dfrac{\phi}{2} = \dfrac{\omega d}{c}$.

Note that $T_{max} = \dfrac{(1-R_1)(1-R_2)}{(1-\sqrt{R_1 R_2})^2}$

$$T_{min} = \frac{(1-R_1)(1-R_2)}{(1+\sqrt{R_1 R_2})^2}$$

Exercise: Evaluate T_{min} & T_{max} for $R_1 = R_2 = .99$.

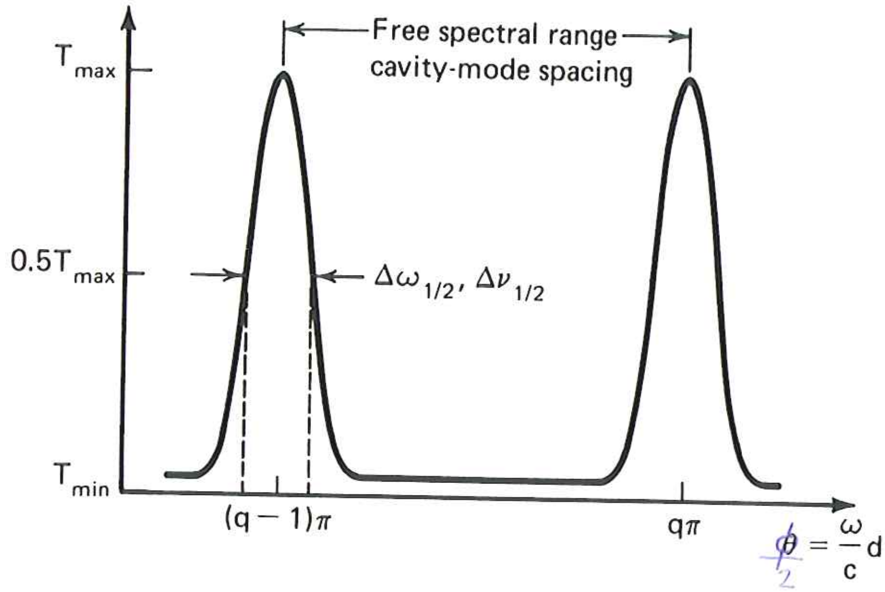


Figure 6-4. *Transmission or bandpass characteristics of a Fabry–Perot cavity.*

The above plot should be an incredible shock!!! Why? A single mirror having $R = .99$ reflects nearly all the light. Our intuition tells us that adding a second reflects still more light. Instead we have shown two parallel mirrors result in T_{max} transmission when

$$\frac{\omega d}{c} = q\pi \quad \text{where } q \text{ is an integer.}$$

$$\text{OR} \quad 2d = q\lambda.$$

i.e. cavity round trip = integer × wavelength

The cavity is then said to be in resonance with the light.

Free Spectral Range ν_{FSR}

$$\nu_{FSR} \equiv \text{peak spacing}$$
$$= \nu_q - \nu_{q-1}$$

$$\boxed{\nu_{FSR} = \frac{c}{2d}}$$

eg. $d = 1$ m. $\nu_{FSR} = \dfrac{3 \times 10^8 \text{ m/sec}}{2 \times 1 \text{ m}}$

$= .15$ GHz.

Exercise: Show that the full width at half max. of the transmission peaks is given by:

$$\Delta\nu_{1/2} = \frac{c}{2d} \frac{1 - (R_1 R_2)^{1/2}}{\pi (R_1 R_2)^{1/4}}$$

eg. $\left.\begin{array}{l} R_1 = R_2 = .99 \\ d = 1 \text{ m.} \end{array}\right\} \Longrightarrow \Delta\nu_{1/2} = 480$ kHz.

Q Factor

The Q or Quality factor describes the sharpness of the peak.

$$Q \equiv \frac{\nu_q}{\Delta\nu_{1/2}}$$

$$Q = \frac{2\pi d}{\lambda} \frac{(R_1 R_2)^{1/4}}{1 - (R_1 R_2)^{1/2}}$$

eg. $\left.\begin{array}{l} R_1 = R_2 = .99 \\ d = 1 \text{ m.} \\ \lambda = 632.8 \text{ nm} \end{array}\right\} \Longrightarrow Q = 9.9 \times 10^8$

Finesse

Finesse $F \equiv \dfrac{\nu_{FSR}}{\Delta\nu_{1/2}}$

$$\boxed{F = \frac{\pi \left(R_1 R_2\right)^{1/4}}{1 - \left(R_1 R_2\right)^{1/2}}}$$

eg. $R_1 = R_2 = .99 \implies F = 313$

Resonance of The Hermite-Gaussian Modes

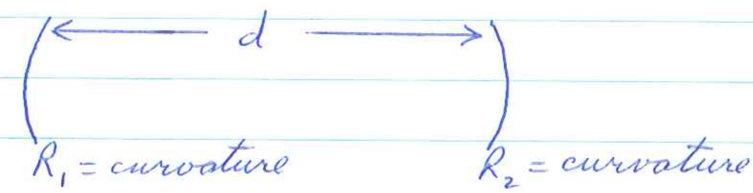
Up to now, only plane waves have been considered to simplify the analysis. We found that a cavity of length d is in resonance if the phase
$\phi(z=d) - \phi(z=0) = q\pi$.

For the TEM_{mn} Hermite-Gaussian mode, this yields:

$$kd - (1+m+n)\,\mathrm{Arctan}\left(\frac{d}{z_0}\right) = q\pi$$

$$\nu_{mnq} = \frac{c}{2d}\left\{ q + \frac{(1+m+n)}{\pi}\,\mathrm{Arctan}\left(\frac{d}{z_0}\right)\right\}$$

For the cavity



one can show $\nu_{mnq} = \dfrac{c}{2d}\left\{ q + \dfrac{(1+m+n)}{\pi}\,\mathrm{Arccos}\sqrt{g_1 g_2}\right\}$

Here $g_{1,2} \equiv 1 - \dfrac{d}{R_{1,2}}$.

The modes are said to be degenerate since different modes eg. $(m=1, n=3, q)$ & $(m=2, n=2, q)$ have the same frequency.

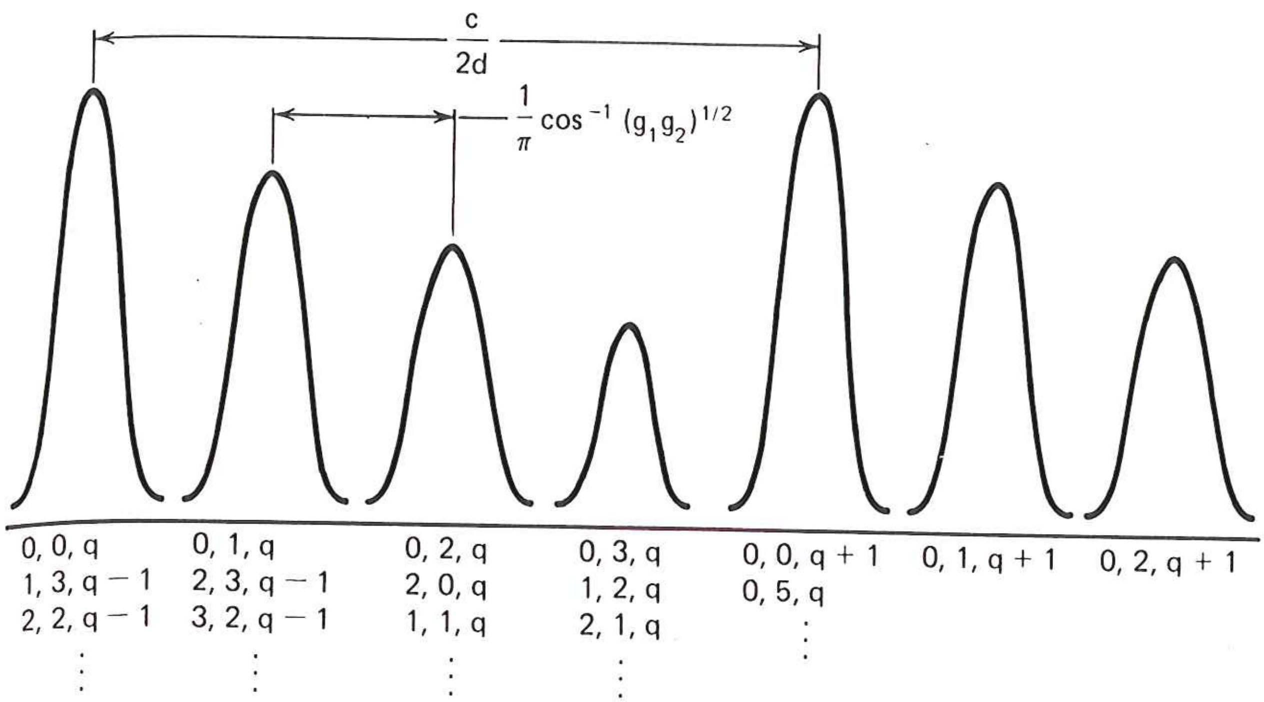
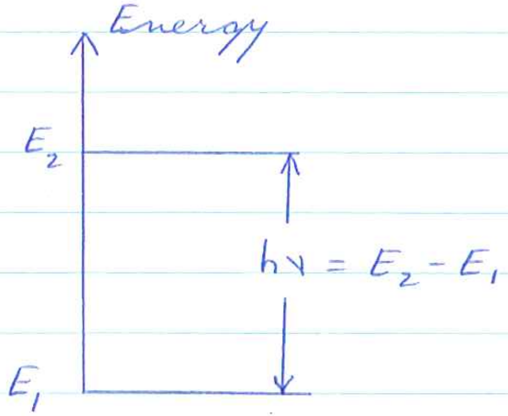


Figure 6-7. *Frequency degeneracy in an optical cavity.*

Einstein A + B Coefficients

Consider a two level atom shown below.



Let $\begin{Bmatrix} N_2 \\ N_1 \end{Bmatrix}$ be density of atoms in state $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$.

The atoms interact with light in 3 ways.

1) Spontaneous Emission: Atoms in the excited state can spontaneously decay to the lower state by emitting a photon $h\nu$.

$$\therefore \frac{dN_2}{dt} = -A_{21} N_2 \qquad A_{21} \text{ is proportionality constant.}$$

$$N_2(t) = N_{20}\, e^{-t/\tau}$$

where $\tau \equiv A_{21}^{-1}$ is called the natural lifetime.

If there are several lower states, this definition generalizes to

$$\tau_j = \sum_{E_k < E_j} A_{jk}^{-1}$$

2) Absorption: Atoms in state 1 absorb light and are excited to state 2.

$$\frac{dN_2}{dt} = + B_{12} N_1 \rho(\nu)$$

where $\rho(\nu)\,d\nu$ is the energy density between frequencies ν and $\nu + d\nu$ and B_{12} is the proportionality constant.

3) Stimulated Emission: This process is the inverse of the previous process. A photon of frequency ν induces an atom in state 2 to decay to state 1 and emit a photon. The emitted photon has the same properties eg. frequency, polarization, direction + phase as the initial photon.

$$\frac{dN_2}{dt} = - B_{21} N_2 \rho(\nu)$$

Taking into account all 3 processes we get:

$$\boxed{\frac{dN_2}{dt} = - A_{21} N_2 + B_{12} N_1 \rho(\nu) - B_{21} N_2 \rho(\nu)}$$

Since the total # of atoms $N_1 + N_2 = \text{const.} \Rightarrow \frac{dN_2}{dt} = - \frac{dN_1}{dt}$.

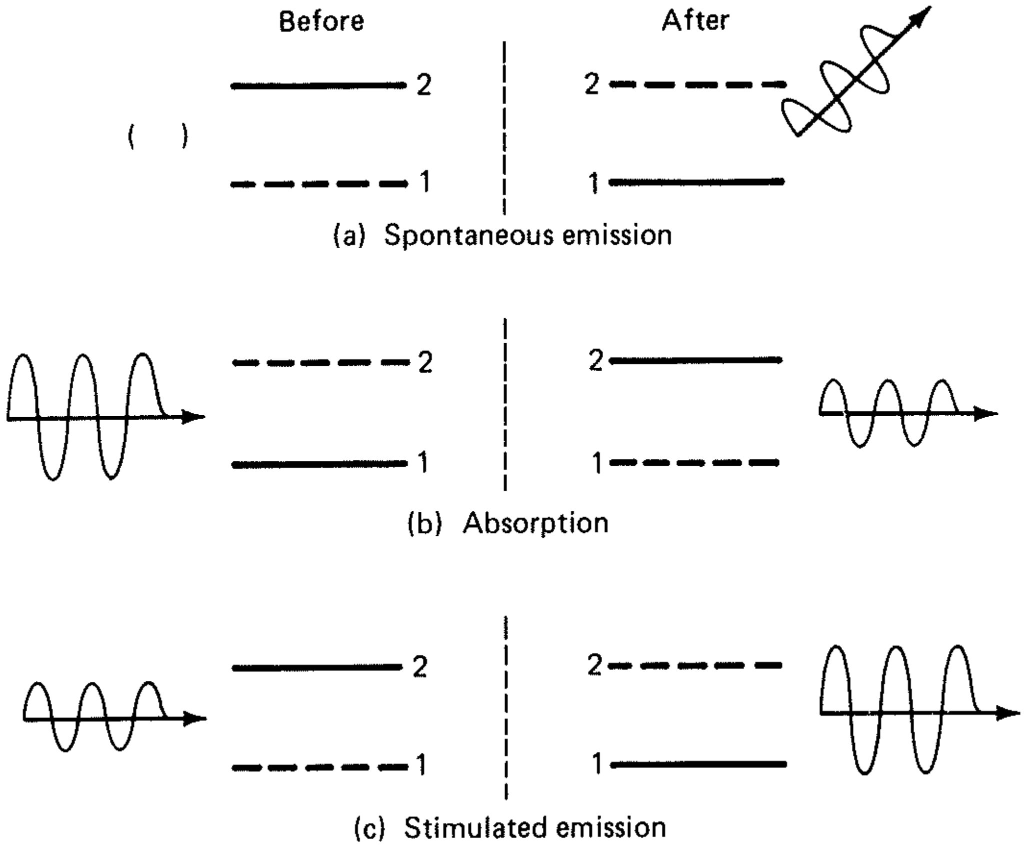


Figure 7-4. *Effect of radiation on an atom.*

Relationship Between Coefficients

At equilibrium $\frac{dN_2}{dt} = 0$ and we get:

$$0 = -A_{21} N_2 + B_{12} N_1 \rho(\nu) - B_{21} N_2 \rho(\nu) \qquad (1)$$

From thermodynamics we know the equilibrium populations are related by

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu/kT} \qquad (2)$$

where k = Boltzmann constant
T = absolute temperature
g_i = degeneracy of state i (i.e. # of states having same energy E_i)

The energy density per unit frequency is given by Planck's black body formula.

$$\rho(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} \qquad (3).$$

Exercise: Rewrite (1) as follows.

$$\frac{A_{21}}{\rho(\nu)} + B_{21} - B_{12} \frac{N_1}{N_2} = 0.$$

Substituting (2) + (3) one obtains:

$$A_{21} \frac{c^3}{8\pi h\nu^3}\left(e^{h\nu/kT}-1\right) + B_{21} - B_{12}\frac{g_1}{g_2} e^{h\nu/kT} = 0.$$

Equating the coefficients of the exponential and remaining terms to zero gives:

$$A_{21} \frac{c^3}{8\pi h\nu^3} - B_{12}\frac{g_1}{g_2} = 0 \qquad (4)$$

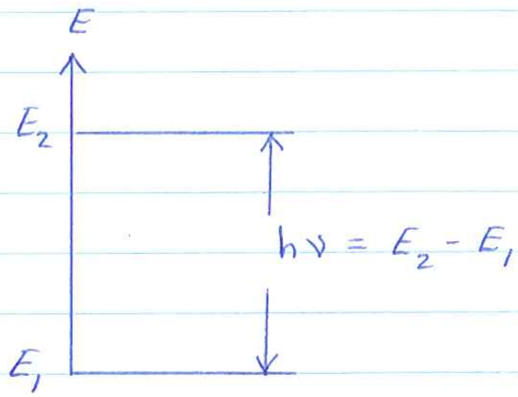
$$-A_{21} \frac{c^3}{8\pi h\nu^3} + B_{21} = 0 \qquad (5)$$

$$(5) \Longrightarrow \quad \boxed{A_{21} = \frac{8\pi h\nu^3}{c^3} B_{21}}$$

$$(4) + (5) \Longrightarrow \quad \boxed{g_1 B_{12} = g_2 B_{21}}$$

Line Shape

We shall examine the frequency profile of radiation emitted by the following atom.



Let τ be the natural or radiative lifetime of state 2. The Heisenberg uncertainty principle then tells us that state 2 has an energy width $\Delta E \sim \frac{\hbar}{\tau}$.

This in turn causes photons emitted when state 2 decays to state 1 to have a frequency width

$$\Delta \nu \sim \frac{1}{2\pi\tau}$$

eg. $\tau = 10$ nsec $\Rightarrow \Delta\nu \sim 16$ MHz.

Note that $\Delta\nu <<$ optical frequency $\sim 5 \times 10^{14}$ Hz.

Classical Picture

The electric field produced by a fluorescing atom is given by:

$$E(t) = E_o e^{-t/2\tau} \cos \omega_o t$$

where ω_o is the transition angular frequency.

Exercise: Why is the exponent $t/2\tau$ rather than t/τ?

The frequency spectrum is found by taking the Fourier transform.

$$E(\omega) = \int_0^\infty E(t) e^{i\omega t} dt.$$

Exercise: Show that $E(\omega) = \frac{E_o}{2} \frac{1/2\tau + i(\omega - \omega_o)}{(1/2\tau)^2 + (\omega - \omega_o)^2}$

What approximation was made + why?

The intensity spectrum is then

$$I(\omega) = |E(\omega)|^2$$

$$= \frac{E_o^2}{4} \frac{1}{(\omega - \omega_o)^2 + (1/2\tau)^2}$$

or $I(\nu) = \dfrac{E_o^2}{4(2\pi)^2} \dfrac{1}{(\nu - \nu_o)^2 + \left(\frac{\Delta\nu}{2}\right)^2}$ where $\dfrac{\Delta\nu}{2} = \dfrac{1}{4\pi\tau}$

$I(\nu)$ is a Lorentzian function centered about $\nu = \nu_0$.

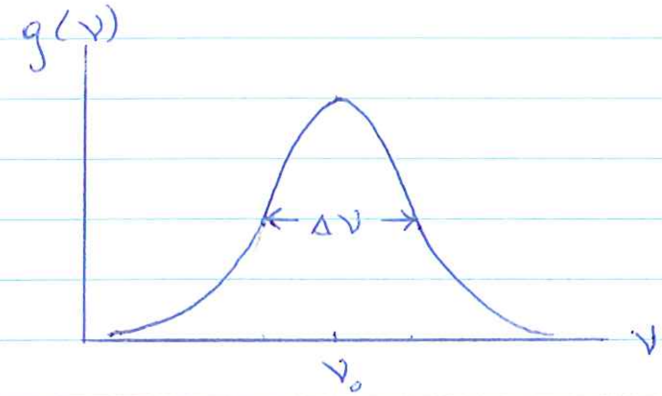
Natural Line Shape

The line shape function $g(\nu)$ is defined such that $g(\nu)\,d\nu$ is the probability an atom emits a photon with frequency between ν and $\nu + d\nu$. Since $g(\nu)$ must have the same functional form as $I(\nu)$, we let:

$$g(\nu) = \frac{K}{(\nu - \nu_0)^2 + \left(\frac{\Delta\nu}{2}\right)^2}.$$

The constant K is found using the total probability a photon is emitted $\int g(\nu)\,d\nu = 1$.

Exercise: Show $\boxed{g(\nu) = \frac{\Delta\nu}{2\pi\left[(\nu - \nu_0)^2 + (\Delta\nu/2)^2\right]}}$

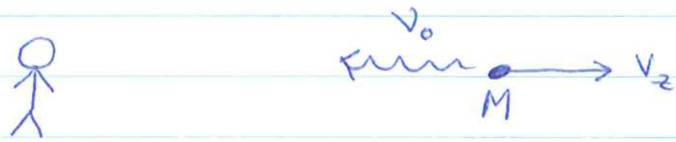


Hence the finite excited state lifetime broadens the line and is called natural broadening. From Quantum Mechanics, we know there is no way to predict when any particular excited atom will decay. Therefore natural broadening affects all atoms identically and is called homogeneous broadening.

Inhomogeneous broadening results when different atoms which in principle can be distinguished, emit different frequencies. eg. 2 isotopes, atoms moving with different velocity.

Doppler Broadening

An atom moving away from an observer with velocity v_z emits a photon.



The observer measures a Doppler shifted frequency

$$\nu_0' = \nu_0 \left(1 - \frac{v_z}{c} \right)$$

The lineshape due to atoms moving with velocity v_z is

$$g(v_z, \nu) = \frac{\Delta\nu_h}{2\pi \left[(\nu - \nu_0')^2 + \left(\frac{\Delta\nu_h}{2} \right)^2 \right]}$$

where $\Delta\nu_h$ is the homogeneous broadening.

$$\therefore g(v_z, \nu) = \frac{\Delta\nu_h}{2\pi \left[\left(\nu - \nu_0 + \nu_0 \frac{v_z}{c} \right)^2 + \left(\frac{\Delta\nu_h}{2} \right)^2 \right]}$$

The lineshape due to all atoms is

$$g(\nu) = \int_{-\infty}^{\infty} g(v_z, \nu)\, P(v_z)\, dv_z$$

where $P(v_z)\,dv_z$ is the fraction of atoms having velocity between v_z and $v_z + dv_z$. For the case of thermal equilibrium $P(v_z)$ is given by the Maxwell Boltzmann distribution function

$$P(v_z) = \left(\frac{M}{2\pi kT}\right)^{1/2} \exp\left(\frac{-M v_z^2}{2kT}\right)$$

$$\therefore \boxed{g(\nu) = \left(\frac{M}{2\pi kT}\right)^{1/2} \int_{-\infty}^{\infty} \frac{\Delta\nu_h}{2\pi\left[\left(\nu - \nu_0 + \nu_0 \frac{v_z}{c}\right)^2 + \left(\frac{\Delta\nu_h}{2}\right)^2\right]} \exp\left(\frac{-M v_z^2}{2kT}\right) dv_z}$$

The above is known as a Voigt profile and must be integrated numerically.

