General Relativity

Physics 5230

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Textbooks

1. Astronomy: From the Earth to the Universe by J. M. Pasachoff, Saunders College, 1991.

*2. Gravitation and Spacetime, by H.C. Ohanian, Norton Company, 1976.

3. Gravitation and Cosmology, S. Weinberg J. Wiley & Sons, 1972.

4. Gravitation, by C. W. Misner, K.S. Thorne and J.A. Wheeler, W. H. Freeman, 1973.

5. Physical Cosmology by P.J.E. Peebles, Princeton University Press, 1971.

6. Black Holes & Time Warps, by K.S. Thorne, W. W. Norton Company, 1994.

7. Scientific American Articles, eg. See Oct. 1994.

Newton's Gravitational Theory

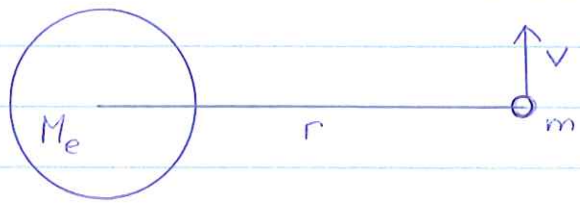
Law of Gravitation

Two masses m_1 & m_2 separated by a distance r exert an attractive force F on each other given by:

$$F = -\frac{G\, m_1 m_2}{r^2}$$

where the gravitational constant $G = 6.674 \times 10^{-8}$ dyne cm^2/gm^2. The Newtonian theory has successfully explained the workings of the Solar System and was used to predict the existence of planet Neptune.

Example: Satellite Orbiting Earth



For circular orbit at radius r we have:

Gravitational Force = Centripetal Force

$$\frac{G M_e m}{r^2} = \frac{m v^2}{r}$$

where satellite speed $v = \frac{2\pi r}{T}$, T = period.

$$\therefore \frac{G M_e}{r^2} = \frac{1}{r}\left(\frac{2\pi r}{T}\right)^2$$

$$T^2 = \frac{(2\pi)^2 r^3}{G M_e}$$

$$\boxed{T = \frac{2\pi r^{3/2}}{(G M_e)^{1/2}}}$$

For satellite near Earth's surface:

$$T = \frac{2\pi (6.4 \times 10^6 \text{ meters})^{3/2}}{(6.67 \times 10^{-11} \text{ Nt m}^2/\text{kg}^2 \times 6 \times 10^{24} \text{ kg})^{1/2}}$$

$= 5085$ secs.

$= 85$ min.

For the moon, period is:

$$T = \frac{2\pi (3.84 \times 10^8 \text{ meters})^{3/2}}{(6.67 \times 10^{-11} \text{ Nt m}^2/\text{kg}^2 \times 6 \times 10^{24} \text{ kg})^{1/2}}$$

$= 2.36 \times 10^6$ sec

$= 27.3$ days.

Potential Energy

The gravitational force $\vec{F}$ is conservative since it can be expressed as:

$$\vec{F} = -\nabla V$$

where $V = -\frac{G m_1 m_2}{r}$ is the potential energy.

Gravitational Potential

Consider a collection of point masses $m_1, m_2, \ldots$ located at positions $\vec{x}_1, \vec{x}_2, \ldots$ These masses exert a gravitational force on mass m at position $\vec{x}$ given by:

$$\vec{F}(\vec{x}) = -Gm \sum_{i=1}^{N} m_i \frac{(\vec{x} - \vec{x}_i)}{|\vec{x} - \vec{x}_i|^3}$$

The gravitational field $\vec{g}(\vec{x})$ is defined to be the force on a unit mass.

$$\therefore \vec{g}(\vec{x}) = -G \sum_{i=1}^{N} m_i \frac{(\vec{x} - \vec{x}_i)}{|\vec{x} - \vec{x}_i|^3}$$