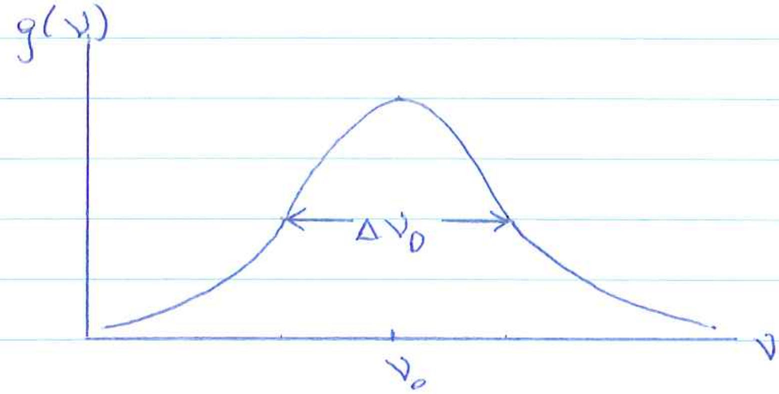
Pure Doppler Limit

In the pure Doppler limit $\Delta\nu_h \to 0$ and

$$\frac{\Delta\nu_h}{2\pi\left[(\nu - \nu_0')^2 + \left(\frac{\Delta\nu_h}{2}\right)^2\right]} \to \delta(\nu - \nu_0')$$

Exercise: Show that $g(\nu) = \left(\frac{4\ln 2}{\pi}\right)^{1/2} \frac{1}{\Delta\nu_D} \exp\left[-4\ln 2\left(\frac{\nu - \nu_0}{\Delta\nu_D}\right)^2\right]$

where $\Delta\nu_D \equiv \left(\frac{8kT\ln 2}{Mc^2}\right)^{1/2} \nu_0$



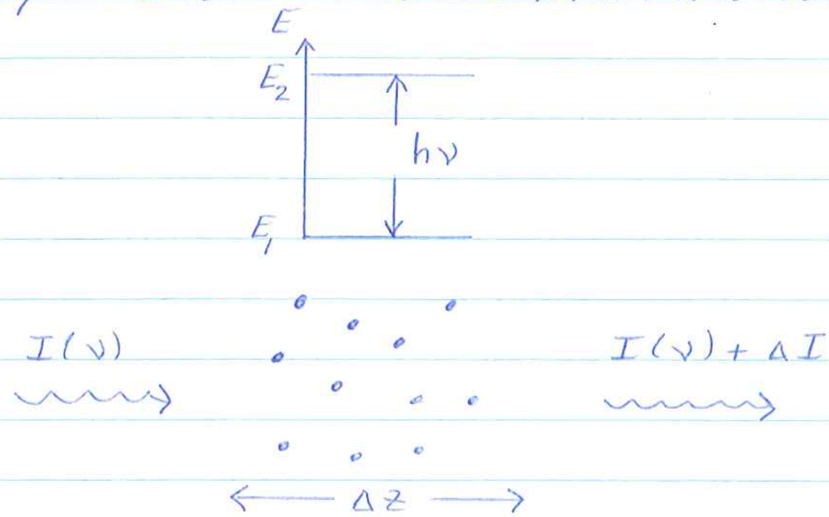
e.g. at room temperature $T = 293\,K$.
Na atom $M = 23 \times 1.67 \times 10^{-24}$ gm.
$\nu_0 \sim 5 \times 10^{14}$ Hz (yellow light)

$$\Rightarrow \Delta\nu_D = \left(\frac{8 \times 1.38 \times 10^{-16} \times 293 \ln 2}{23 \times 1.67 \times 10^{-24} \, (3 \times 10^{10})^2} \right)^{1/2} 5 \times 10^{14}$$
$$= 1.3 \text{ GHz}$$

Note that $\Delta\nu_D \gg \Delta\nu_{nat}$ in the previous example.

Light Amplification

Let $I(\nu)$ be the intensity of a polarized light beam having frequency ν. The light propagates through a gas of 2 level atoms as shown below.



ΔI = photon energy × transition rate per unit volume × trans. probability (lineshape) × Δz

(stimulated emission, absorption + spont. emission)

The contribution of spontaneous emission to the beam intensity is negligible since it is emitted isotropically. Furthermore, spontaneous emission must be much less than stimulated emission in order for the light to be amplified.

$$\therefore \Delta I = h\nu \left[B_{21}\, \rho(\nu) N_2 - B_{12}\, \rho(\nu) N_1 \right] g(\nu)\, \Delta z$$

Atoms experience a radiation density $\rho(\nu) = \dfrac{I(\nu)}{c}$.

$$\therefore \frac{dI(\nu)}{dz} = \frac{h\nu}{c} \left(B_{21} N_2 - B_{12} N_1 \right) g(\nu)\, I(\nu)$$

Exercise: Using the relations between the A and B coefficients, show the preceding eqn. can be written as

$$\boxed{\frac{dI(\nu)}{dz} = \gamma(\nu, I)\, I(\nu) \qquad (1)}$$

where the gain $\gamma(\nu, I) \equiv A_{21} \frac{\lambda^2}{8\pi} g(\nu) \left(N_2 - \frac{g_2}{g_1} N_1 \right)$

Hence the stimulated emission cross section $\sigma_{SE} = A_{21} \frac{\lambda^2}{8\pi} g(\nu)$

" " absorption " " $\sigma_{AB} = A_{21} \frac{\lambda^2}{8\pi} g(\nu) \frac{g_2}{g_1}$

Defining $\Delta N \equiv N_2 - \frac{g_2}{g_1} N_1$ the gain becomes $\gamma(\nu, I) = \sigma_{SE}(\nu) \Delta N$

The intensity increases if $\gamma > 0$ or $\Delta N > 0$. If $g_1 = g_2$, this implies $N_2 > N_1$ i.e. more atoms are in the excited state than in the lower state. The atomic population is then said to be inverted.

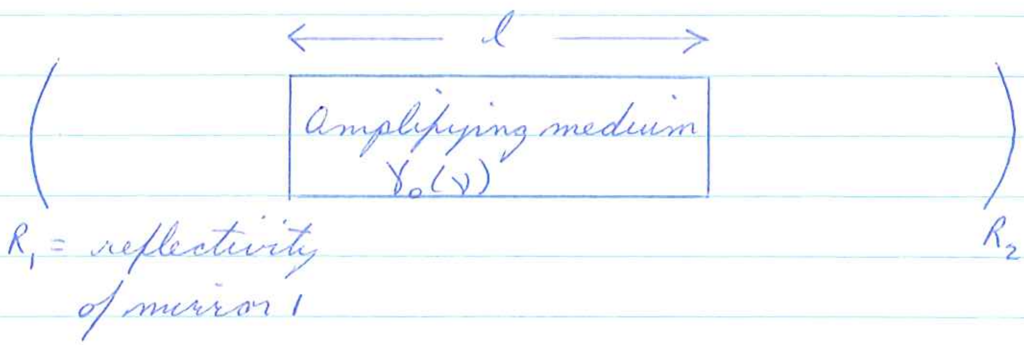
Equation (1) is nonlinear, since the populations N_2 & N_1 (and hence the gain) strongly depend on intensity. At low intensities such that the atomic populations are not significantly perturbed $\gamma(\nu, I) \simeq \gamma(\nu, 0) \equiv \gamma_0(\nu)$ which is called the small signal gain.

$(1) \Rightarrow I(\nu, z) = G\, I(\nu, 0)$ where $G \equiv e^{\gamma_0(\nu) z}$ is

called the small signal power gain. We shall show that as the intensity increases, ΔN and hence γ decrease leading to gain saturation.

Laser Oscillation

Basic Laser



Light is partially transmitted by at least one mirror to obtain an output beam. In order for the laser oscillation to be self sustaining (i.e. not die away), the roundtrip gain must exceed unity.

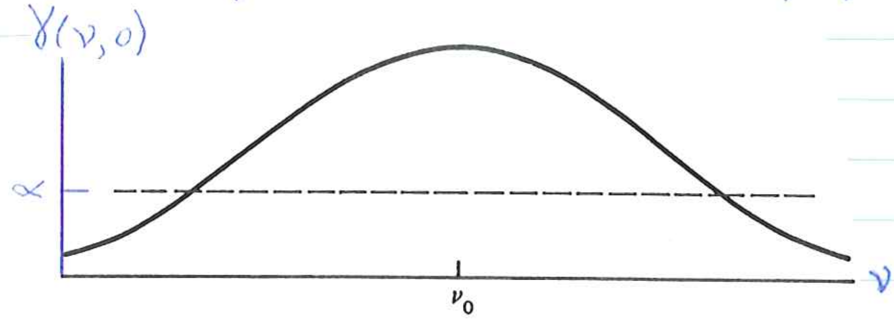
$$\therefore \quad R_1 R_2 \, e^{\gamma_0(\nu) 2l} \geq 1$$

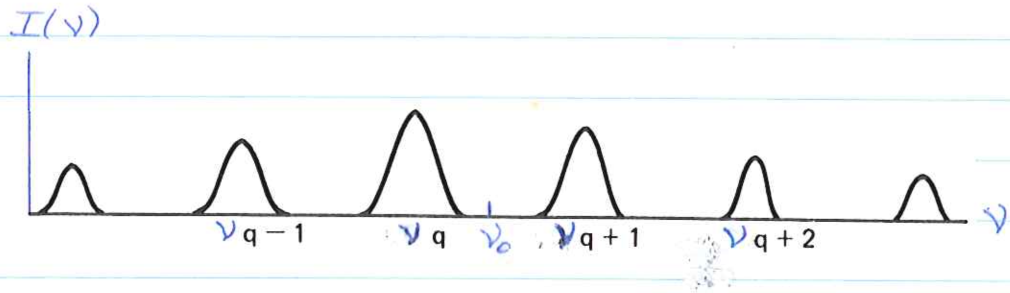
$$\text{or} \quad \gamma_0(\nu) \geq \frac{1}{2l} \ln\left(\frac{1}{R_1 R_2}\right) \equiv \frac{\alpha d}{l}$$

α is called the loss per unit length.

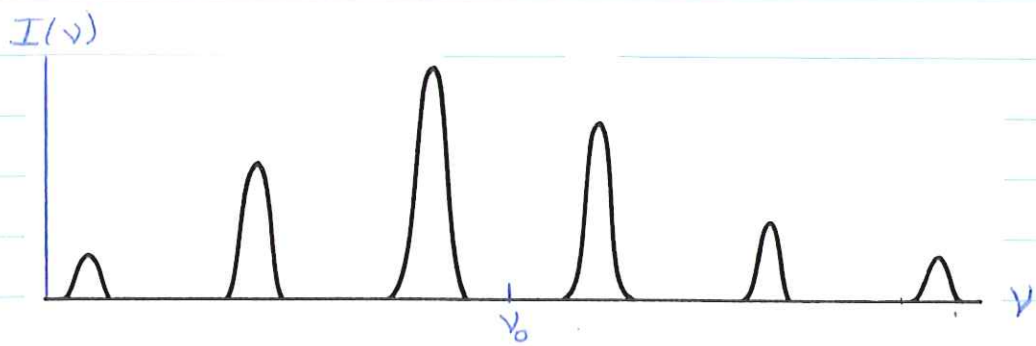
Evolution of Laser Oscillation

Stage 1: Photons are generated by spontaneous emission. Light resonant with the cavity modes ν_q for which gain γ > loss α are amplified.





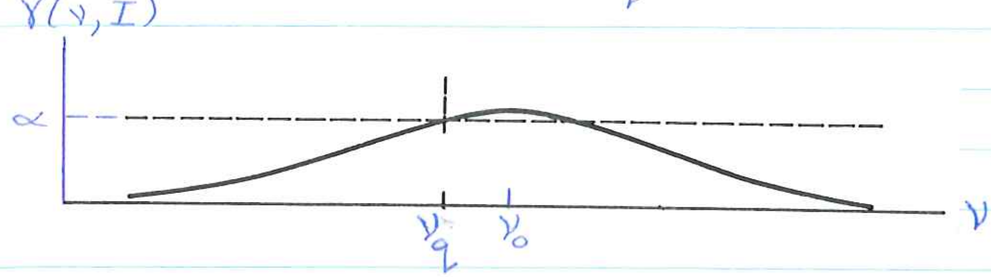
Stage 2: Stimulated emission amplifies the cavity modes nearer to gain center ν_0 more than modes further away.

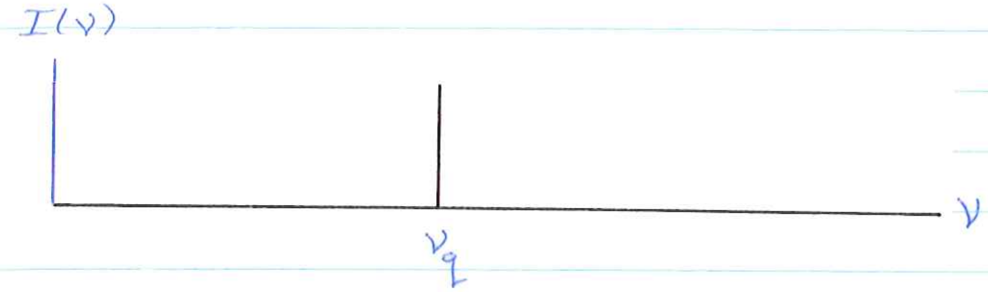


What happens next depends on whether transition is homogeneously or inhomogeneously broadened.

Stage 3 - Homogeneous Broadening

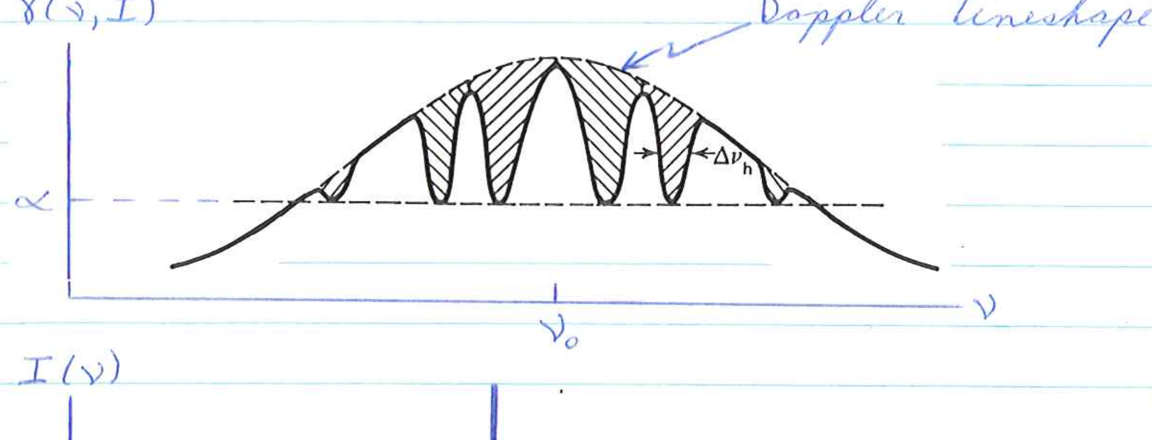
Photons compete for gain from all atoms in gain medium. Stimulated emission therefore increases the intensity of the mode closest to ν_0 more than the other modes. As the intensity increases, the gain decreases until it reaches the loss value for mode ν_q.

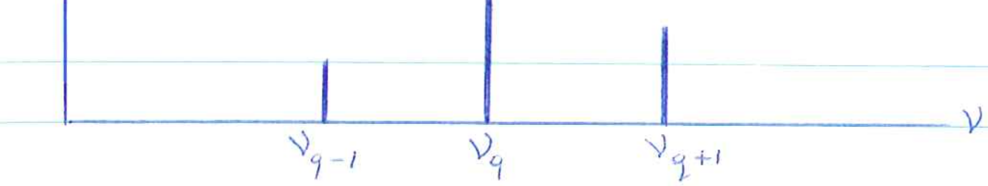




Stage 3 - Inhomogeneous Broadening

Photons can only extract gain from excited atoms whose frequency is within homogeneous linewidth $\Delta\nu_h$ of ν_q. The intensity increases until the gain falls to the threshold value for each cavity mode.





Each cavity mode is said to burn two holes in the gain profile since it extracts energy from atoms having different velocities on its forward and reverse passes through the cavity. The laser is said to be multimode since it lases at more than one frequency.

Gain Saturation In A Homogeneous Broadened Transition

We shall show how the gain saturates for a homogeneous broadened transition as the laser intensity increases.
We consider the following model.

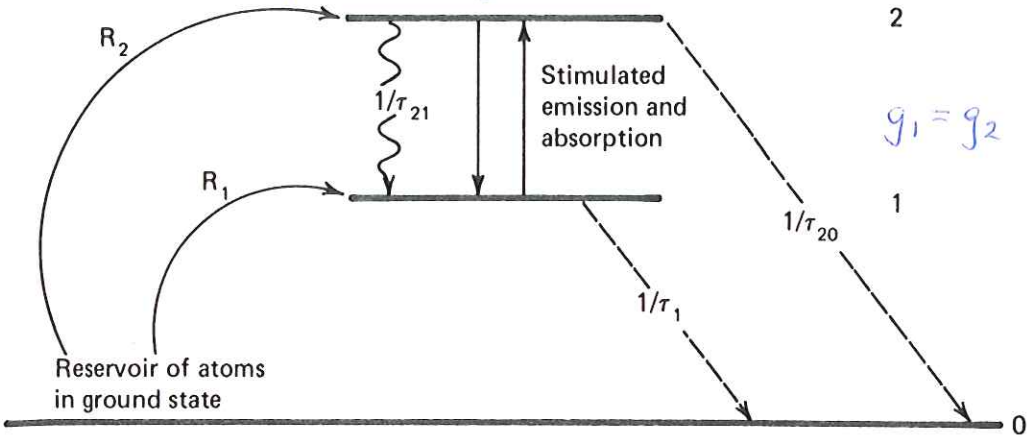


Figure 8-4. *Generalized pumping scheme of a laser.*

R_1 and R_2 are the rates at which atoms are excited to states 1 + 2 respectively. $1/\tau_{ij}$ is the decay rate due to radiative decay and collisions from state i to state j. The rate equations for the population densities N_1 and N_2 are:

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} - B_{21}\, g(\nu)\rho(\nu) N_2 + B_{12}\, g(\nu)\rho(\nu) N_1$$

$$\frac{dN_1}{dt} = R_1 - \frac{N_1}{\tau_1} + \frac{N_2}{\tau_{21}} + B_{21}\, g(\nu)\rho(\nu) N_2 - B_{12}\, g(\nu)\rho(\nu) N_1$$

where $\dfrac{1}{\tau_2} = \dfrac{1}{\tau_{20}} + \dfrac{1}{\tau_{21}}$.

Exercise: Show that for steady state lasing ($N_1 + N_2$ constant)

$$R_2 = \left[\frac{1}{\tau_2} + \frac{\sigma(\nu) I(\nu)}{h\nu}\right] N_2 - \frac{\sigma(\nu) I(\nu)}{h\nu} N_1$$

$$R_1 = -\left[\frac{1}{\tau_{21}} + \frac{\sigma(\nu) I(\nu)}{h\nu}\right] N_2 + \left[\frac{1}{\tau_1} + \frac{\sigma(\nu) I(\nu)}{h\nu}\right] N_1$$

where $\sigma \equiv A_{21} \frac{\lambda^2}{8\pi} g(\nu)$ is the stimulated emission cross section.

Exercise: For a homogeneous broadened line $g(\nu) = \dfrac{\Delta\nu/2\pi}{(\nu-\nu_0)^2 + \left(\frac{\Delta\nu}{2}\right)^2}$

Show that $\sigma(\nu) = \sigma(\nu_0)\, \bar{g}(\nu)$ where $\bar{g}(\nu) \equiv \frac{\pi \Delta\nu}{2} g(\nu)$.

Using the above equations, one finds an expression for the gain

$$\gamma(\nu, I) = (N_2 - N_1)\, \sigma(\nu)$$

$$= \frac{\left[R_2\, \tau_2 (1 - \tau_1/\tau_{21}) - R_1 \tau_1\right] \sigma(\nu)}{1 + (\tau_1 + \tau_2 - \tau_1 \tau_2/\tau_{21}) \dfrac{\sigma(\nu) I(\nu)}{h\nu}}$$

This has the form $\boxed{\gamma(\nu, I) = \dfrac{\gamma(\nu, 0)}{1 + \dfrac{I(\nu)}{I_s} \bar{g}(\nu)}}$ where

the small signal gain $\gamma_0(\nu) \equiv \gamma(\nu, 0) = \left[R_2\, \tau_2(1 - \tau_1/\tau_{21}) - R_1 \tau_1\right] \sigma(\nu)$

& the saturation intensity $I_s = \dfrac{h\nu}{\tau_1 + \tau_2 - \tau_1\tau_2/\tau_{21}} \; \dfrac{1}{\sigma(\nu_0)}$

The laser intensity is given by:

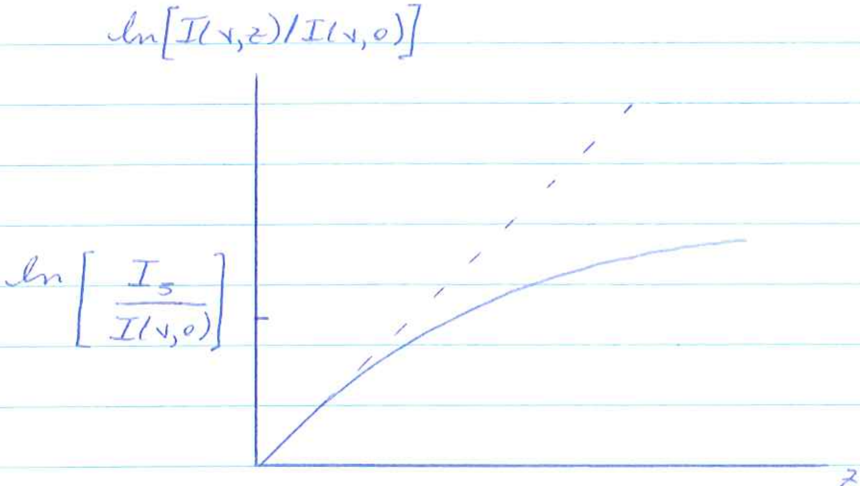
$$\frac{dI(\nu)}{dz} = \gamma(\nu, I)\ I(\nu)$$

$$= \frac{\gamma_0(\nu)}{1 + \frac{I(\nu)}{I_s}\bar{g}(\nu)}\ I(\nu)$$

$$\int_{I(0)}^{I(z)} \left(\frac{1}{I} + \frac{\bar{g}}{I_s}\right) dI = \int_0^z \gamma_0\, dz.$$

$$\boxed{\ln\left[\frac{I(\nu,z)}{I(\nu,0)}\right] + \frac{\bar{g}}{I_s}\left(I(z) - I(0)\right) = \gamma_0(\nu) z.}$$

This equation must be solved numerically.



Exercise: When $I(\nu,0) \gg I_s$ show that $I(\nu,z) = I(\nu,0) + \frac{\gamma_0(\nu) I_s z}{\bar{g}(\nu)}$

Hence at high intensities, $I(\nu,z)$ is limited by the gain length z as one would expect.

Gain Saturation In An Inhomogeneous Broadened Transition

For a homogeneous broadened transition we just found the gain to be given by:

$$\gamma(\nu, I) = \frac{\gamma_0(\nu)}{1 + \frac{I(\nu)}{I_s}\bar{g}(\nu)}$$

where the small signal gain $\gamma_0(\nu) = \Delta N \, A_{21} \frac{\lambda^2}{8\pi} g(\nu)$,

the lineshape $g(\nu) = \frac{\Delta\nu/2\pi}{(\nu-\nu_0)^2 + (\Delta\nu/2)^2}$ and $\bar{g}(\nu) = \pi \frac{\Delta\nu}{2} g(\nu)$.

Exercise: Show that the gain can be written as

$$\gamma(\nu, I) = \Delta N \, A_{21} \frac{\lambda^2}{8\pi} \frac{\Delta\nu}{2\pi} \frac{1}{(\nu-\nu_0)^2 + (\Delta\nu_H/2)^2}$$

where $\boxed{\Delta\nu_H \equiv \Delta\nu\sqrt{1 + \frac{I(\nu)}{I_s}}}$.

For an inhomogeneous line, we must add the "homogeneous gains" at the various center frequencies ν_0. If $\rho(\nu)$ is the inhomogeneous lineshape, than the inhomogeneous gain is:

$$\gamma(\nu, I) = \Delta N \, A_{21} \frac{\lambda^2}{8\pi} \frac{\Delta\nu}{2\pi} \int_0^\infty \frac{\rho(\nu_0) \, d\nu_0}{(\nu-\nu_0)^2 + (\Delta\nu_H/2)^2}$$

The Lorentzian can be approximated as a delta function when $\Delta\nu_H <<$ inhomogeneous linewidth.

i.e. $$\frac{\Delta\nu_H/2\pi}{(\nu-\nu_0)^2 + (\Delta\nu_H/2)^2} \longrightarrow \delta(\nu-\nu_0)$$

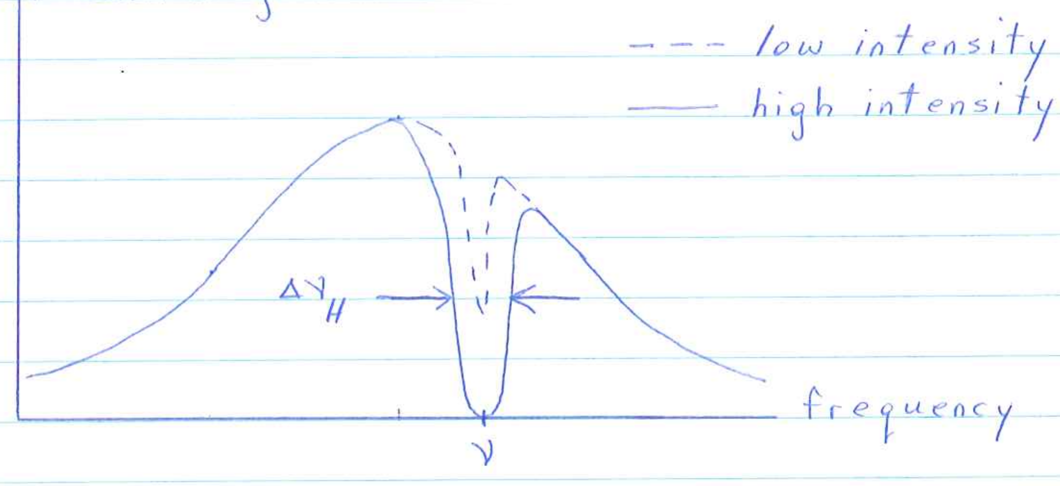
$$\therefore \; \gamma(\nu, I) = \Delta N \, A_{21} \frac{\lambda^2}{8\pi} \, \rho(\nu) \, \frac{\Delta\nu}{\Delta\nu_H}$$

$$= \gamma_0(\nu) \frac{\Delta\nu}{\Delta\nu_H}$$

$$\therefore \; \boxed{\gamma(\nu, I) = \frac{\gamma_0(\nu)}{\sqrt{1 + \frac{I(\nu)}{I_S}}}}$$

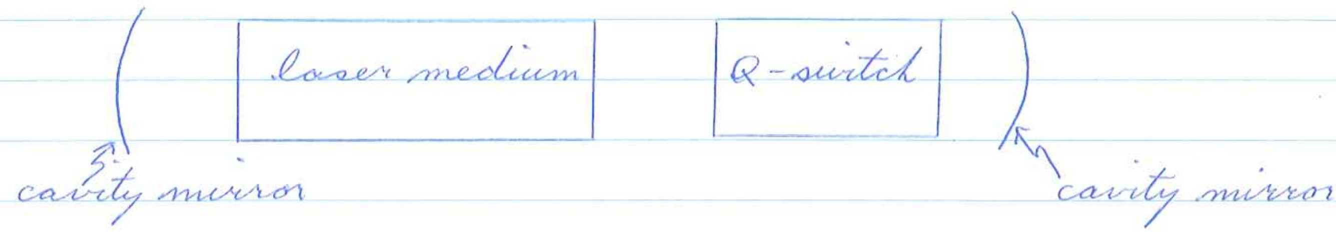
We note that the gain decreases less rapidly than for the homogeneous broadened line. The gain is still suppressed as the intensity increases but the width of the hole $\Delta\nu_H$ burned into the gain profile also increases. (i.e. line is power broadened)

Gain For Inhomogeneous Line

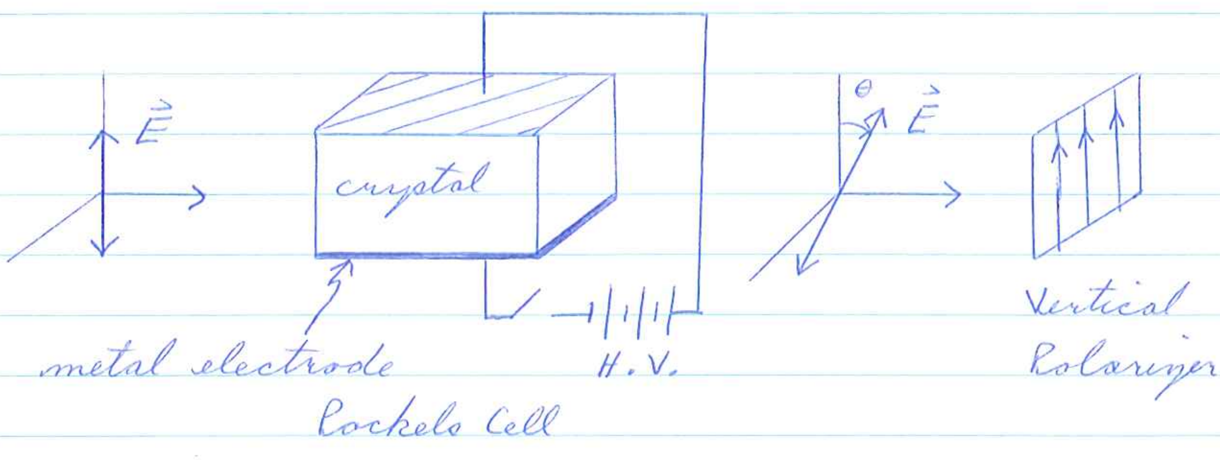


Q Switching

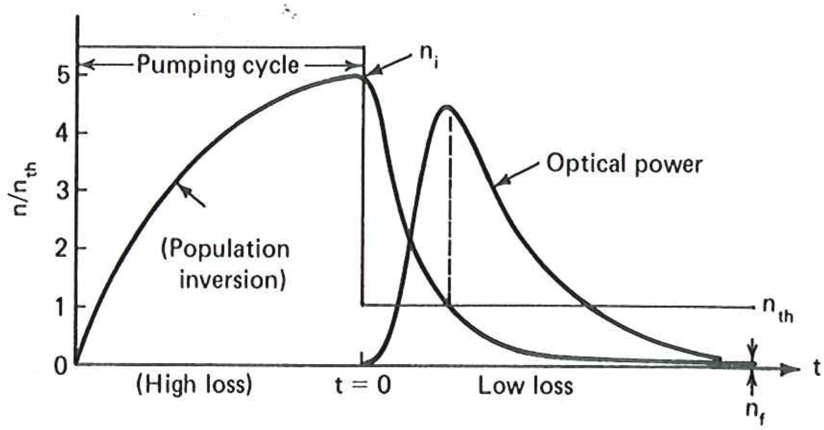
Many lasers are pulsed. To obtain maximum power, it is desirable to suppress lasing until the population inversion is maximized. This is done using a so-called Q switch.



A Q switch is a fast shutter that changes the Q or quality of the cavity. The shutter consists of a Pockels cell followed by a linear polarizer.



The Pockels cell consists of a crystal that rotates the light linear polarization axis by angle θ. The light is then attenuated as it passes through the vertical polarizer raising the lasing threshold. When a high (lethal!) voltage is applied to the Pockels cell, the angle $\theta = 0$ and a lasing pulse builds up rapidly.



Quantitative Analysis

We shall determine expressions for the energy and duration of the laser pulse. The laser intensity is given by

$$\frac{dI}{dz} = (\gamma - \alpha) I$$

where γ (α) is the gain (loss) per unit length.

$$\frac{dI}{dt} \cdot \frac{dt}{dz} = (\gamma - \alpha) I$$

$$\frac{dI}{dt} = c(\gamma - \alpha) I$$

Since intensity $\propto$ # photons in cavity ϕ_p, we obtain

$$\frac{d\phi_p}{dt} = c(\gamma - \alpha)\phi_p \qquad (1)$$

When a photon is emitted, the population difference between the upper and lower lasing levels n, decreases by 2.

$$\therefore \frac{dn}{dt} = -2c\gamma\phi_p \qquad (2)$$

Combining equations (1) & (2) we find that

$$\frac{d\phi_p}{dn} \cdot \frac{dn}{dt} = c(\gamma - \alpha)\phi_p$$

$$\frac{d\phi_p}{dn}\left(-2c\gamma\phi_p\right) = c(\gamma-\alpha)\phi_p$$

$$\frac{d\phi_p}{dn} = -\frac{(\gamma - \alpha)}{2\gamma}$$

Now $\gamma \propto n$ the population inversion.
When $\gamma = \alpha$, the lasing begins and the laser is said to be at threshold. $\therefore \gamma_{th} = \alpha \propto n_{th}$

$$\therefore \boxed{\frac{d\phi_p}{dn} = -\frac{n - n_{th}}{2n} \qquad (3)}$$

The photon number in the cavity starts at zero when the Q switch opens. ϕ_p increases until the inversion n falls to the threshold value n_{th}.

$$\therefore \phi_{p\,max} = -\frac{1}{2}\int_{n_i}^{n_{th}} \frac{n - n_{th}}{n}\, dn$$

$n_i \leftarrow$ initial inversion

$$= -\frac{1}{2}(n_{th} - n_i) + \frac{1}{2} n_{th} \ln(n_{th}/n_i)$$

The maximum output power P_{max} is given by

P_{max} = loss rate through mirrors × photon energy × # photons in cavity

$$P_{max} = c\alpha_0 h\nu \, \phi_{p_{max}}$$

where $\alpha_0 = \frac{1}{2d} \ln\left(\frac{1}{R_1 R_2}\right)$. d is the cavity length and R_1 & R_2 are the mirror reflectances.

$$\therefore \boxed{P_{max} = \frac{c}{2d} \ln\left(\frac{1}{R_1 R_2}\right) \frac{h\nu}{2}\left[(n_i - n_{th}) - n_{th} \ln\left(\frac{n_i}{n_{th}}\right)\right]} \quad (4)$$

After the inversion n falls below threshold n_{th}, the lasing medium absorbs photons. Hence the final value of the inversion n_f is given by:

$$0 = -\frac{1}{2} \int_{n_i}^{n_f} \frac{n - n_{th}}{n} \, dn$$

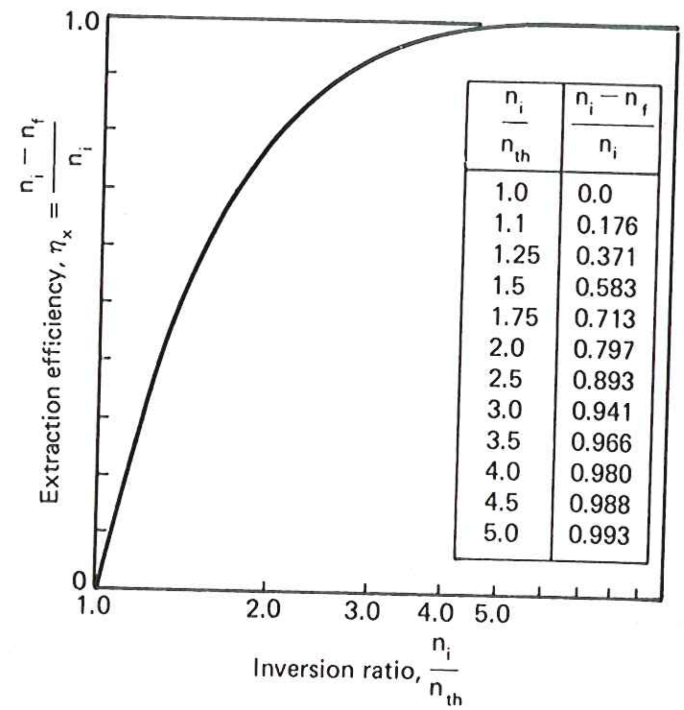
$$= -\frac{1}{2}(n_f - n_i) + \frac{n_{th}}{2} \ln\left(\frac{n_f}{n_i}\right)$$

$$\boxed{0 = (n_i - n_f) - n_{th} \ln\left(\frac{n_i}{n_f}\right)} \quad (5)$$

This equation must be solved numerically.

One may define the extraction efficiency η as the inversion fraction that ~~generates~~ contributes photons to the laser beam.

i.e. $\eta = \frac{n_i - n_f}{n_i}$



$\frac{n_i}{n_{th}}$	$\frac{n_i - n_f}{n_i}$
1.0	0.0
1.1	0.176
1.25	0.371
1.5	0.583
1.75	0.713
2.0	0.797
2.5	0.893
3.0	0.941
3.5	0.966
4.0	0.980
4.5	0.988
5.0	0.993

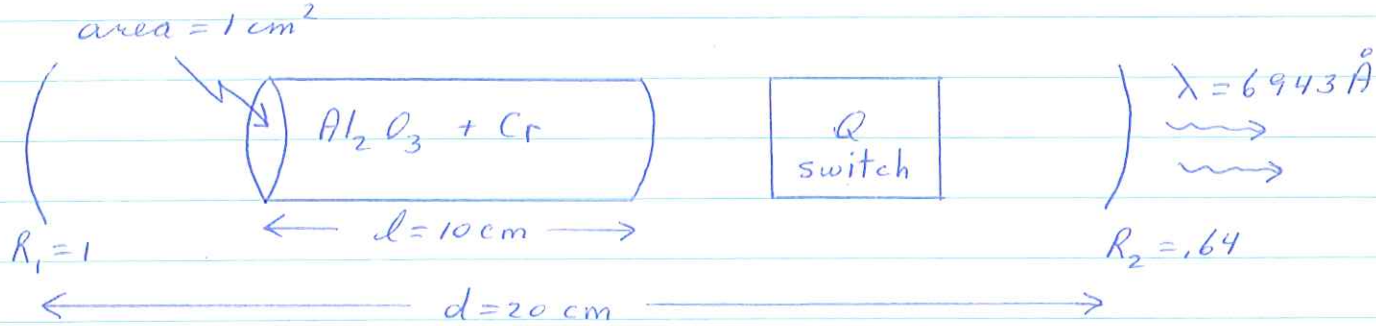
The above graph illustrates the value of Q switching since an extraction efficiency of 98% is achieved if the initial inversion ratio $n_i / n_{th} = 4$.

Pulse Energy $E = \frac{n_i - n_f}{2} h\nu$

Pulse Duration $\Delta t \simeq \frac{E}{P_{max}}$

$$\boxed{\Delta t \simeq \frac{2d}{c}\left[\ln\left(\frac{1}{R_1 R_2}\right)\right]^{-1} \frac{n_i - n_f}{(n_i - n_{th}) - n_{th}\ln(n_i/n_{th})}}$$

Example: Ruby laser



$[Cr] = 1.58 \times 10^{19}$ atoms/cm^3

Unpumped rod has transmission at 6943 Å of $T = 13.5\%$

Now $T = \exp\{-[Cr]\,\sigma_{abs}\,l\}$. Hence absorption cross section

$$\sigma_{abs} = \frac{-\ln T}{[Cr]\,l}$$
$$= \frac{-\ln .135}{1.58 \times 10^{19}\ \text{cm}^{-3} \times 10\ \text{cm}}$$
$$= 1.27 \times 10^{-20}\ \text{cm}^2$$

If degeneracies of upper + lower lasing levels are equal then $\sigma_{stim} = \sigma_{abs}$.

Assuming the only **loss** is due to mirror transmission, we find threshold gain $\gamma_{th} = \frac{1}{2l} \ln\left(\frac{1}{R_1 R_2}\right)$

$$= \frac{1}{2 \times 10\ \text{cm}} \ln\left(\frac{1}{1 \times .64}\right)$$
$$= .0223\ \text{cm}^{-1}$$

$\therefore$ inversion threshold $N_{th} = \gamma_{th} / \sigma_{stim}$

$$= \frac{.0223 \text{ cm}^{-1}}{1.27 \times 10^{-20} \text{ cm}^2}$$

$$= 1.76 \times 10^{18} \text{ cm}^{-3}$$

$\therefore$ $n_{th} = N_{th} \times \text{volume}$

$$= 1.76 \times 10^{18} \text{ cm}^{-3} \times 1 \text{ cm}^2 \times 10 \text{ cm}$$

$$= 1.76 \times 10^{19} \text{ atoms}$$

If rod is pumped to 4 times threshold before Q switch is opened then $n_i = 4 n_{th}$

$$= 7.05 \times 10^{19} \text{ atoms}$$

From graph we find extraction efficiency $\eta = 98\%$.