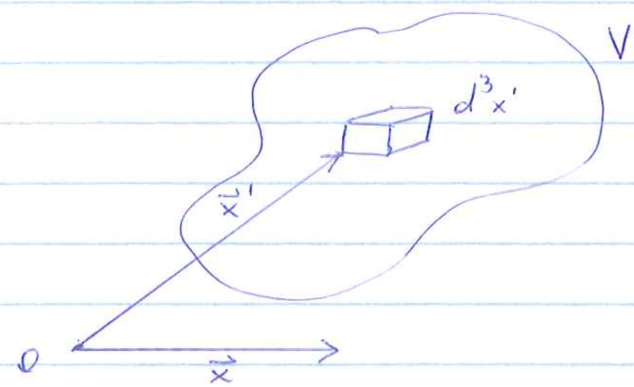
This can be expressed as:

$$\vec{g}(\vec{x}) = -\nabla \Phi(\vec{x})$$

where $\Phi(\vec{x}) = -\sum_{i=1}^{N} \frac{G m_i}{|\vec{x} - \vec{x}_i|}$ is called the Gravitational Potential.

Continuous Mass Distribution

Consider a volume V enclosing a mass distribution described by mass density $\rho(\vec{x}')$.



Consider an infinitesimal volume d^3x' located at $\vec{x}'$.
Mass inside d^3x' is $\rho(\vec{x}')\,d^3x'$

Gravitational Potential of mass $\rho(\vec{x}')\,d^3x'$ at $\vec{x}$ is:

$$d\Phi(\vec{x}) = -\frac{G\,\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$$

Gravitational Potential at $\vec{x}$ due to all mass inside V is:

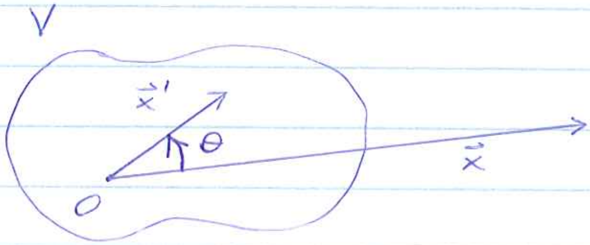
$$\Phi(\vec{x}) = -\int \frac{G\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$$

Recalling electricity & magnetism, we see that $\Phi(\vec{x})$ satisfies the Poisson equation:

$$\boxed{\nabla^2\Phi(\vec{x}) = 4\pi G\,\rho(\vec{x})}$$

Multipole Expansion of Gravitational Potential

Consider a continuous mass distribution inside volume V.



Gravitational Potential at $\vec{x}$ is:

$$\Phi(\vec{x}) = -G \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$$

Now $\dfrac{1}{|\vec{x}-\vec{x}'|} = \displaystyle\sum_{\ell=0}^{\infty} \frac{r'^{\ell}}{r^{\ell+1}} P_\ell(\cos\theta)$ where $r' \equiv |\vec{x}'|$, $r \equiv |\vec{x}|$

$\cos\theta = \dfrac{\vec{x}\cdot\vec{x}'}{r\, r'}$ and $P_\ell(\cos\theta)$ is Legendre Polynomial.

Exercise: Show $\Phi(\vec{x}) = -\dfrac{GM}{r} - \dfrac{G\,\vec{x}\cdot\vec{D}}{r^3} - \dfrac{G}{2}\displaystyle\sum_{k\ell} \frac{Q^{k\ell} x^k x^\ell}{r^5} + \dots$

where: $M \equiv \int \rho(\vec{x}')\, d^3x'$ is total mass.

$\vec{D} \equiv \int \vec{x}'\rho(\vec{x}')\, d^3x'$ is dipole moment

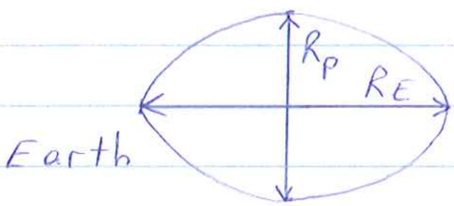
$Q^{k\ell} \equiv \int (3x'^k x'^\ell - r'^2 \delta_{k\ell})\rho(\vec{x}')\, d^3x'$ is Quadrupole Moment Tensor

For simplicity, the origin is normally chosen to be at the center of mass $\Rightarrow \vec{D} = 0$. Therefore distant observers feel only the monopole term. As they get closer to the mass distribution, the Quadrupole term becomes significant.