$\therefore$ final inversion $n_f = (1-\eta) n_i$

$$= .02 \times 7.05 \times 10^{19}$$

$$= 1.4 \times 10^{18} \text{ atoms}$$

Pulse Energy $E = \dfrac{n_i - n_f}{2} h\nu$

$$= \frac{.98}{2} \times 7.05 \times 10^{19} \times 6.64 \times 10^{-34} \text{ J.sec} \times \frac{3 \times 10^{8} \text{ m/sec}}{6.943 \times 10^{-7} \text{ m}}$$

$$= 10 \text{ joules}$$

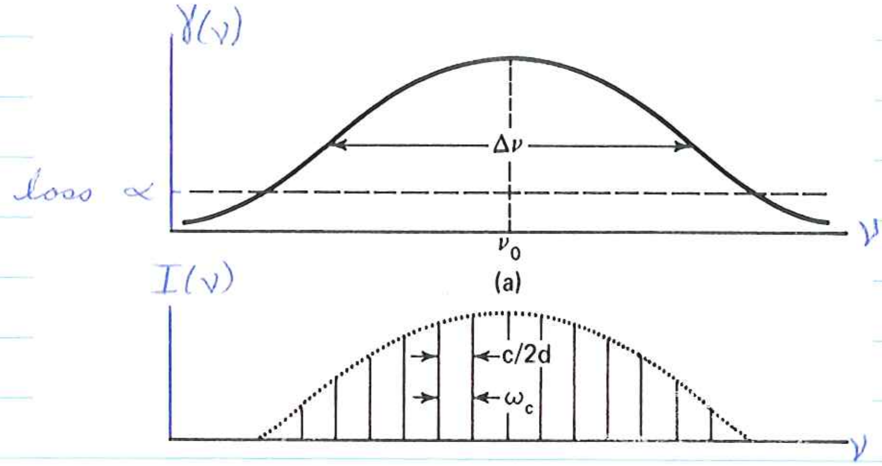
Pulse Duration $\Delta t = \dfrac{2d}{c} \left[\ln\left(\dfrac{1}{R_1 R_2}\right) \right]^{-1} \dfrac{n_i - n_f}{(n_i - n_{th}) - n_{th} \ln(n_i / n_{th})}$

$$\simeq 7 \text{ nsec.}$$

The actual pulse length is longer since speed of light in ruby < c.

Mode Locking

Consider an inhomogeneously broadened laser that lases multimode.



We shall show how such a laser can generate very short pulses.

The total electric field at time t for a laser having N modes centered at ω_o is:

$$E_{TOT}(t) = \sum_{n=-\frac{(N-1)}{2}}^{n=\frac{(N-1)}{2}} E_n \, e^{i[(\omega_o + n\omega_c)t + \phi_n]}$$

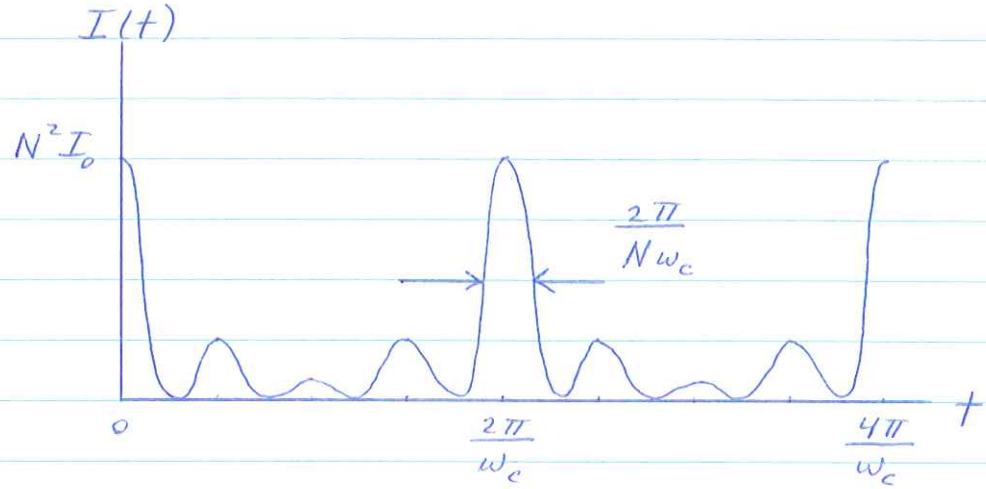
We shall assume that $\phi_n = \phi \ \forall n$ i.e. the mode phases are locked together. For simplicity we shall also assume that $E_n = E_o \ \forall n$.

Exercise: Show $E_{TOT} = E_o \, e^{i(\omega_o t + \phi)} \dfrac{\sin N\omega_c t/2}{\sin \omega_c t/2}$

Laser Intensity $I \propto E_{TOT} E_{TOT}^*$

$$I(t) = I_0 \left(\frac{\sin N\omega_c t/2}{\sin \omega_c t/2} \right)^2 \quad \text{where } I_0 \propto E_0^2.$$

$I(t)$ is shown below for the case $N=4$.



The graph shows a short pulse every $2\pi/\omega_c$ seconds whose intensity is N^2 times the single mode intensity.

Pulse width $\tau_p = \dfrac{2\pi}{N\omega_c}$

$$= \frac{1}{N\nu_c}$$

# of modes $N = \dfrac{\Delta\nu}{\nu_c} \implies \boxed{\tau_p \simeq \dfrac{1}{\Delta\nu}}$

Note: The remaining $(N-1)$ less intense peaks are absent if the electric field amplitudes of modes near the gain center ν_0 are larger than those further from ν_0.

Phase Locking Mechanism

Mode locking produces a short bunch of photons bouncing back and forth in the laser cavity. Hence, mechanisms to lock the mode phases are equivalent to those that shorten the laser pulse such as only briefly switching on the gain. This frequently occurs automatically due to gain saturation. The initial photons are amplified while "stragglers" are not. Hence a short mode locked pulse results.

Table 11.1. Some Laser Systems, Their Gain Linewidth $\Delta\nu$, and the Length of Their Pulses in the Mode-Locked Operation

Laser Medium	$\Delta\nu$ Hz	$(\Delta\nu)^{-1}$ Seconds	Observed Pulse Duration, Seconds
He-Ne ($0.6328\ \mu m$) CW	1.5×10^{9}	6.66×10^{-10}	6×10^{-10}
Nd: YAG ($1.06\ \mu m$) CW	1.2×10^{10}	8.34×10^{-11}	7.6×10^{-11}
Ruby ($0.6943\ \mu m$) pulsed	6×10^{10}	1.66×10^{-11}	1.2×10^{-11}
Nd^{3+}: glass ($1.06\ \mu m$) pulsed	3×10^{12}	3.33×10^{-13}	4×10^{-13}
Rhodamine 6G ($\sim0.6\ \mu m$)	5×10^{12}	2×10^{-13}	4×10^{-13}

Examples of lasers

1a) Ruby Laser

Laser oscillation at optical wavelengths was first achieved by T. Maiman in 1960 using ruby (5% Cr^{+3} + Al_2O_3) The active laser constituent is the Cr^{+3} ion whose energy levels are shown below.

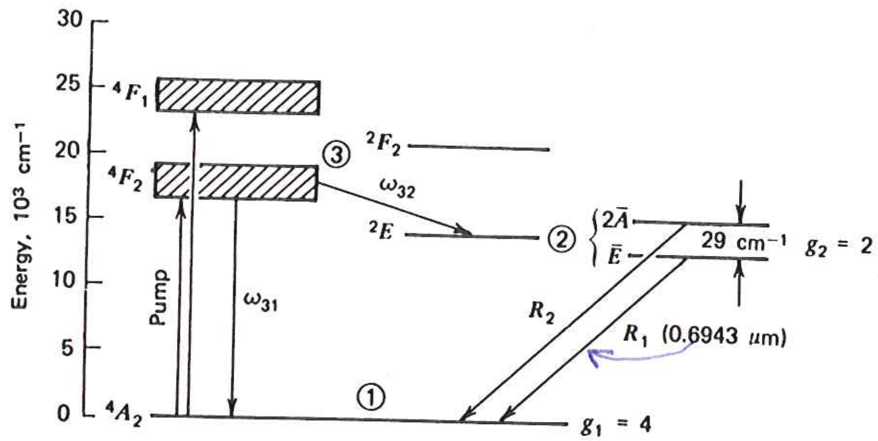
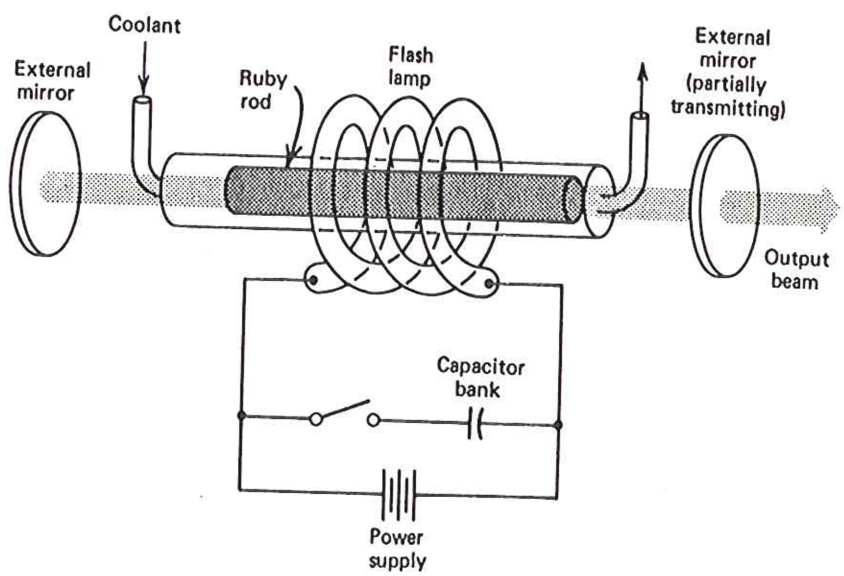


Figure 10.2 Energy levels pertinent to the operation of a ruby laser. (After Ref. 2.)

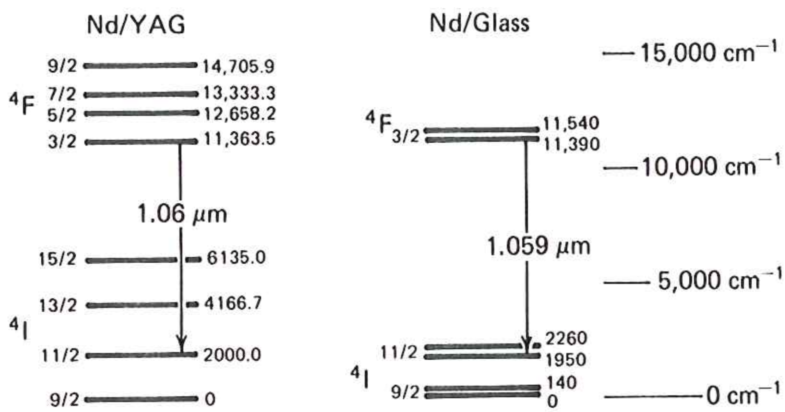
The ruby rod is pumped by a flashlamp and lases at 6943 Å.



The laser is pulsed at a repetition rate of ~1Hz. It is no longer commonly used having been replaced by the more powerful YAG laser.

1b) Nd: YAG/Glass laser

The Nd^{+3} is the active constituent of these lasers. Nd is embedded in either a YAG (Yittrium Aluminium Garnet) or glass. The substrate shifts the energy levels slightly as shown below.

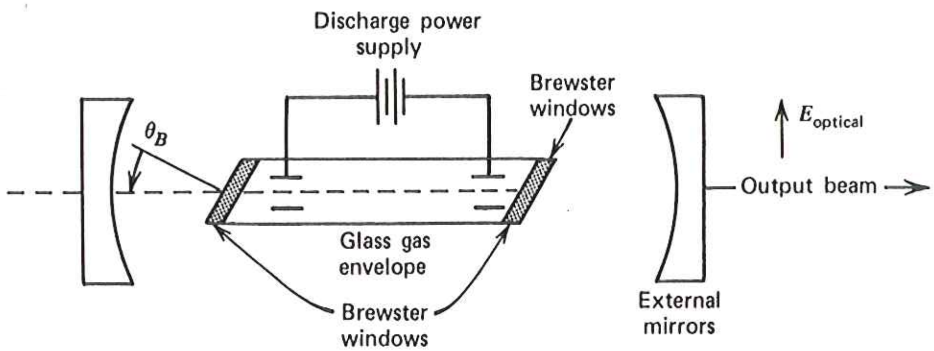


Like the ruby laser, the Nd: YAG/Glass laser is flashlamp pumped. It operates either continuously (CW) or in pulsed mode at the infrared wavelength of 1.06 μm. This wavelength may be frequency doubled in order to pump a dye laser. The worlds largest laser at Livermore is an amplified Nd: Glass laser which is used to study laser fusion. Typical parameters of a commercially available laser are:

Repetition rate = 10 Hz.
Pulse duration = 7 nsec.
Pulse energy = 1 J.

2) Gas Lasers.

A typical gas laser is shown below.



A population inversion is created by an electric discharge in the gas cell.

2a) HeNe Laser.

The cheapest and therefore most widely used laser is the HeNe laser. A gas mixture of He:Ne = (5:1 - 20:1) is used. Electrons excite the He^+ 2S states which are nearly degenerate with the Ne^+ 2S + 3S levels. These states are therefore readily populated by collisions. Lasing is possible at many wavelengths, the most common being the red line at 6328 Å with a power of a few milliwatts.

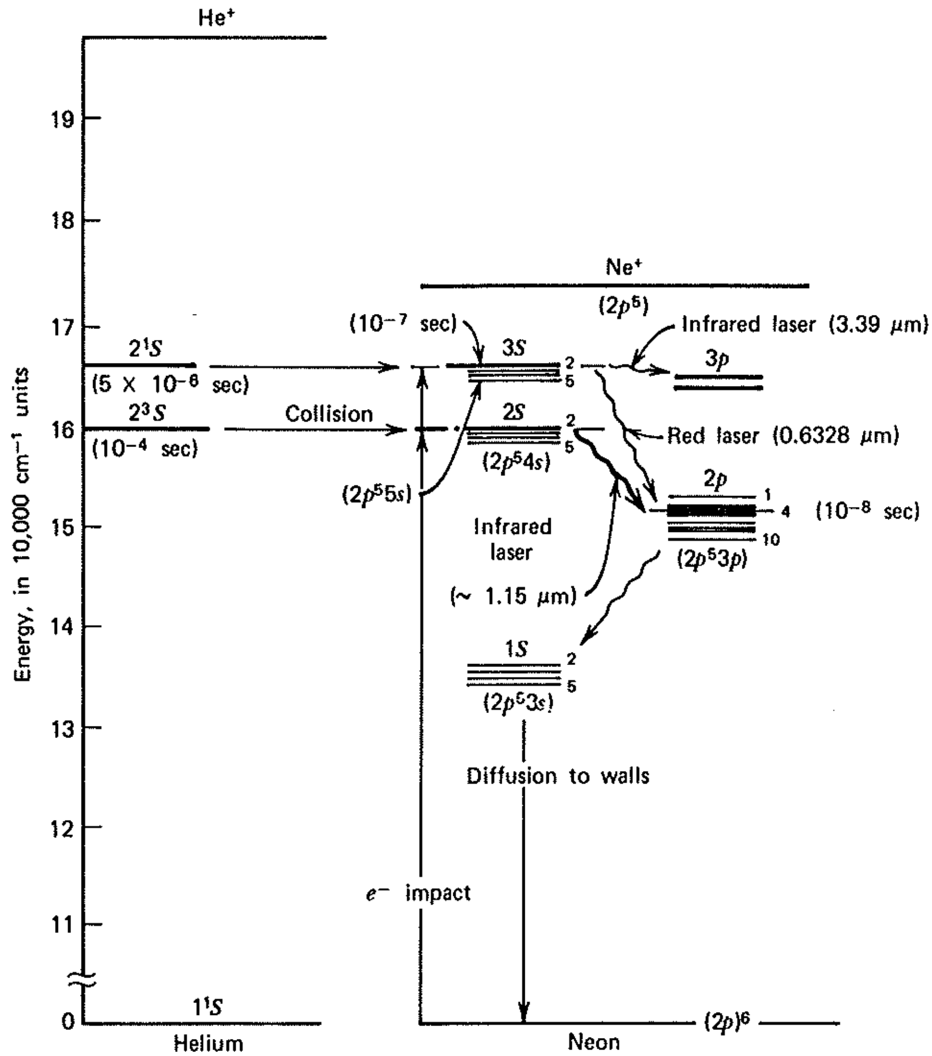


Figure 10.15 He-Ne energy levels. The dominant excitation paths for the red and infrared laser transitions are shown. (After Ref. 11.)

2b) Ar$^+$ Laser

Ar$^+$ (also Kr$^+$) lasers are run CW. They are commonly used to pump dye lasers. The Ar$^+$ laser is also used for eye surgery. These lasers lase at many wavelengths as is shown below.

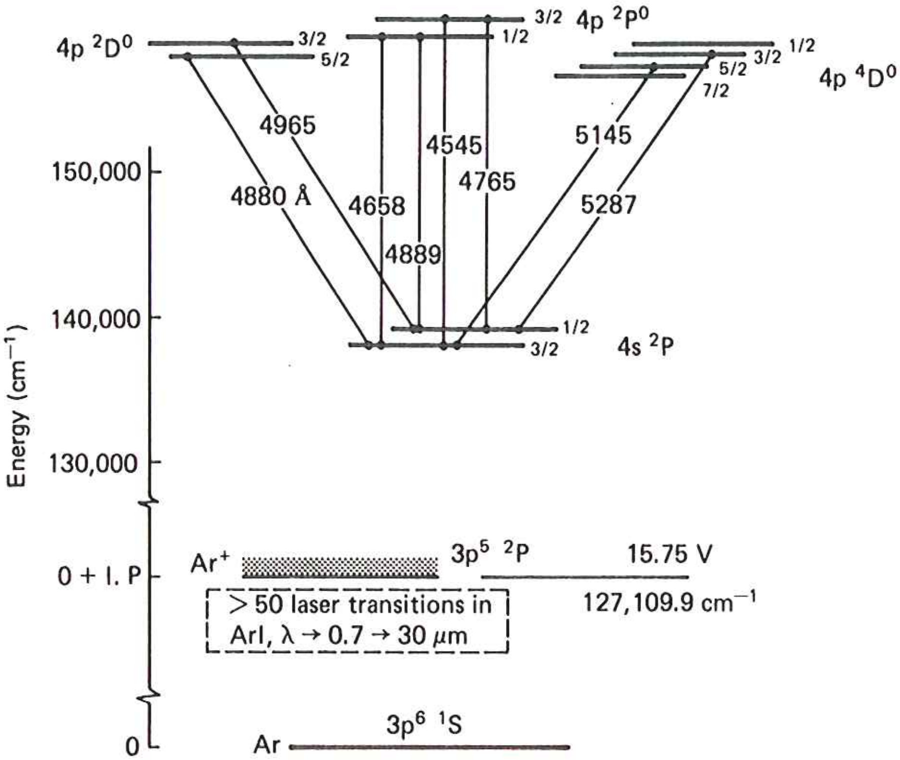
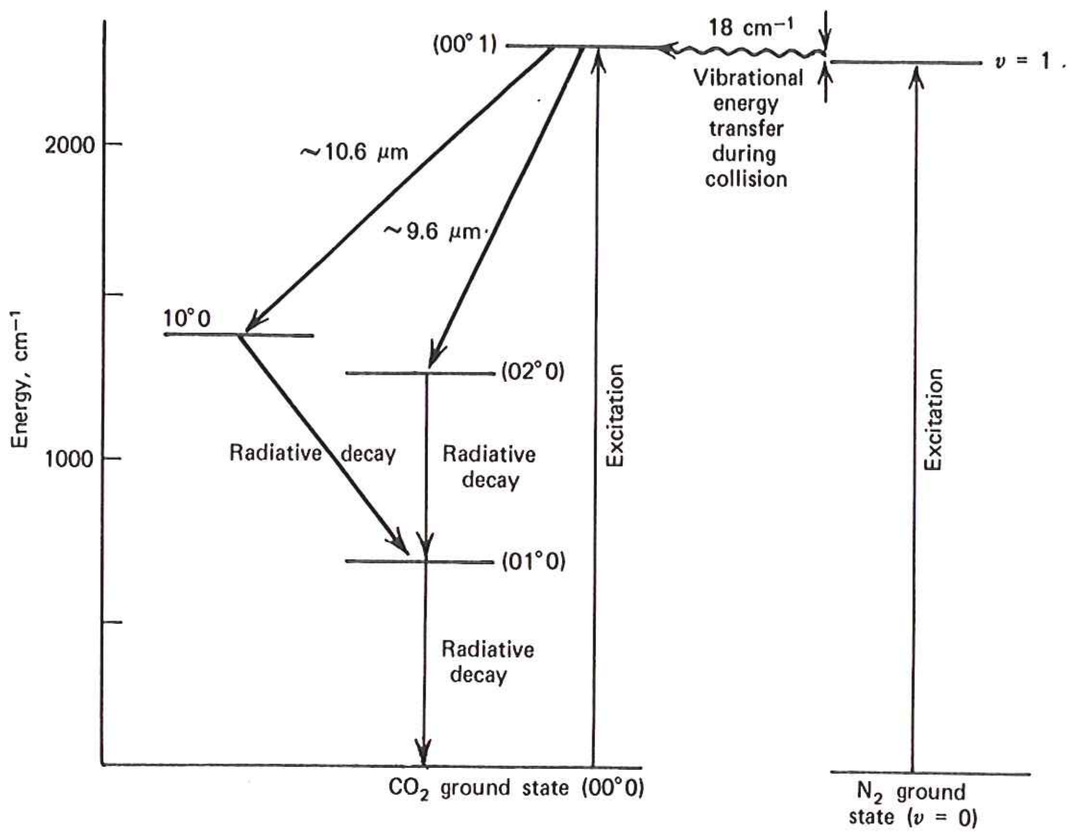


Figure 10-12. *Energy-level diagram for the Ar$^+$(ion) laser.*

Powers of tens of watts are readily obtained at a single wavelength using tubes ~1½ meter in length.

2c) CO_2 Laser

This laser uses a gas mixture $CO_2 / N_2 / He = 1:2:3$ (press. ratio) Electrons excite the CO_2 + N_2. The N_2 excited state is nearly degenerate with a CO_2 state leading to significant collisional population of the CO_2 excited state.



Helium controls the electron temperature to maximize lasing and helps transfer heat from the plasma to the walls of the laser tube.