Exercise: Show $Q^{kl} = 0$ for a spherically symmetric mass distribution.

Example: Earth

The Earth's polar and equatorial diameters differ slightly giving rise to a quadrupole moment.



$R_E = 6.38 \times 10^6$ meters

$R_P = R_E - 2.15 \times 10^4$ meters

the quadrupole term affects the orbits of satellites.

Equivalence of Inertial & Gravitational Mass

The gravitational force closely resembles the Coulomb force.

$$F_{Coul} = \frac{k\, q_1 q_2}{r^2} \qquad F_{Grav} = \frac{G\, m_1 m_2}{r^2}$$

Hence, m can be thought of as a gravitational charge or gravitational mass m_g. The force experienced by a particle having gravitational mass m_g interacting with potential Φ is:

$$\vec{F} = -m_g \nabla \Phi$$

From Newton's Law $\vec{F} = m_I \vec{a}$ where m_I is the particle's inertial mass.

$$\therefore\ m_I \vec{a} = -m_g \nabla \Phi$$

$$\vec{a} = -\frac{m_g}{m_I} \nabla \Phi$$

Galilean Principle of Equivalence: Galileo observed that all objects fall to Earth (neglecting air effects) with the same acceleration. $\therefore m_I = m_g$.

Indeed, all experiments to date confirm this result.

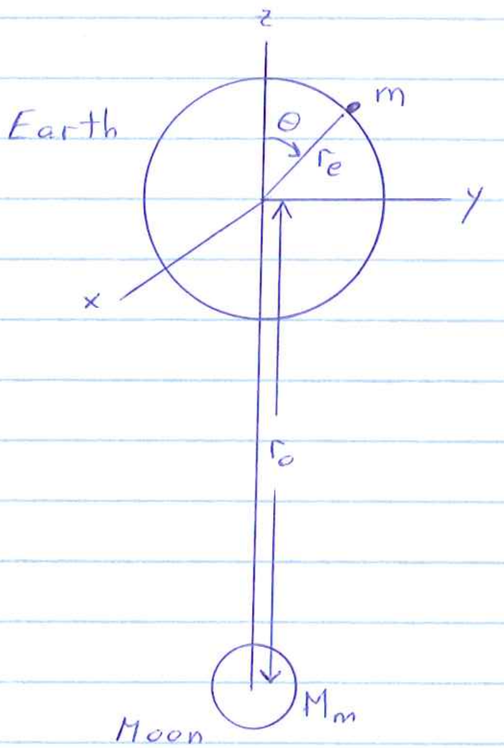
TABLE 1.3 EQUALITY OF m_I AND m_G

Experimenter(s)	**Year**	**Method**	$\|m_I - m_G\|/m_I$
Galileo[13]	~1610	pendulum	$<2 \times 10^{-3}$
Newton[14]	~1680	pendulum	$<10^{-3}$
Bessel[15]	1827	pendulum	$<2 \times 10^{-5}$
Eötvös[16]	1890	torsion-balance	$<5 \times 10^{-8}$
Eötvös et al.[17]	1905	torsion-balance	$<3 \times 10^{-9}$
Southerns[18]	1910	pendulum	$<5 \times 10^{-6}$
Zeeman[19]	1917	torsion-balance	$<3 \times 10^{-8}$
Potter[20]	1923	pendulum	$<3 \times 10^{-6}$
Renner[21]	1935	torsion-balance	$<2 \times 10^{-10}$
Dicke et al.[22]	1964	torsion-balance; sun	$<3 \times 10^{-11}$
Braginsky et al.[23]	1971	torsion-balance; sun	$<9 \times 10^{-13}$

* Pronounced ut-vush (u as in turn).

Tidal Force

Ocean tides are caused by the gravitational force exerted by the moon. Consider an object of mass m on the Earth's surface.



Gravitational Potential of mass m interacting with moon is:

$$V = -\frac{G M_m m}{\left(r_o^2 + r_e^2 - 2 r_o r_e \cos\theta\right)^{1/2}}$$

$$= \underbrace{-\frac{G M_m m}{r_o}}_{\text{Pot. form at Earth's center}} \underbrace{- \frac{G M_m m r_e \cos\theta}{r_o^2}}_{\text{Pot. for m on surface of circular Earth}} \underbrace{- \frac{G M_m m r_e^2}{r_o^3} P_2(\cos\theta)}_{\text{Pot. responsible for tides (i.e. non-circular Earth deformation)}} + \ldots$$

Tidal height h is found by considering the potential.

$$V(h,\theta) = \underbrace{mgh}_{\text{interaction of m with Earth}} - \underbrace{\frac{GM_m m\, r_e^2}{r_0^3} P_2(\cos\theta)}_{\text{tidal potential due to moon.}}$$

We now find the equipotential surface $V(h,\theta)=\text{const.}$ For simplicity we set constant to zero.

Exercise: Why may we set constant to zero?

$$\therefore\ 0 = mgh - \frac{GM_m m\, r_e^2}{r_0^3} P_2(\cos\theta)$$

$$\boxed{h(\theta) = \frac{GM_m}{g r_0^3} r_e^2 P_2(\cos\theta)}$$

Max. tidal range $\Delta h = h(\theta=0) - h(\theta=\pi/2)$

$$= \frac{3}{2}\frac{GM_m r_e^2}{2 g r_0^3}$$

$$= \frac{3}{2}\frac{(6.67\times10^{-11})(7.4\times10^{21})(6.4\times10^{6})^2}{2\times 9.8\,(3.84\times10^{8})^3}$$

$= 0.53$ meters

Exercise: Find tidal range on Earth due to sun.

Actual tides can be much larger due to constructive interference of sun + moon tides and resonance effects in bays. eg. Δh(Bay of Fundy) ~ 16 meters

Special Relativity

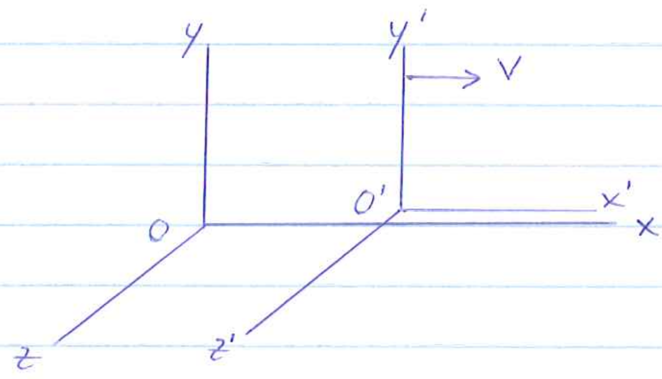
Maxwell Equations

$$\nabla \cdot \vec{E} = 4\pi \rho \qquad\qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad\qquad \nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

In order for these equations to hold for all observers, the different spatial & time coordinates of two observers are related by the Lorentz transformation.

Lorentz Transformation



$$x' = (x - vt)\gamma \qquad\qquad t' = (t - vx/c^2)\gamma$$

$$y' = y$$

$$z' = z \qquad\qquad \gamma \equiv (1 - v^2/c^2)^{-1/2}$$

We define the so called space-time interval

$$ds^2 \equiv c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Exercise: Show $ds^2 = ds'^2$

Contravariant Vector

$$x^\mu \equiv (x^0, x^1, x^2, x^3)$$
$$\equiv (ct, x, y, z)$$

Exercise: Show Lorentz transformation can be expressed as:

$$x'^\mu = a^\mu{}_\nu x^\nu$$

$$\text{where } a^\mu{}_\nu \equiv \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Metric Tensor

$$\eta_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Covariant Vector

$$x_\mu \equiv \eta_{\mu\nu} x^\nu$$

Exercise: Show $x_0 = ct$, $x_1 = -x$, $x_2 = -y$, $x_3 = -z$

Exercise: Show $ds^2 = \sum_{\mu=0}^{3} dx^\mu dx_\mu$

Einstein introduced the convention that repeated Greek indices are automatically summed over. i.e. $ds^2 = dx^\mu dx_\mu$.