The laser lases at many wavelengths centered around 10.6 + 9.6 μm. The efficiency with which electrical energy is converted to light is as high as 60%. This compares to less than 1% for Nd:YAG, Ar^+ lasers. The CW output powers may be as large as 100 kW which is useful for welding and cutting metals.

2d) Excimer Lasers

Excimers are molecules which are bound in excited states but are unstable in their electronic ground states. eg. ArF, KrF, XeCl

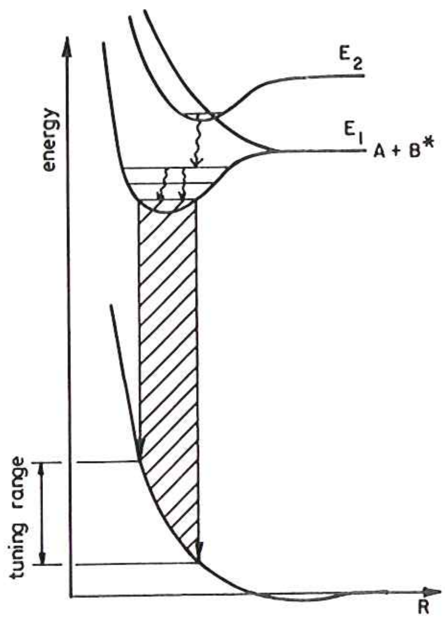


Fig.7.27. Schematic potential diagram of an excimer molecule

The excited state is populated using an electron beam or discharge. Lasing occurs between the molecular excited and ground states. These lasers are becoming increasingly important because of their ability to generate high powers of UV light.

Table 7.3. Survey on excimer laser characteristics

Excimer	Wavelength [nm]	Energy/pulse [mJ]	Peak power [MW]	Average power [W]
ArF	193	200	> 10	55
KrCl	222	70	> 3,5	0.4
KrF	248	350	> 15	6
XeBr	282	17	> 2	0.1
XeCl	308	90	> 9	4
XeF	351	90	> 5	5

Dye Laser

The disadvantage of the preceding lasers is that their output cannot be tuned to any desired wavelength. This problem is overcome with the dye laser. The spectrum from 400 – 1200 nm (+ growing!) is attainable using various dyes and pump lasers.

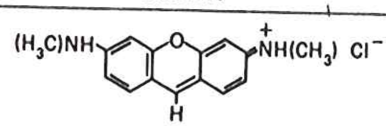
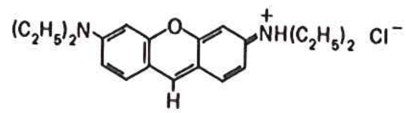
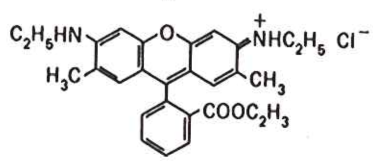
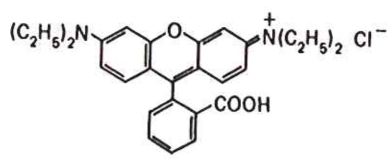
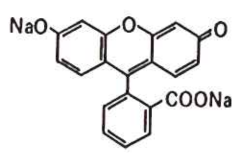
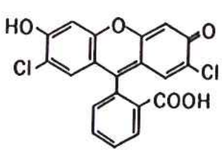
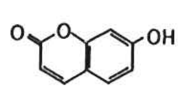
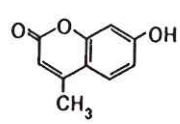
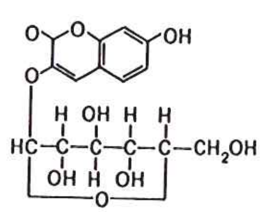
Dye	Structure	Solvent	Wavelength
Acridine red		EtOH	Red 600–630 nm
Puronin B		MeOH H_2O	Yellow
Rhodamine 6G		EtOH MeOH H_2O DMSO Polymethyl-methacrylate	Yellow 570–610 nm
Rhodamine B		EtOH MeOH Polymethyl-methacrylate	Red 605--635 nm
Na-fluorescein		EtOH H_2O	Green 530–560 nm
2,7-Dichloro-fluorescein		EtOH	Green 530–560 nm
7-Hydroxy-coumarin		H_2O (pH ~ 9)	Blue 450–470 nm
4-Methylem-belliferone		H_2O (pH ~ 9)	Blue 450–470 nm
Esculin		H_2O (pH ~ 9)	Blue 450–470 nm

Figure 10-6. *Molecular structure, laser wavelength, and solvents for some laser dyes. (After Ref. 8.)*

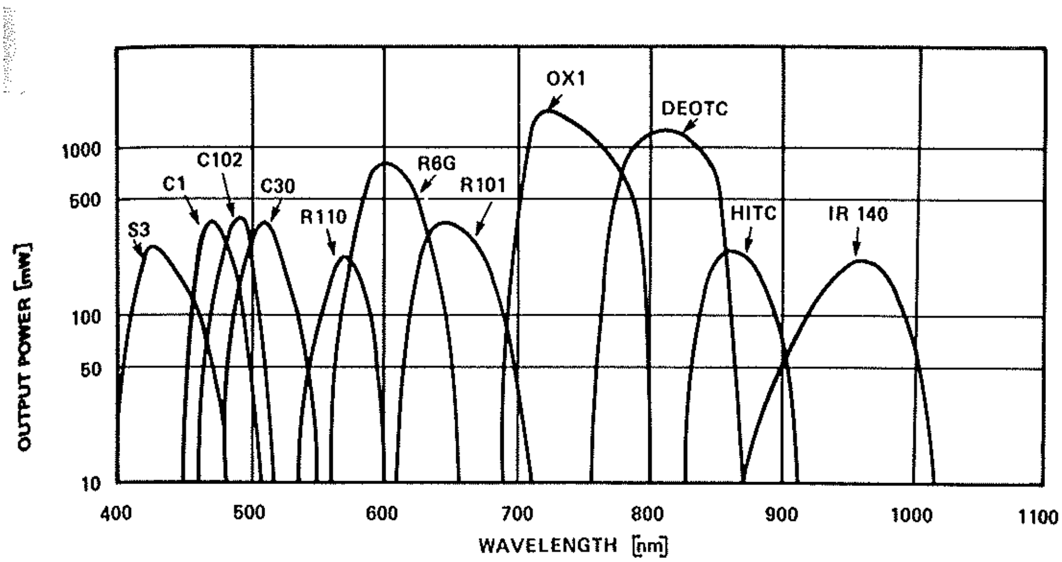
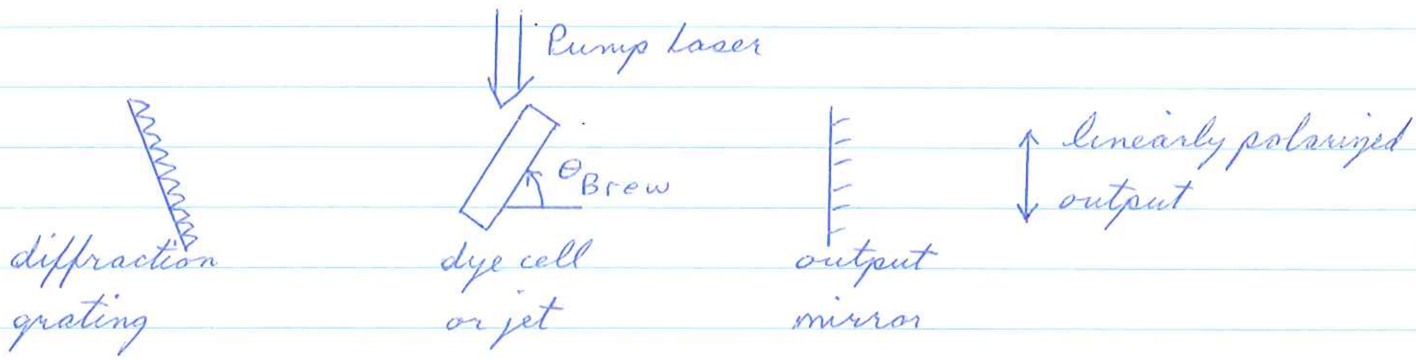


Fig.7.22. Spectral gain profiles of different laser dyes illustrated by the output power of cw dye lasers (from Coherent Radiation information sheet)

Table 7.2. Characteristic properties of dye lasers with different excitation sources

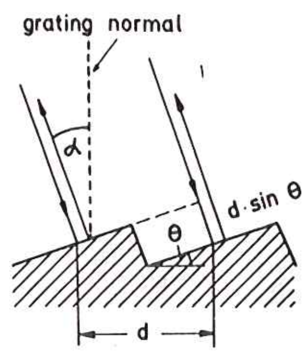
Pump	Tuning range [nm] with different dyes	Average output power [w]	Peak power [w]	Pulse duration [ns]	Linewidth [nm]
N_2 laser	350 - 1000	0.1 - 1	10^4-10^5	1 - 10	Fourier-limited
Excimer multiline laser	320 - 980	< 0.4	10^4-10^6	1 - 10	Fourier-limited
Flash-lamp	400 - 800	0.1 - 100	10^5	10^2-10^5	multimode: 0.1-0.01 single-mode: 10^{-4}
cw argon laser	400 - 800	0.1 - 10	maximum cw power reported:40 W	cw	< 1 MHz if well stabilized
cw krypton laser	400 - 800	0.1 - 1	-	cw	
YAG laser $\lambda/2$=530 nm $\lambda/3$=355 nm	400 - 800	0.1 - 1	10^4-10^6	5 - 30	0.01

1) Grating Dye Laser



Exercise: Why is the dye cell or jet positioned at Brewsters angle?

Frequently, one end mirror of the cavity is replaced by a diffraction grating. The grating is positioned in the so called Littrow configuration where light is reflected back into the direction of the incident light.



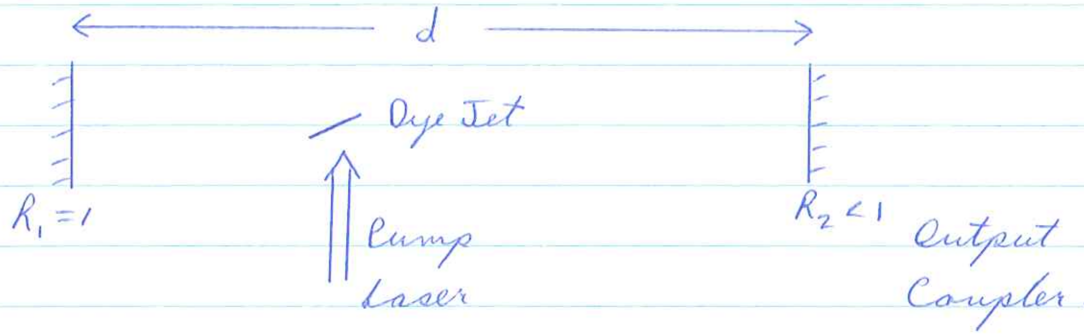
Constructive interference occurs if: $2d\sin\theta = m\lambda$ where

θ is called the blaze angle.
d = groove spacing
m = diffraction order.

The diffracted beam has a nonnegligible angular spread.

Hence the wavelength can be tuned by tilting the grating. Note that the blaze angle must be specified for the desired spectral range.

2) Standing Wave Dye Laser



Exercise: Why is the above called a "standing wave" dye laser?

The lasing wavelengths satisfy $m\lambda = 2d$.

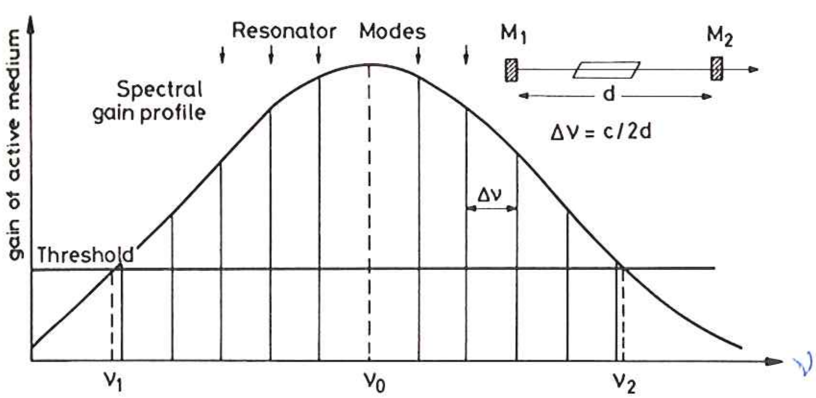


Fig.6.12. Longitudinal resonator modes within the spectral gain profile of a laser transition

The laser is obviously multimode. For many high precision experiments, as pure a frequency as possible is required. Hence, single mode operation is needed.

Etalon

An etalon is just two plane parallel plates of glass, which form an optical cavity or Fabry Perot interferometer. It is used as a wavelength selective transmission filter to narrow the laser linewidth.

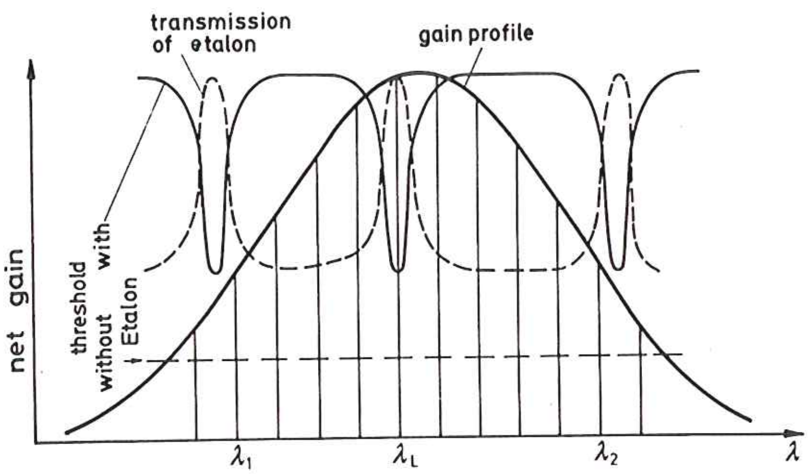


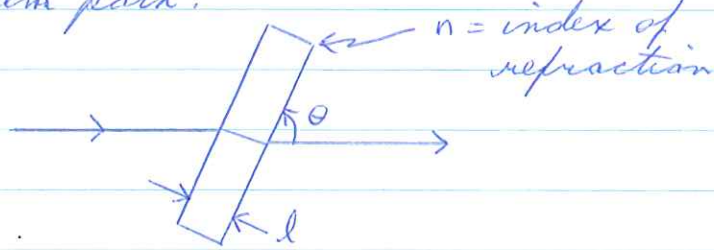
Fig.6.15. Gain profile, resonator modes, and spectral transmission of the etalon tuned for single-mode operation

Sometimes two etalons of different widths must be used to get single mode operation.

Question: How is the single mode tuned?

Answer: Change the cavity length.

This is commonly done by inserting a piece of glass in the beam path.



The cavity length then changes by $(n-1)\frac{l}{\cos\theta}$. Hence the cavity modes can be tuned by adjusting θ. When the laser is tuned, the etalon must be adjusted simultaneously or the power will be reduced until eventually the laser switches lasing to another mode. This is called mode hopping.

Etalons are adjusted by mounting one mirror on a piezoelectric crystal. Such a crystal changes its length when a voltage ($\sim$kV!) is applied. Naturally, the tilting of the tuning plate and the piezo voltage must be carefully coordinated for proper tuning. This is done by the electronics and gets complicated!

3) Ring Dye Laser.

In a standing wave laser, not all the gain is used, since stimulated emission occurs only at the high intensity regions in the dye. This problem is overcome by the ring laser. The ring laser is a "circular" cavity where light may oscillate in only one direction - clockwise or counterclockwise. Ring lasers produce powers roughly twice as large as standing wave lasers. The price is increased complexity.

Box = wavemeter

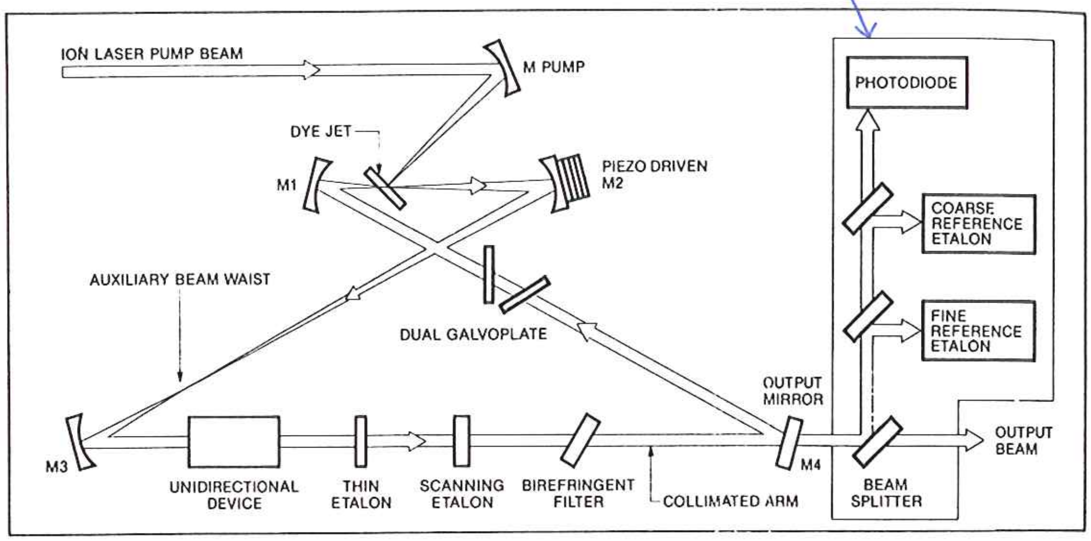


Fig.7.26. Ring dye laser (Courtesy Spectra Physics)

4) Ti / Sapphire

Recently, a solid material titanium sapphire has been discovered that is tunable from 700–900 nm when pumped by an Ar^+ laser. This crystal yields twice as much power as dyes in this region. It also does not degrade in time. Hence there is no disposal of carcinogenic chemical dye waste. Obviously, research is seeking other solid materials to replace dyes completely.

Diode Lasers

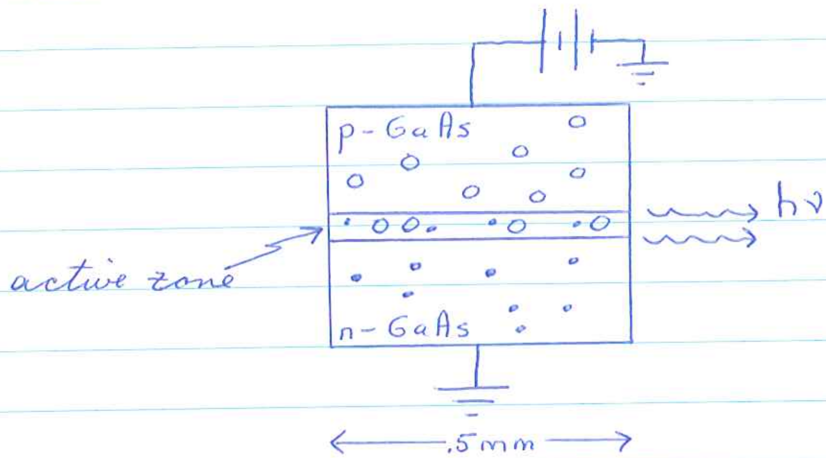
Diode or semiconductor lasers were first made in 1962. A diode laser is simply a pn junction where electrons and holes recombine to emit light. They can be mass produced using the same technology used to make computer chips. Hence they potentially could cost pennies as compared to >$20 K for a dye laser! An enormous research effort is presently underway to:

1) increase the output power from the current 1-10 mwatt
& 2) extend the wavelength range from the infrared into the visible spectrum.

Material	Oscillation Wavelength (micrometers)	
GaAs	0.837 (4.2°K)	0.843 (77°K)
InP		0.907 (77°K)
InAs		3.1 (77°K)
InSb	5.26 (10°K)	
PbSe	8.5 (4.2°K)	
PbTe	6.5 (12°K)	
$Ga(As_xP_{1-x})$	0.65–0.84	
$(Ga_xIn_{1-x})As$	0.84–3.5	
$In(As_xP_{1-x})$	0.91–3.5	
GaSb		1.6 (77°K)
$Pb_{1-x}Sn_xTe$	9.5–28 (~12°K) ↓ ↘ $x = 0.15$ $x = 0.27$	
$Ga_{1-x}Al_xAs$	0.69–0.85	
InGaP	0.5–0.7	

Laser Operation

A schematic diagram of a gallium arsenide (GaAs) laser is shown below.



p - GaAs ≡ GaAs doped with electron acceptor such as Zn
n - GaAs ≡ " " donor " " Te

The ends of the material are highly polished such that the material acts as its own cavity.

The energy levels are as shown below.