Exercise: Beginning with $ds^2 = ds'^2$, show that

$$a^{\alpha}{}_{\mu}\, \eta_{\alpha\beta}\, a^{\beta}{}_{\nu} = \eta_{\mu\nu}$$

Scalar

This is a one component object that is invariant under a Lorentz transformation.

eg. ds^2 or $d\tau = \frac{\sqrt{ds^2}}{c}$

$$= \frac{(c^2 dt^2 - dx^2 - dy^2 - dz^2)^{1/2}}{c}$$

$$= dt \sqrt{1 - \frac{v^2}{c^2}}$$

Note that in a particle's rest frame, $v = 0$ and $d\tau = dt$.
$\therefore$ $d\tau$ is called the proper time interval or the time interval measured by a clock that moves along with the particle.

Generalized Definition of Contravariant Vector

This is a 4 component object A^μ that transforms under a Lorentz transformation as:

$$A'^\mu = a^\mu{}_\nu A^\nu$$

Examples.

1) $u^\mu \equiv \dfrac{dx^\mu}{d\tau}$ 4 velocity

$$= \frac{d}{d\tau}(ct, x, y, z)$$

$$= \left(c\frac{dt}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)\frac{dt}{d\tau}$$

$$= (c, v_x, v_y, v_z)\gamma$$

2) $p^\mu = m_0 u^\mu$ energy momentum 4 vector
m_0 = particle rest mass

$$= (m_0 c, m_0 v_x, m_0 v_y, m_0 v_z)\gamma$$

$$= \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

Lorentz Tensor

A Lorentz tensor of rank r is an object having 4^r components which transforms under a Lorentz transformation as:

$$A'^{\alpha\beta\ldots\gamma} = a^\alpha{}_\mu a^\beta{}_\nu \ldots a^\gamma{}_\chi A^{\mu\nu\ldots\chi}$$

eg. Any object constructed by taking the product of two vectors is a tensor such as: $x^\mu x^\nu$, $x^\mu p^\nu - x^\nu p^\mu$, $m_0 u^\mu u^\nu$.

Generalized Definition of Covariant Vector

$$A_\nu = \eta_{\nu\mu} A^\mu$$

The metric tensor can be used to "lower" or "raise" an index.

e.g. $A_\alpha{}^\beta{}_\delta = \eta_{\alpha\mu} \eta_{\delta\nu} A^{\mu\beta\nu}$

$$A^\mu = \eta^{\mu\nu} A_\nu \quad \text{where} \quad \eta^{\mu\nu} \equiv \eta_{\mu\nu}$$

Trace of Tensor

$$A \equiv Tr(A^{\mu\nu})$$

$$\equiv \eta_{\mu\nu} A^{\mu\nu}$$

Note that the Trace is a Lorentz scalar.

e.g. $Tr(u^\mu u^\nu) = u^\mu u_\mu$

$$= (c, v_x, v_y, v_z)\gamma \cdot (c, -v_x, -v_y, -v_z)\gamma$$
$$= (c^2 - \vec{v}^2)\gamma^2$$
$$= c^2$$

Exercise: Show $Trace(p^\mu p^\nu) = m_0^2 c^2$

Gradient Operator

$$\partial^\mu \equiv \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$$
$$= \left(\frac{\partial}{\partial x^0}, -\frac{\partial}{\partial x^1}, -\frac{\partial}{\partial x^2}, -\frac{\partial}{\partial x^3} \right)$$

Exercise: Show ∂^μ transforms as a contravariant vector.

$$\partial_\mu \equiv \eta_{\mu\nu} \partial^\nu$$
$$= \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right)$$

Exercise: Show $\partial^\mu \partial_\mu = \partial'^\mu \partial'_\mu$ i.e. is a Lorentz scalar

Energy Momentum Tensor

$T^{\mu\nu} \equiv n_0 m_0 u^\mu u^\nu$

m_0 = particle rest mass
n_0 = " number density in reference frame that moves with particle

A moving observer measures: particle mass $m = m_0 \gamma$
" " density $n = n_0 \gamma$

$T^{\mu\nu}$ is said to be a symmetric tensor since $T^{\mu\nu} = T^{\nu\mu}$.

Significance of Components

$$T^{00} = n_0 m_0 u^0 u^0$$
$$= n_0 m_0 c^2 \gamma^2$$
$$= n m c^2$$

$\therefore T^{00}$ is energy density.

$$T^{k0} = T^{0k} = n_0 m_0 u^0 u^k$$
$$= n_0 m_0 c \gamma v^k \gamma$$
$$= n m v^k c$$
$$= n p^k c$$

$\therefore T^{k0} \propto$ momentum density

$$T^{kl} = T^{lk} = n_0 m_0 v^l \gamma v^k \gamma$$
$$= n m v^k v^l$$
$$= n p^k v^l$$

$\therefore T^{kl} \propto$ flux density of momentum p^k emitted in l direction

Exercise: Show that $T^{\mu\nu}$ satisfies the conservation law

$$\partial_\mu T^{\mu\nu} = 0.$$

Relativistic Electrodynamics

Exercise: Show Maxwell equations imply:

$$\vec{B} = \nabla \times \vec{A}$$
$$\vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

Current 4 Vector

$J^{\mu} \equiv (c\rho, J_x, J_y, J_z)$

Exercise: Show the continuity equation $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

can be expressed as $\partial_{\mu} J^{\mu} = 0$.

Vector Potential 4 Vector

$A^{\mu} \equiv (\Phi, A_x, A_y, A_z)$

Exercise: Show the Lorentz gauge $\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0$

can be expressed as $\partial_{\mu} A^{\mu} = 0$.

Electromagnetic Field Tensor

$F^{\mu\nu} \equiv \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$

$F^{\mu\nu}$ is said to be antisymmetric since $F^{\mu\nu} = -F^{\nu\mu}$.

Exercise: Show $F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$

Exercise: Show Maxwell's equations are equivalent to:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

$$\partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0$$

Energy Momentum Tensor For Electromagnetic Field

$$T^{\mu\nu} = \frac{-1}{4\pi}\left\{ F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4}\eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right\}$$

Exercise: Derive the following.

a) $T^{00} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$

b) $T^{0k} = \frac{1}{4\pi}(\vec{E} \times \vec{B})^k$

c) $Tr(T^{\mu\nu}) = 0.$

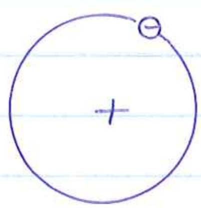
Equivalence Principle

All of nature observed so far is described by the following four forces:

1) Electromagnetic
2) Strong Nuclear
3) Weak Nuclear
4) Gravitational

Electromagnetic

Consider a hydrogen atom.



Coulomb Binding Energy of Ground State = 13.6 eV.

Do the gravitational and inertial masses of the H atom differ by $\Delta m = \frac{13.6\,\text{eV}}{c^2}$?

$$\frac{\Delta m}{m_H} \sim \frac{(13.6\,\text{eV})(1.6\times10^{-12}\,\text{erg}/\text{eV})/c^2}{(1.67\times10^{-24}\,\text{gm})}.$$

$$\sim 10^{-8}$$

Experiments testing Galileo's Equivalence Principle have used objects made of platinum, aluminum, wood etc. and found $\Delta m/m < 10^{-12}$.

These objects have very different electromagnetic energies and we therefore conclude that the equivalence principle holds in the presence of electromagnetic forces.