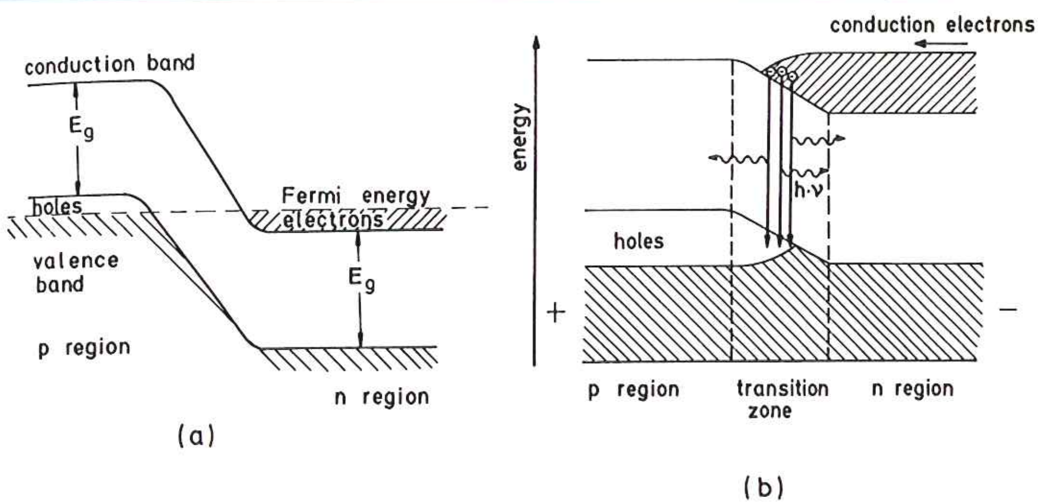
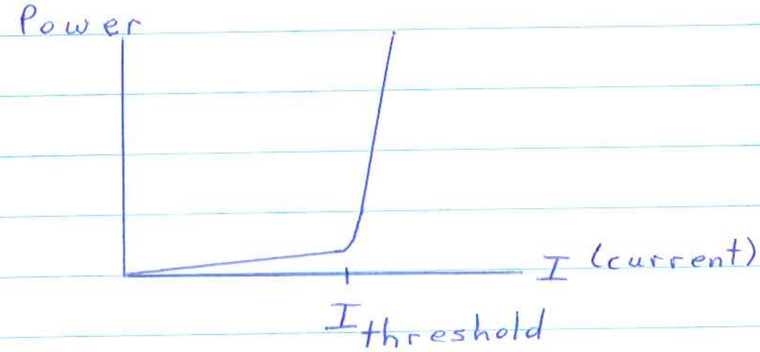


Fig.7.2a,b. Schematic level diagram of a semiconductor diode laser. (a) Unbiased, (b) with a forward voltage applied

When a voltage is applied across the pn junction, electrons and holes recombine in the transition zone emitting light. The photon energy equals the band gap E_g which in turn depends on the materials used. The number of photons increases linearly with current until a threshold value is reached where stimulated emission causes the photon number to grow exponentially as shown below.



Wavelength Tuning

The gain profile of a diode laser typically extends for only a few Å or cm⁻¹. The laser runs single mode if the transition is homogeneously broadened. However, as the current increases, the gain curve changes and the laser will mode hop. This is usually accompanied by a change in output

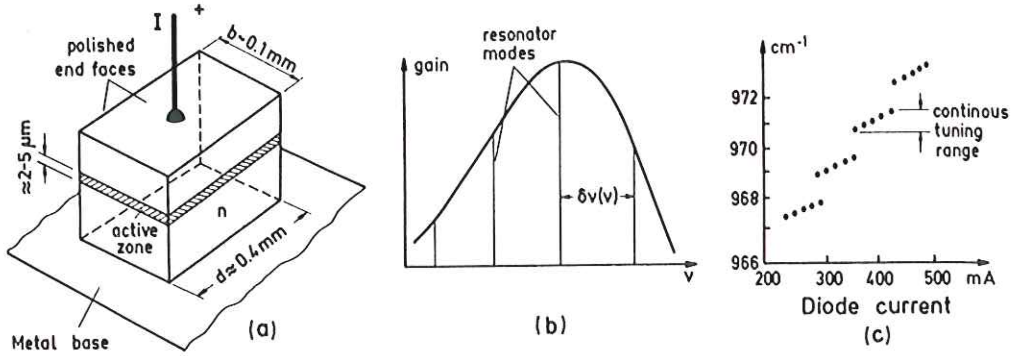


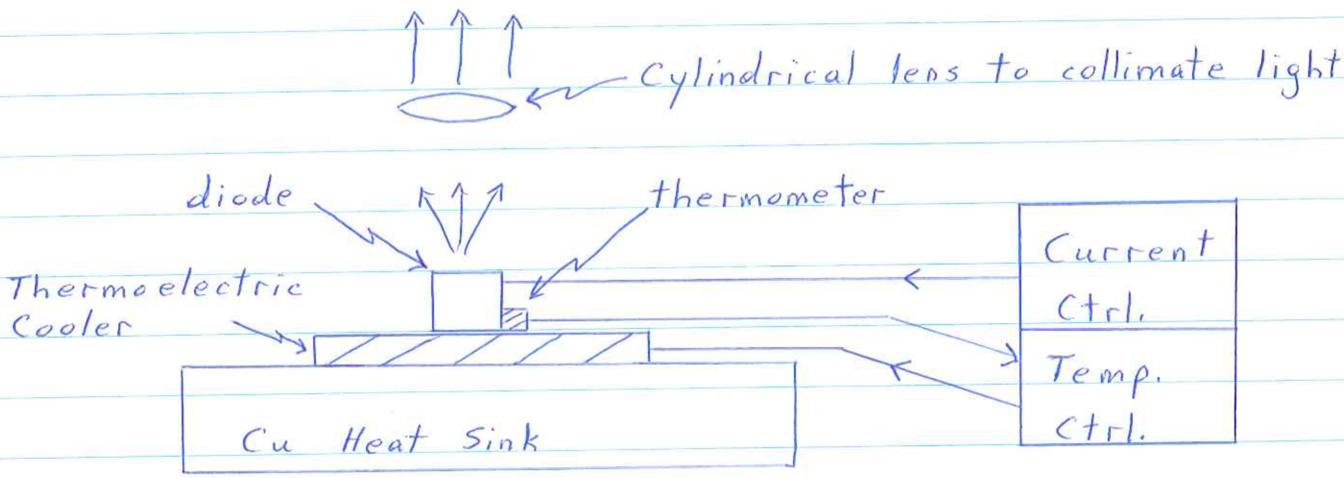
Fig.7.4a-c. Schematic diagram of a diode laser. (a) Injection laser structure. (b) Mode spectrum within the gain profile. (c) Mode hops of a quasi-continuously tunable cw Pb Sn Te diode laser in an He cryostat. The laser frequency is tuned by changing the diode current.

power since different modes have different gains.

a) Temperature Tuning

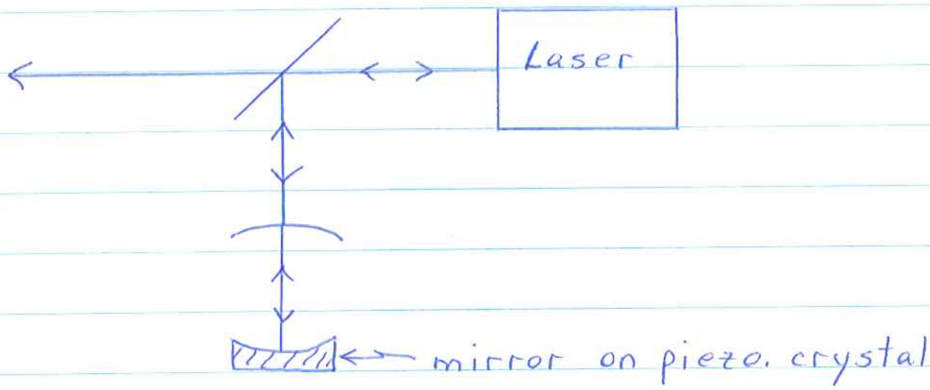
The laser wavelength can be tuned by changing the diode temperature which alters the materials index of refraction and hence the cavity optical length. Typically, the frequency changes by 1 MHz for each 1 mK change in temperature. Temperature can be controlled using a thermoelectric cooler.

Diode Laser Setup
For Spectroscopy



b) External Cavity

A laser diode may also be locked to an external cavity.



The external cavity acts as a laser end mirror having high reflectivity at the cavity resonance frequency. The laser is tuned by changing the cavity length which is conveniently done by mounting one mirror on a piezoelectric crystal. Recently, an external cavity has been shown to reduce the laser linewidth to a few kHz!!

Conclusion

Diode lasers are not a mature technology. For up to date articles see Scientific American Fall 1991.

Nonlinear Optics

A nonlinear medium is one whose polarization is given by

$$P = \epsilon_0 \left[\chi^{(1)} E + \chi^{(2)} E^2 + \chi^{(3)} E^3 + \cdots \right]$$

where $\chi^{(k)}$ is the kth order susceptibility. Such a material can mix frequencies as we now show.

Consider the sum of two fields at frequencies ω_1 + ω_2.

$$E_{TOT} = E_1 \cos \omega_1 t + E_2 \cos \omega_2 t$$

$$E_{TOT}^2 = E_1^2 \cos^2 \omega_1 t + E_2^2 \cos^2 \omega_2 t + 2 E_1 E_2 \cos \omega_1 t \cos \omega_2 t$$

$$= \frac{1}{2}(E_1^2 + E_2^2) + \frac{E_1^2}{2} \cos 2\omega_1 t + \frac{E_2^2}{2} \cos 2\omega_2 t$$

$$+ E_1 E_2 \left[\cos(\omega_1 + \omega_2)t + \cos(\omega_1 - \omega_2)t \right]$$

Hence the first nonlinear term of P acts as a source of waves at frequencies $2\omega_1$, $2\omega_2$, $\omega_1 + \omega_2$ + $\omega_1 - \omega_2$. Nonlinear materials are especially useful to generate coherent light in the UV where lasers are unavailable.

General Form

The susceptibilities are in general tensors since $\vec{P}$ + $\vec{E}$ are vectors.

$$\therefore P_i = \epsilon_0 \left[\chi_{ij}^{(1)} E_j + \chi_{ijk}^{(2)} E_j E_k + \chi_{ijkl}^{(3)} E_j E_k E_l + \cdots \right]$$

$$= \epsilon_0 \chi_{ij}^{(1)} E_j + P_{NL\,i}$$

nonlinear term

Sum Frequency Generation

We shall show how two waves at frequencies ω_1 + ω_2 can generate a wave at frequency $\omega_3 = \omega_1 + \omega_2$.

Exercise: Derive the following equation from Maxwells equations assuming 1) no free charges or currents
2) nonmagnetic material
3) $\epsilon = \epsilon_0 (1 + \chi_L)$ where $\chi^{(1)}_{ij} = \chi_L \delta_{ij}$

$$\nabla^2 \vec{E}_\perp = \mu_0 \epsilon \frac{\partial^2 \vec{E}_\perp}{\partial t^2} + \mu_0 \frac{\partial^2 \vec{P}_{NL\perp}}{\partial t^2} \quad (1)$$

where $\perp$ denotes component of $\vec{E}$, $\vec{P}_{NL}$ such that $\nabla \cdot \vec{E}_\perp = 0$. For simplicity we consider the waves to be plane waves travelling in the z direction.

$$\vec{E}_1(z,t) = \vec{E}_1(z) \cos(\omega_1 t - k_1 z)$$

$$\vec{E}_2(z,t) = \vec{E}_2(z) \cos(\omega_2 t - k_2 z)$$

$$\vec{E}_3(z,t) = \vec{E}_3(z) \cos(\omega_3 t - k_3 z)$$

Here $k_i^2 = \mu_0 \epsilon_i \omega_i^2$ since the index of refraction and hence ϵ may depend on frequency. $\therefore \epsilon \to \epsilon_i$

The incident waves 1 + 2 create the following nonlinear polarization at frequency $\omega_1 + \omega_2$.

$$P_{NL\,i} = \epsilon_0 \chi^{(2)}_{ijk} \; E_{1j}(z) \, E_{2k}(z) \cos\left[(\omega_1 + \omega_2)t - (k_1 + k_2) z\right]$$

Exercise: Show the following.

$$\nabla^2 E_{3i}(z,t) = \left[\frac{d^2 E_{3i}(z)}{dz^2} - 2ik_3 \frac{dE_{3i}(z)}{dz} - k_3^2 E_{3i}(z) \right] \frac{e^{i(\omega_3 t - k_3 z)}}{2} + c.c.$$

complex conjugate

We shall assume the wave amplitude $E_3(z)$ grows slowly over one wavelength ($\lambda_3 = 2\pi/k_3$) such that

$$\frac{d^2 E_{3i}}{dz^2} << 2k_3 \frac{dE_{3i}}{dz}.$$

Hence wave equation (1) becomes the following:

$$\frac{1}{2}\left[-2ik_3 \frac{dE_{3i}(z)}{dz} - k_3^2 E_{3i}(z) \right] e^{i(\omega_3 t - k_3 z)} + c.c.$$

$$= -\frac{\mu_0 \epsilon_3 \omega_3^2}{2} \left[E_{3i}(z) e^{i(\omega_3 t - k_3 z)} + c.c. \right] - \frac{\mu_0 (\omega_1 + \omega_2)^2}{2} \epsilon_0 \chi^{(2)}_{ijk} E_{1j}(z) E_{2k}(z) \left[e^{i[(\omega_1+\omega_2)t - (k_1+k_2)z]} + c.c. \right]$$

$$\frac{dE_{3i}(z)}{dz} = \frac{-i\omega_3}{2} \sqrt{\frac{\mu_0}{\epsilon_3}} \epsilon_0 \chi^{(2)}_{ijk} E_{1j}(z) E_{2k}(z) e^{i\Delta k z}$$

where $\Delta k \equiv k_3 - k_1 - k_2$ and $\omega_3 = \omega_1 + \omega_2$.

Integrating this equation and assuming $E_3(z) << E_1(z), E_2(z)$ (i.e. very little energy is converted from $\omega_1, \omega_2 \to \omega_3$) we get:

$$E_{3i}(L) = \frac{-i\omega_3}{2} \sqrt{\frac{\mu_0}{\epsilon_3}} \epsilon_0 \chi^{(2)}_{ijk} E_{1j} E_{2k} \frac{e^{i\Delta k L} - 1}{i\Delta k}$$

Power in wave $$P = \underbrace{A}_{\text{beam area}} \; \underbrace{\frac{c}{n}}_{\text{speed}} \; \underbrace{\frac{\epsilon E^2}{2}}_{\text{energy density}}$$

Exercise: Show power in wave at frequency ω_3 is:

$$P(\omega_3) = \frac{A}{8}\sqrt{\frac{\mu_0}{\epsilon_3}}\;\omega_3^2\left(\epsilon_0\chi^{(2)}_{ijk}\right)^2 E_{1j}^2\,E_{2k}^2\,L^2\left[\frac{\sin \Delta kL/2}{\Delta kL/2}\right]^2$$

Taking $\epsilon_1 = \epsilon_2 = \epsilon_3$ we get:

$$P(\omega_3) = \frac{1}{2A}\left(\frac{\mu_0}{\epsilon_0}\right)^{3/2}\frac{\omega_3^2}{n^3}\left(\epsilon_0\chi^{(2)}_{eff}\right)^2 P(\omega_1)\,P(\omega_2)\,L^2\left[\frac{\sin \Delta kL/2}{\Delta kL/2}\right]^2$$

For second harmonic generation $\omega_1 = \omega_2 = \omega$, $\omega_3 = 2\omega$.

$$\boxed{P(2\omega) = \frac{2}{A}\left(\frac{\mu_0}{\epsilon_0}\right)^{3/2}\frac{\omega^2}{n^3}\left(\epsilon_0\chi^{(2)}_{eff}\right)^2 P^2(\omega)\,L^2\left[\frac{\sin \Delta kL/2}{\Delta kL/2}\right]^2}$$

Hence the efficiency of frequency conversion is greater at higher powers and when $\Delta k = 0$.

Phase Matching

The second harmonic power is largest when $\Delta k = 0$. This is equivalent to $k_3 = k_1 + k_2$ which is a statement of photon momentum conservation.

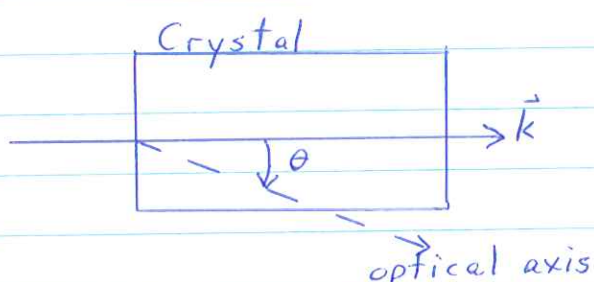
Exercise: Show $k(2\omega) = 2k(\omega) \Longrightarrow n(2\omega) = n(\omega)$

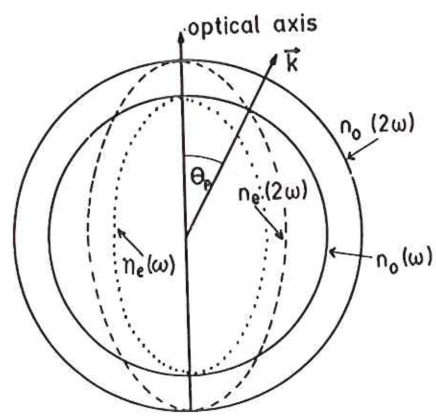
Therefore the waves at frequencies ω & 2ω have the same phase velocity. They remain in phase and hence the criteria $\Delta k = 0$ is called the phase matching condition.

$n(2\omega) = n(\omega)$ can be achieved in some uniaxial crystals. The refraction index of the extraordinary ray is given by:

$$\frac{1}{n^2} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

where θ is the angle between $\vec{k}$ and the optical axis. Hence by cutting the crystal at the right angle θ, $n_e(2\omega) = n_o(\omega)$ as shown below.





Example.

KDP ≡ Potassium dihydrogen phosphate is used to generate doubled + tripled YAG light.

Step 1: $2 \times 1.06\,\mu m \longrightarrow 532\,nm$ (green)

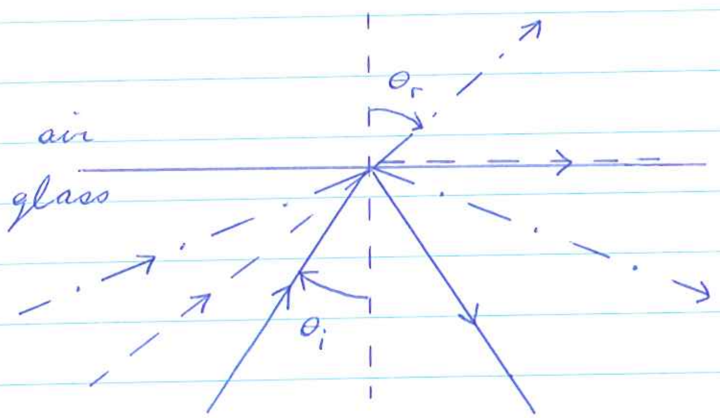
Step 2: $1.06\,\mu m + 532\,nm \longrightarrow 355\,nm$ (blue)

The conversion efficiency ~50%. The doubled + tripled YAG light are used to pump dye lasers which then operate all through the visible spectrum.

Optical Fibers

Optical Fiber ≡ Glass highway for light transportation

Total Internal Reflection



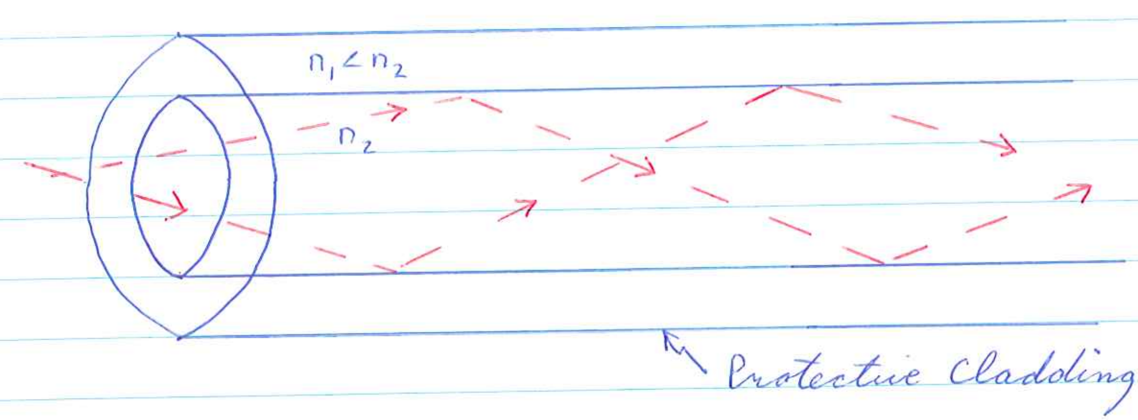
Snell's Law $\quad n_g \sin\theta_i = n_{air} \sin\theta_r$

Light is transmitted into air if $n_g \sin\theta_i > n_{air}$

$$\sin\theta_i > \frac{n_{air}}{n_g} = \frac{1}{1.5}$$

$$\therefore \theta_i > 41.8^\circ$$

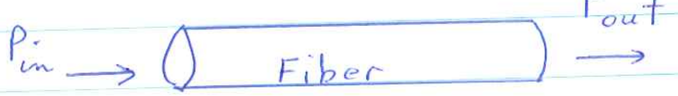
An optical fiber confines "nearly" on axis rays to central core.



Problem:

Glass attenuates light. In 1967, a good fiber had loss of ~1000 dB/km.