Strong Nuclear Force

Exercise: Considering the mass difference between an alpha particle (He nucleus) and 2 protons + 2 neutrons show that:

$$\frac{\Delta m}{m} \sim 10^{-2}$$

$\therefore$ equivalence principle holds in the presence of strong nuclear forces.

Weak Nuclear Force

Exercise: Considering the beta decay of a nucleus having molecular weight of about 100, show that:

$$\frac{\Delta m}{m} \sim 10^{-5}$$

$\therefore$ equivalence principle holds in the presence of weak nuclear forces.

Gravity

The gravitational self-energy of a mass m having a size r is $\sim \frac{Gm^2}{r}$.

$$\therefore \frac{\Delta m}{m} \sim \frac{Gm}{rc^2}.$$

For a hydrogen atom $\frac{\Delta m}{m} \sim \frac{6.67\times10^{-11} \times 1.67\times10^{-27}}{1\times10^{-10}\ (3\times10^{8})^2}$

$$\sim 10^{-44}$$

Experiments are nowhere near this accurate.

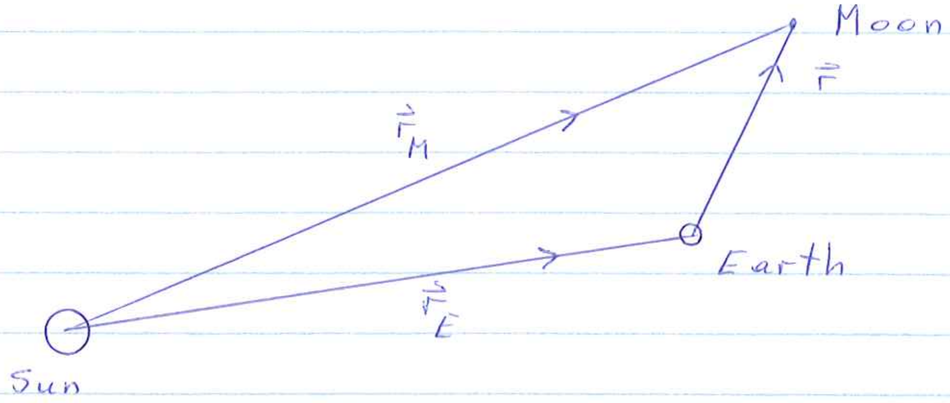
For Earth $\frac{\Delta m}{m} \sim \frac{6.67\times10^{-11} \times 6\times10^{24}}{6.4\times10^{+6}\ (3\times10^{8})^2}$

$$\sim 7\times10^{-10}$$

Exercise: Estimate $\frac{\Delta m}{m}$ for the moon.

Galileo's experiment cannot be done for a heavy mass such as the Earth. We can however consider the pull of the sun on the moon and the Earth. If $\eta \equiv \Delta m/m \neq 0$ for gravity, then the sun may pull more strongly on the Earth by a factor of $1+\eta$ than on the moon.

Interaction of Earth, Moon & Sun



Potential Energy is:

$$V(r) = -\frac{G m_E m_M}{|\vec{r}|} - G m_S \left(\frac{m_E}{|\vec{r}_E|} + \frac{m_M}{|\vec{r}_M|} \right)$$

Assuming sun pulls harder on Earth than on Moon by factor $1+\eta$ we get:

$$V(r) = -\frac{G m_E m_M}{r} - G m_S \left(\frac{m_E}{r_E}(1+\eta) + \frac{m_M}{r_M} \right)$$

Define $\vec{R}$ to be center of mass of Earth-Moon system.
i.e. $(m_M + m_E)\vec{R} = m_M \vec{r}_M + m_E \vec{r}_E$.