Decibel

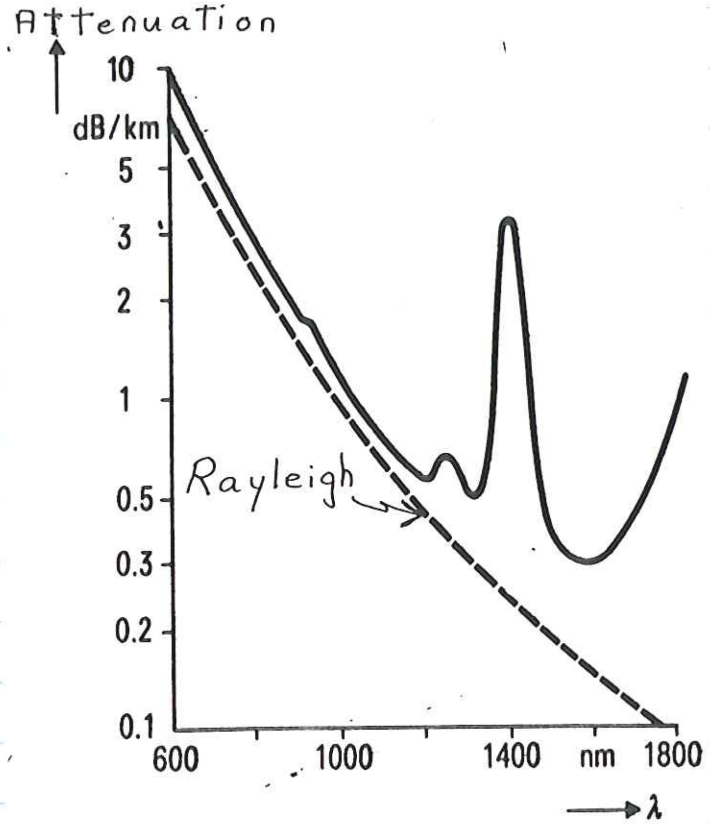


$$\text{Loss (decibels)} = -10 \log_{10}\left(\frac{P_{out}}{P_{in}}\right)$$

Loss	P_{out} / P_{in}
1 dB	80 %
3	50 %
10	10 %
100	10^{-9}
1000	10^{-99}

Material	Attenuation (dB/km)
Window glass	10 000
Optical glass (spectacles, objectives)	300
Pure water at $\lambda_0 = 0.5\,\mu m$	90
Silica glass optical fibres:	
Corning (USA) 1970	20
1972	4
1973	2
Fujikura (Japan) at $\lambda_0 = 0.83\,\mu m$	1
$\lambda_0 = 1.10\,\mu m$	0.5
Several $\lambda_0 = 1.30\,\mu m$	< 0.5
Approximate present limit	0.25

Wavelength Dependence of Attenuation

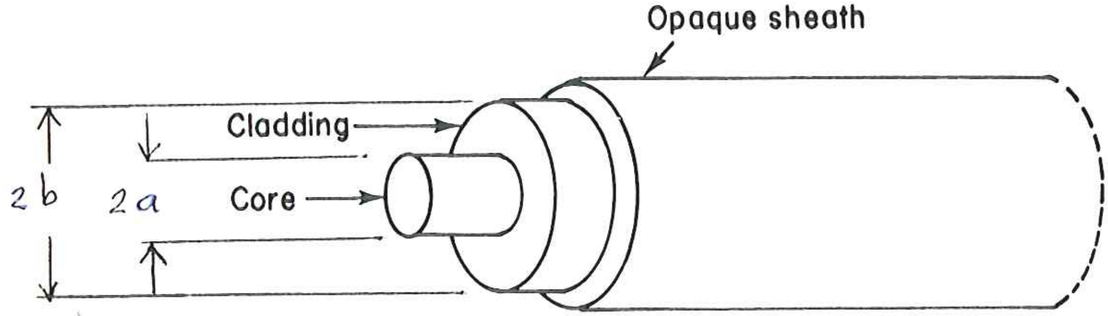


Causes of Light Attenuation

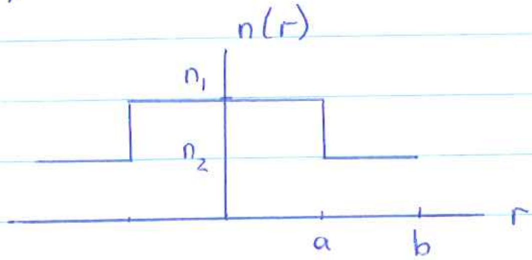
1. Rayleigh Scattering
2. Absorption by impurities such as metallic traces, H_2O
3. Scattering from Fiber Imperfections

Optimum wavelengths are: 800 – 900 nm, 1.3 μm, 1.5 μm } laser diodes + detectors readily available

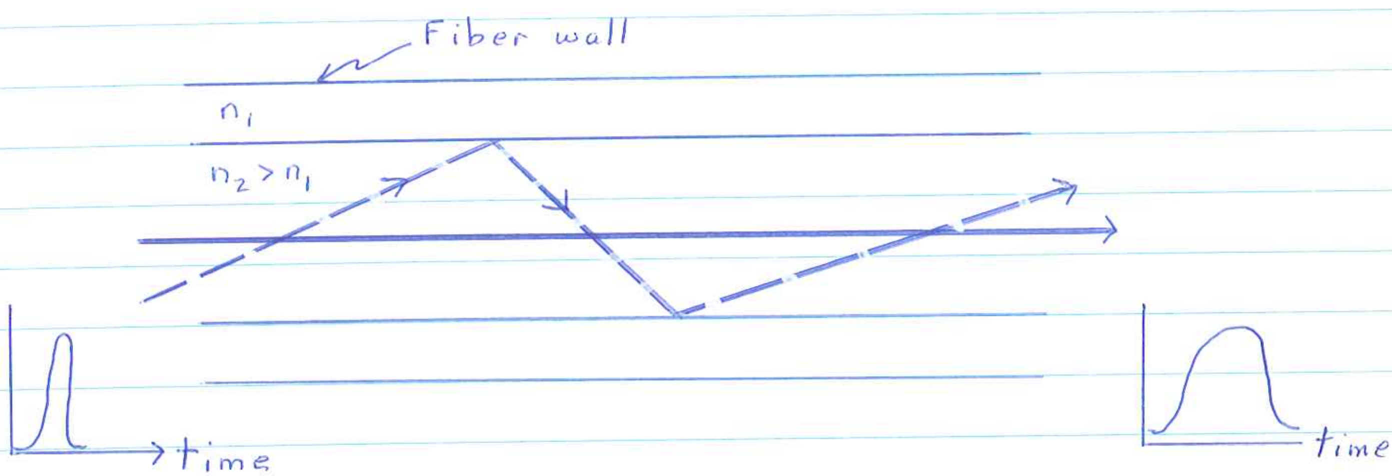
Types of Fibers



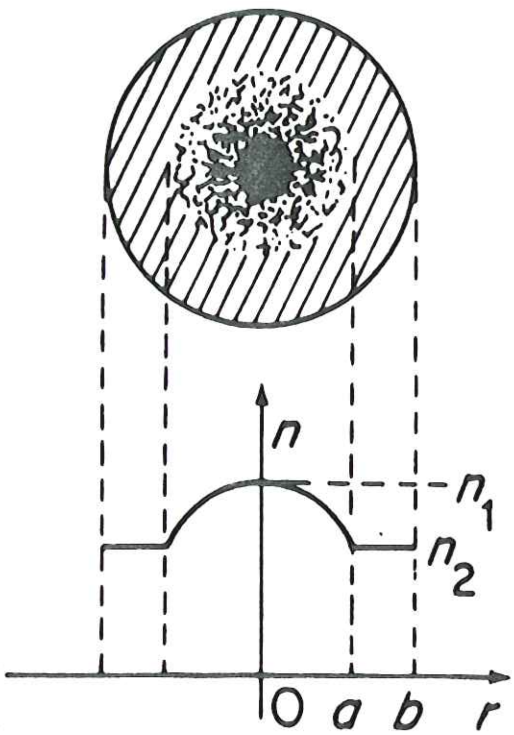
1) Step Index Fiber

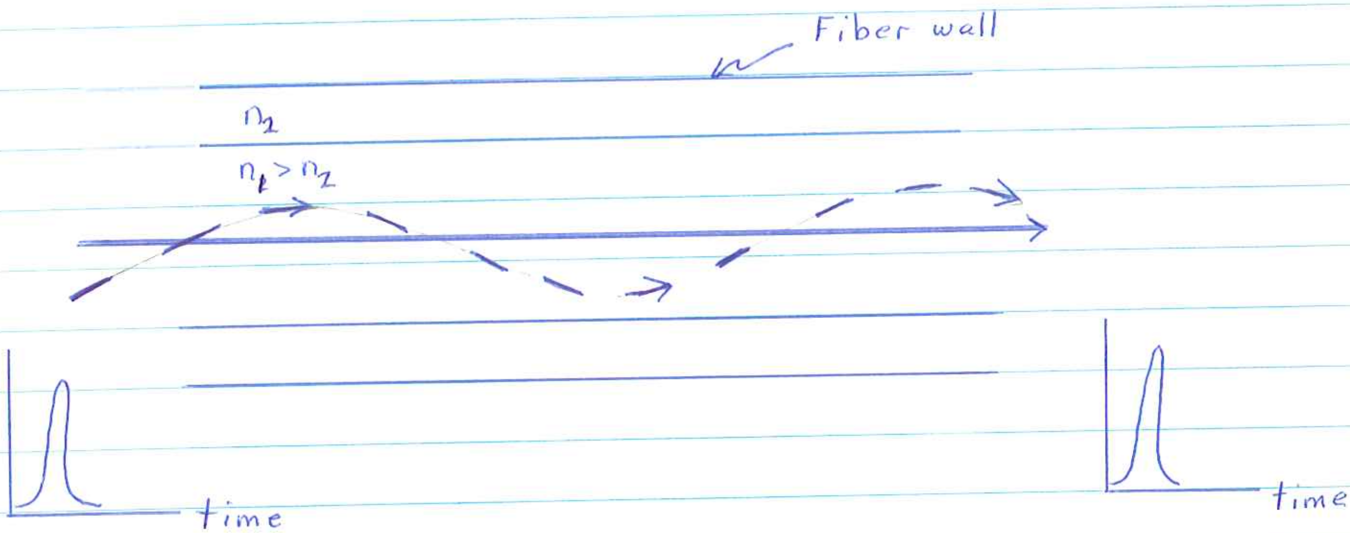


This fiber is easy to make. The disadvantage is that off axis rays take longer to travel through the fiber resulting in pulse broadening.



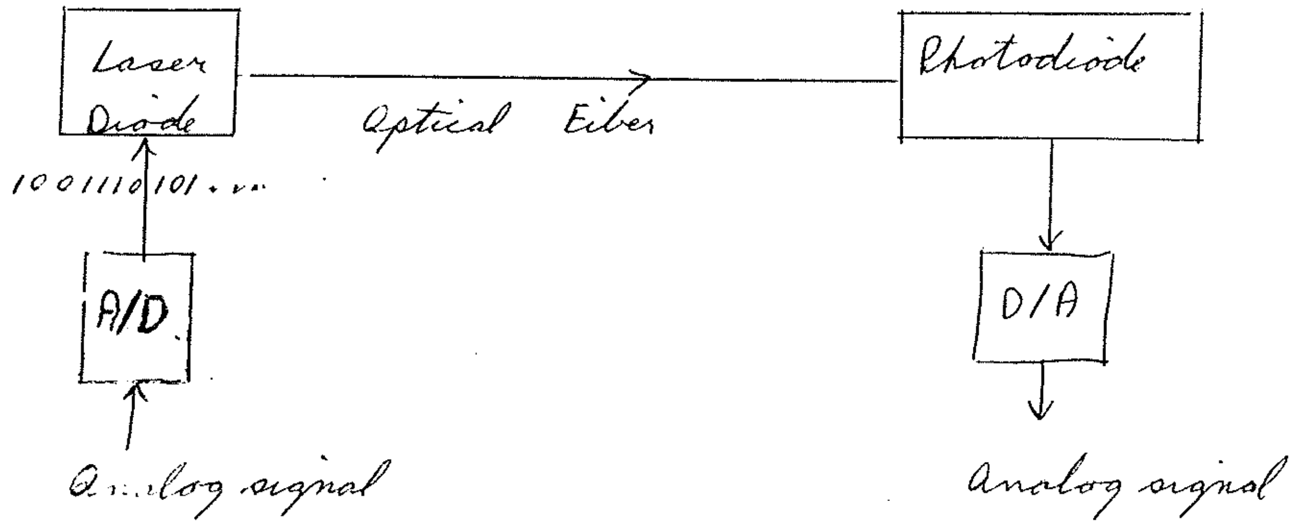
2) Graded Index Fiber



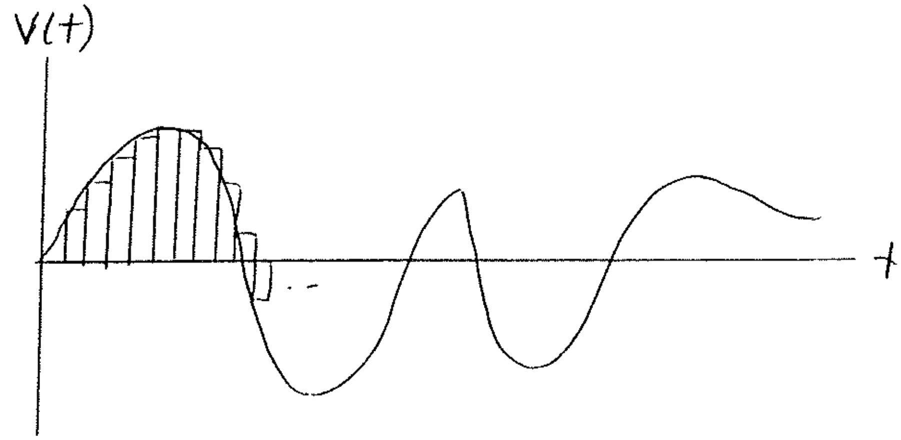


Off axis modes have higher speed than on axis rays. Hence there is less pulse broadening and information may flow faster. i.e. higher bandwidth

Digital Signal Transmission



Digitization



Voice bandwidth 300-3000 Hz.

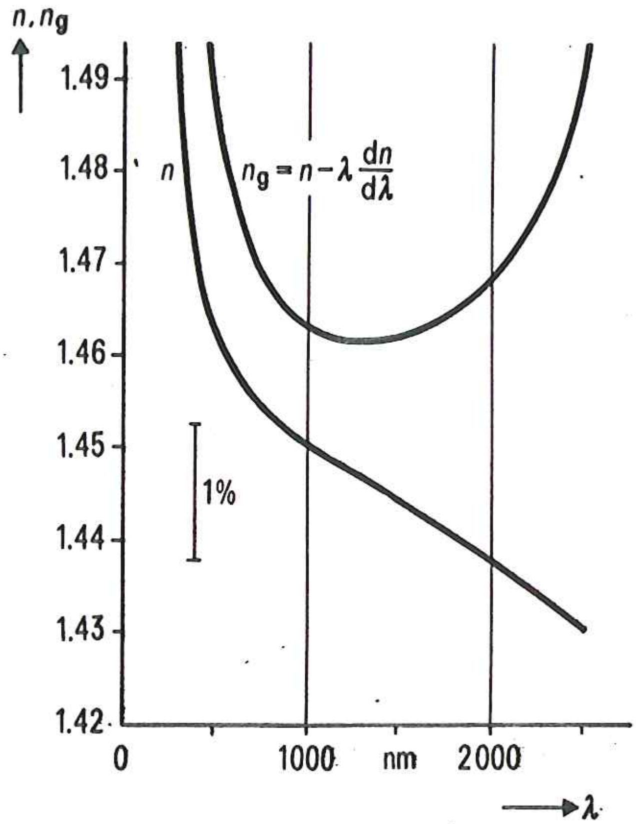
Sampling frequency $= 2 \times f_{max}$
~ 8000 Hz.

Quantize signal using 8 bits. (V(t) assigned number from 1- $2^8 = 256$.

$\therefore$ 1 telephone conversation requires 8×8000
$= 64$K bit/sec.

Maximum Data Rate

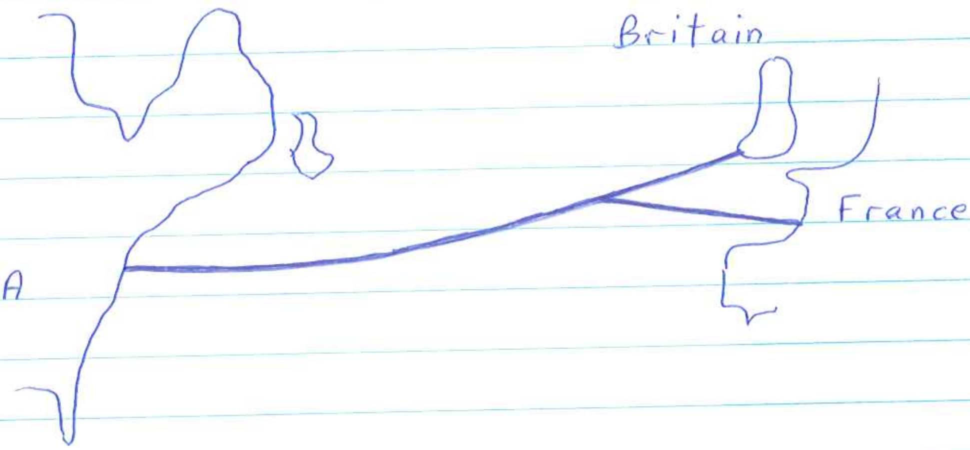
The max. data rate is limited by chromatic dispersion ($n = n(\lambda)$) and the finite linewidth of the laser. i.e. different colours travel at different speeds.



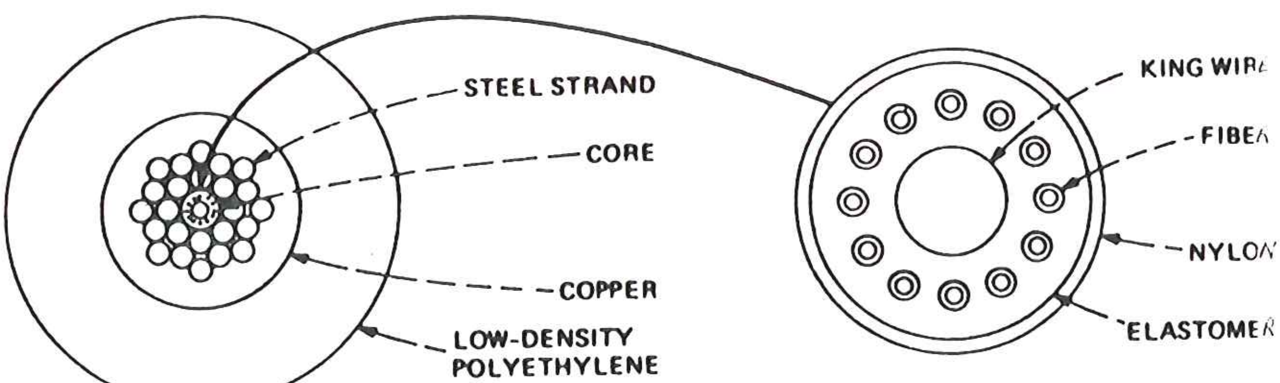
The dispersion is minimized when n_g has zero slope at 1.3 μm.

"Present" max. data rate = 10 GBit/sec.

TAT 8 Cable (1988)



- 12 single mode fibers
- 7680 telephone channels (274 MB/sec) per fiber
- 35 km between repeater amplifiers
- 6500 km. length



CABLE STRUCTURE:

STRAND DIAMETER = 9.47 mm (0.373 in.)
CONDUCTOR OD (COPPER) = 10.46 mm (0.412 in.)
INSULATION OD = 21 mm (0.827 in.)

CABLE CORE:

KING WIRE OD = 0.71 mm (0.028 in.)
NUMBER OF FIBERS = 12
FIBER OD (COATED) = 250 μm (0.010 in)
NYLON THICKNESS = 0.1 mm (0.004 in)
CORE OD = 2.97 mm (0.117 in.)

Advantages of Optical Fibers vs. Cu Coax Cables

1. Higher carrier frequency in optical fiber.

$$\frac{\text{visible freq.}}{\text{microwave freq.}} = \frac{10^{14}\ Hz}{10^{11}\ Hz} = 1000$$

Coax cable - 1 GBit/sec
Optical fiber - 10 GBit/sec.

2. Well suited for digital signals - light is on/off.

3. Longer distance between repeaters ~50 km.

4. No EM interference - crosstalk, lightning

5. Good security

What's wrong with Satellites?

Not enough room in geostationary orbit.
- must be separated to avoid interference when using same frequency band.
- orbit over N.A. is full for 4 & 6 GHz bands.

- time delay $\overbrace{\text{is}}^{\sim .5\ \text{sec}}$ a nuisance

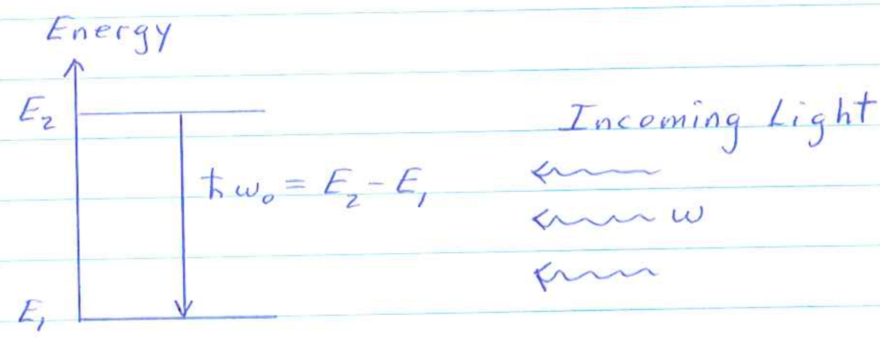
Laser Cooling of Atoms

Frequencies of atomic transitions are commonly measured when making high precision measurements. The precision is usually limited by the Doppler linewidth. Laser cooling uses lasers to slow or even stop the atoms motion. The linewidth then equals the natural width which is much less than the Doppler width.

Light interacts with atoms via the so called radiation pressure and the dipole or gradient forces.

Radiation Pressure Force

Consider a two level atom interacting with light of frequency ω.



$$\vec{F}_{rad} = \hbar\vec{k}\ \frac{\Gamma}{2}\ \frac{\omega_1^2/2}{(\omega-\omega_0)^2 + \frac{\omega_1^2}{2} + \frac{\Gamma^2}{4}}$$

where: $\hbar\vec{k}$ = photon momentum

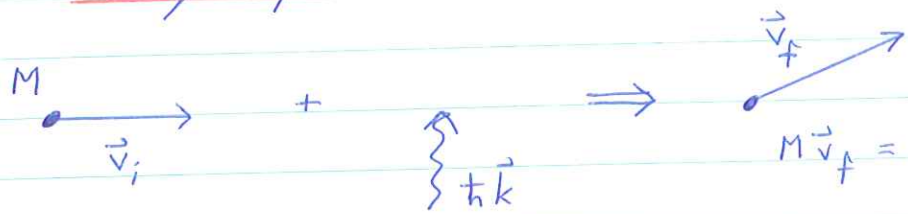
Γ = transition linewidth

$= \tau^{-1}$ (τ = excited state lifetime)

$\omega_1 = \dfrac{\vec{d}\cdot\vec{\mathcal{E}}}{\hbar}$ Rabi frequency (d = electric dipole mom.)

$\omega - \omega_0$ = detuning of laser from atomic resonance

Origin of Force



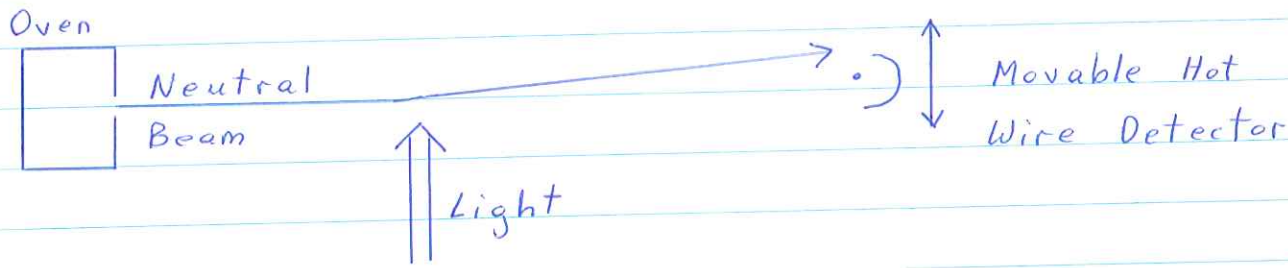
$$M\vec{v}_f = M\vec{v}_i + \hbar\vec{k}$$

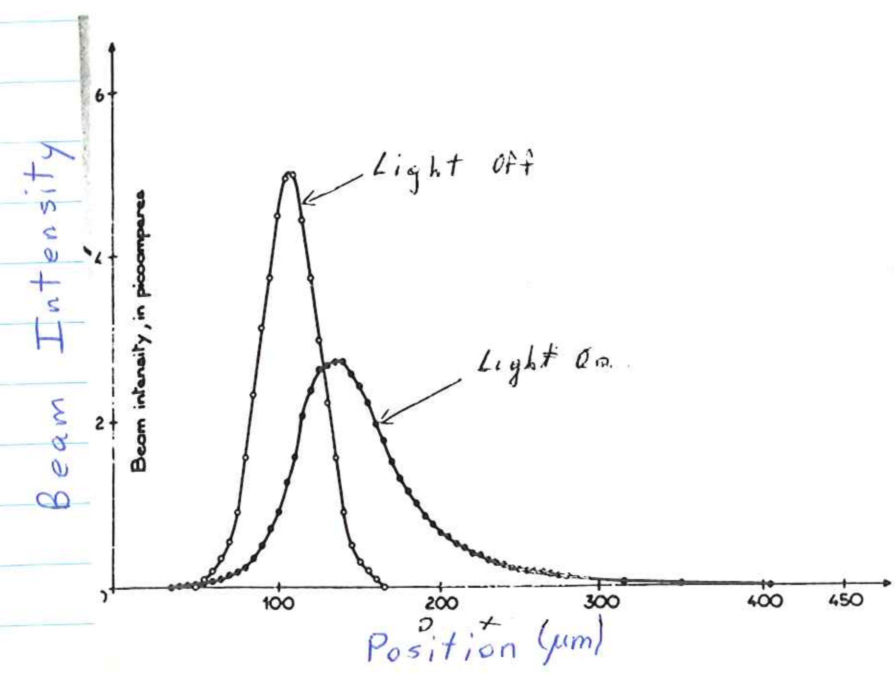
After absorbing and emitting N photons, atom has momentum

$$M\vec{v}_f = \underbrace{M\vec{v}_i}_{\text{initial momentum}} + \underbrace{N\hbar\vec{k}}_{\text{incoming photon beam}} - \underbrace{\sum_{i=1}^{N}\hbar\vec{k}_i'}_{\text{reradiated photon momentum}}$$

When excited atom decays, photon is emitted in random direction. $\Longrightarrow \sum_{i=1}^{N}\hbar\vec{k}_i' = 0$

Laser Deflection of Atomic Beam





Dipole or Gradient Force

$$\vec{F}_D = -\frac{\hbar(\omega-\omega_0)}{2}\,\frac{\nabla(\omega_1^2/2)}{(\omega-\omega_0)^2+\frac{\omega_1^2}{2}+\frac{\Gamma^2}{4}}$$

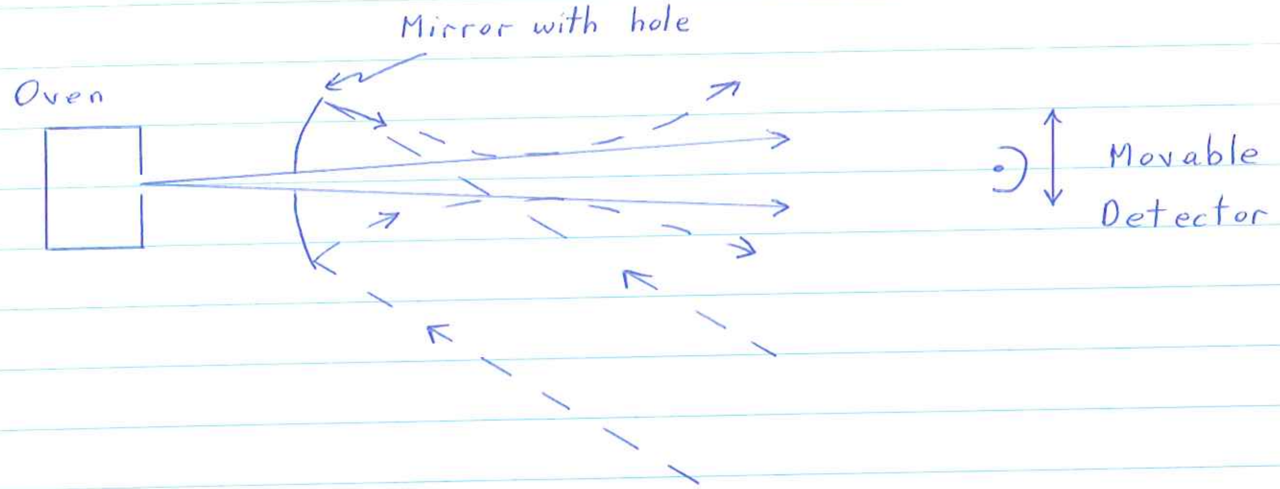
where: $\omega-\omega_0$ = detuning of laser from atomic resonance
$\omega_1 = \vec{d}\cdot\vec{\mathcal{E}}/\hbar$ Rabi frequency
Γ = transition linewidth

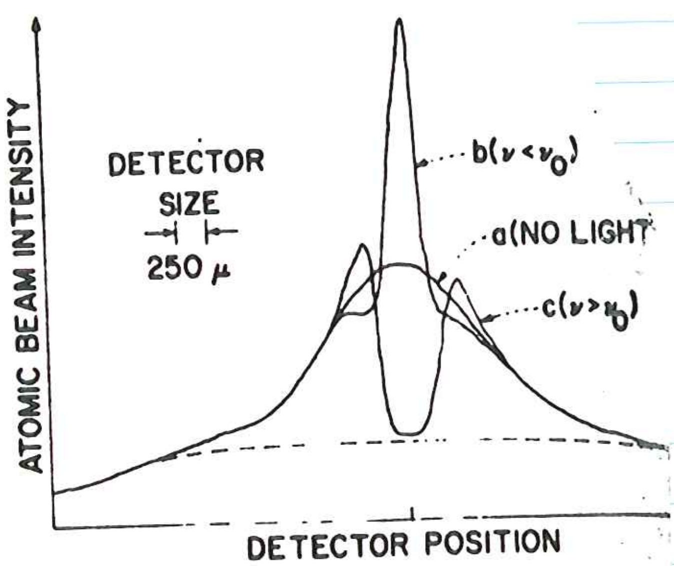
Note that $\omega_1^2 \propto$ laser intensity.
$\therefore\ \omega < \omega_0 \Rightarrow$ atom is attracted to laser
$\omega > \omega_0 \Rightarrow$ " " repelled " "

Origin of Force

The laser electric field polarizes the atom which then acts as an electric dipole. The dipole force then arises when this induced dipole interacts with the field.

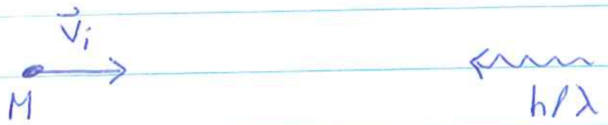
Focussing an Atomic Beam





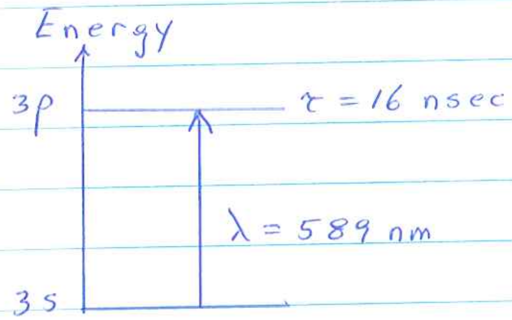
∴ lasers can focus atomic beams greatly increasing the beam flux. This may be used to improve experimental accuracy. This focussing ability is also of interest for the manufacture of microscopic devices.

Stopping a Beam



\# photons required to stop atom $N = \dfrac{M v_i}{h/\lambda}$.

Example: Na atom



$M_{Na} = 23 \times 1.67 \times 10^{-24}$ gm.

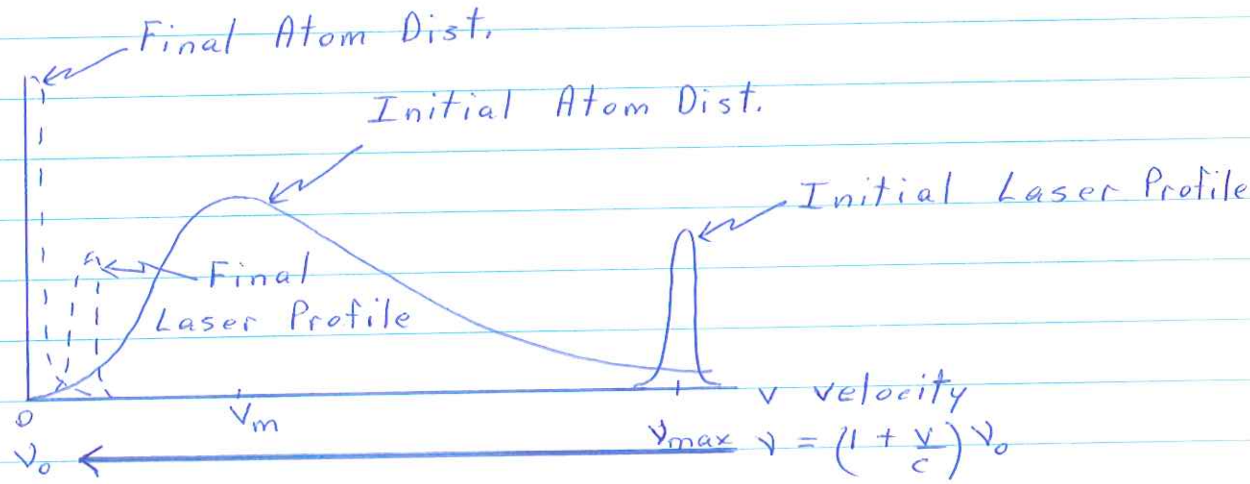
$v_i \simeq 600$ m/sec

$\Longrightarrow N \sim 20{,}000$ photons

Exercise: Show that 1) stopping time ~ 1 msec
2) stopping distance ~ 30 cm.

Problem

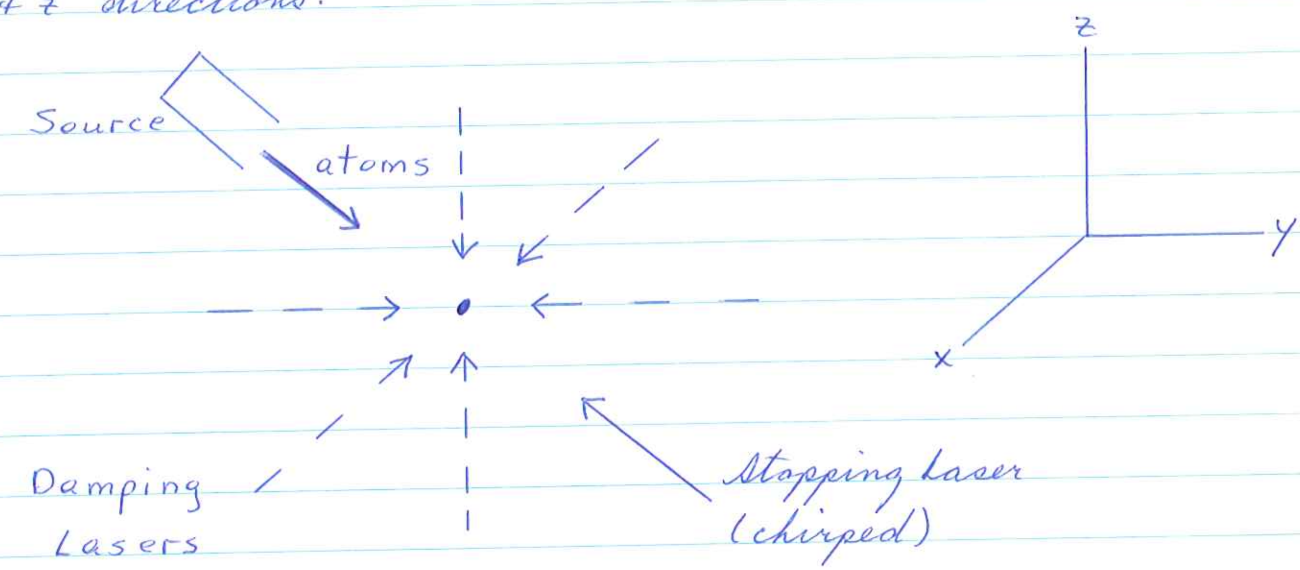
The laser cannot be absorbed by all atoms.

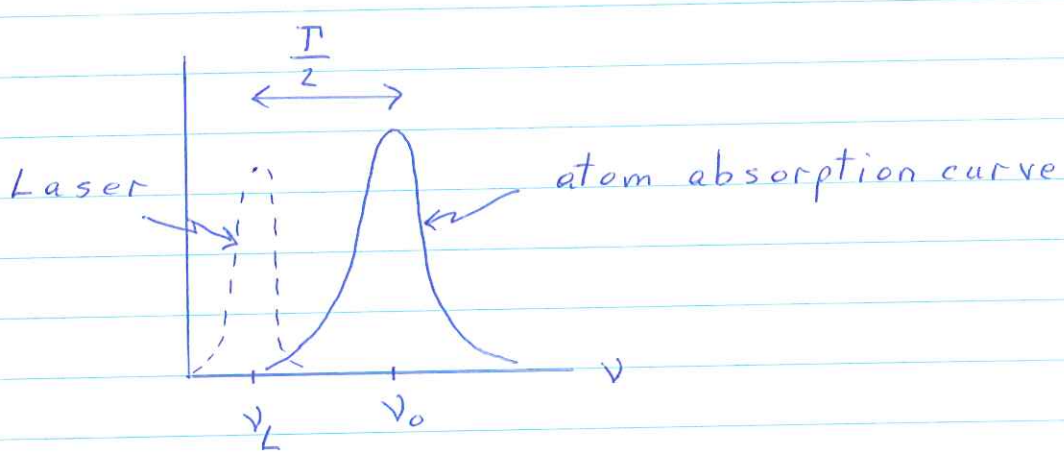


One can vary or chirp the laser frequency from ν_{max} to ν_0 to stop all atoms.

Optical Molasses

For atoms to stay trapped, their motion must be damped in 3 dimensions. This can be done using pairs of lasers in x, y + z directions.





If the atom moves to right, laser coming from right is Doppler shifted into resonance. Atom absorbs a photon and is kicked back. If the laser is tuned onto resonance, light is absorbed and the atoms heat up. The minimum temp. is attained when $\nu_L - \nu_0 = \frac{\Gamma}{2}$. $\Rightarrow kT_{min} = \frac{h\Gamma}{2}$.

eg. for Na $\Gamma \sim 10\,MHz \Rightarrow T_{min} \sim 240\,\mu K$.

Laser Isotope Separation

Isotopes of the same atom are difficult to separate because of their nearly identical chemical properties. For example transitions at frequencies of $\sim 10^{14}$ Hz may differ by less than 1 part in 10^6. Laser such as cavity stabilized ring dye lasers have a linewidth less than the isotope shift. A specified isotope can therefore be excited and even photoionized. The resulting isotopic ions can then be deflected out of the neutral beam as shown below.

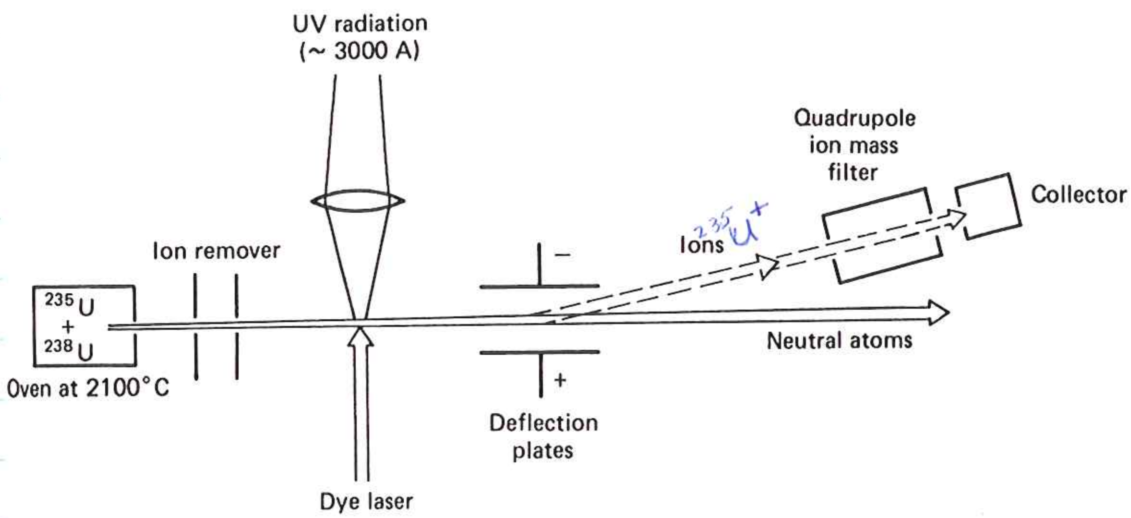


Fig. 24.2 Experimental arrangement of uranium isotope separation by the two-step photoionization method.

LIDAR

LIDAR ≡ light detection and ranging

Lidar is proving to be extremely useful to study the atmosphere. A laser pulse is sent into the sky, and backscattered light is detected. The following information is learned.

Time for scattered light to reach detector after laser pulse ⟹ height of scattering species such as aerosols, molecules, pollutants etc.

size of scattering signal ⟹ density of scatterer

wavelength dependence ⟹ identify type of scatterer

polarization dependence of scattered light ⟹ crystalline structure eg. water droplet or ice crystal

A. Carswell uses a setup similar to that below to measure ozone concentration at heights ~ 70 km!

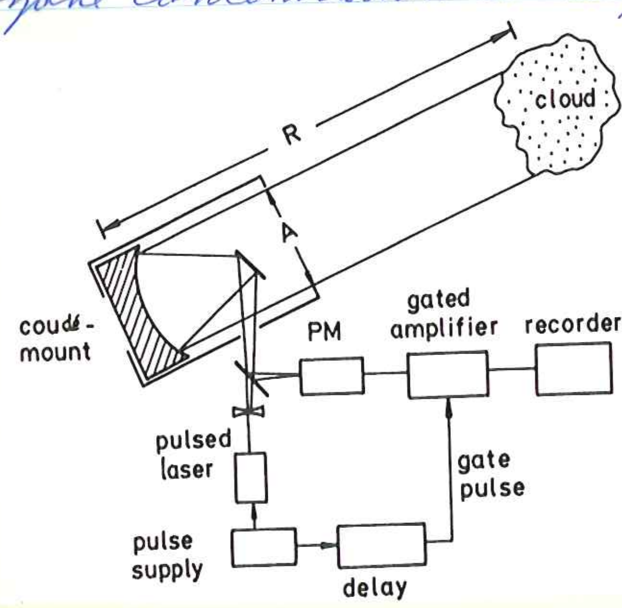


Fig.14.5. Schematic diagram of LIDAR (ligth detection and ranging)