Exercise: Show that $\vec{r}_E = \vec{R} - \frac{m_M}{m_M + m_E}\vec{r}$

$$\vec{r}_M = \vec{R} + \frac{m_E}{m_M + m_E}\vec{r}$$

Exercise: Expand r_E^{-1} + r_M^{-1} in a power series of R^{-1} where $\hat{n} \equiv \frac{\vec{R}}{R}$ to get:

$$\frac{1}{r_E} \simeq \frac{1}{R}\left\{1 + \frac{m_M}{m_M+m_E}\frac{\vec{r}\cdot\hat{n}}{R} + \left(\frac{m_M}{m_M+m_E}\right)^2 \frac{\left(-r^2+(\vec{r}\cdot\hat{n})^2\right)}{2R^2}\right\}$$

$$\frac{1}{r_M} \simeq \frac{1}{R}\left\{1 - \frac{m_E}{m_M+m_E}\frac{\vec{r}\cdot\hat{n}}{R} + \left(\frac{m_E}{m_M+m_E}\right)^2 \frac{\left(-r^2+(\vec{r}\cdot\hat{n})^2\right)}{2R^2}\right\}$$

$$\begin{aligned}
\therefore V(r) = &-\frac{G m_E m_M}{r} \\
&-\frac{G m_S m_E (1+\eta)}{R}\left\{1 + \frac{m_M}{m_M+m_E}\frac{\vec{r}\cdot\hat{n}}{R} + \left(\frac{m_M}{m_M+m_E}\right)^2 \frac{\left(-r^2+(\vec{r}\cdot\hat{n})^2\right)}{2R^2}\right\} \\
&-\frac{G m_S m_M}{R}\left\{1 - \frac{m_E}{m_M+m_E}\frac{\vec{r}\cdot\hat{n}}{R} + \left(\frac{m_E}{m_M+m_E}\right)^2 \frac{\left(-r^2+(\vec{r}\cdot\hat{n})^2\right)}{2R^2}\right\} \\
= &-\frac{G m_E m_M}{r} - \frac{G m_S}{R}\left[(1+\eta)m_E + m_M\right] \\
&-\frac{G m_S m}{R^2}\,\vec{r}\cdot\hat{n}\,\eta \\
&-\frac{G m_S m}{2R^3}\left[-r^2 + (\vec{r}\cdot\hat{n})^2\right]
\end{aligned}$$

where $\frac{1}{m} = \frac{1}{m_M} + \frac{1}{m_E}$ and m is called the reduced mass.

Tides in lunar orbit are caused by the last two terms. Note that $\vec{r} \cdot \hat{n}$ is asymmetric under $\vec{r} \to -\vec{r}$ while $\left[-r^2 + (\vec{r} \cdot \hat{n})^2\right]$ remains unchanged.

Effect of $-\dfrac{G m_S m}{R^2} \vec{r} \cdot \hat{n}\, \eta$

Effect of $-\dfrac{G m_S m}{2R^3} \left[-r^2 + (\vec{r} \cdot \hat{n})^2\right]$

The ratio of the tidal effect due to η to the Earth - Moon interaction is:

$$\frac{\dfrac{G m_S m}{R^2}\, r\, \eta}{\dfrac{G m_E m_M}{r}} \simeq \eta \frac{m_S m}{m_E m_M} \left(\frac{r}{R}\right)^2$$

$$\simeq \eta \frac{m_S}{m_E} \left(\frac{r}{R}\right)^2 \quad \text{using } m \simeq m_M$$

$$\simeq 7 \times 10^{-10} \frac{2 \times 10^{30}\,\text{kg.}}{6 \times 10^{24}\,\text{kg.}} \left(\frac{1}{400}\right)^2$$

$$\simeq 1.5 \times 10^{-9}$$

Hence the η term shifts the moon's position by:

$$\Delta r \sim \eta r$$

$$\simeq 1.5 \times 10^{-9} \times 4 \times 10^{5}\,\text{km}$$

$$\simeq 60\,\text{cm}.$$

A resonance effect actually amplifies this to about 10 meters. (PRL <u>36</u> (1976) p. 551-555)
The distance to the moon has been measured to 5 cm accuracy and the tidal effect due to η has not been observed.

$\therefore$ the equivalence principle holds in the presence of gravitational forces.

An Inertial Observer

An inertial observer is not affected by a gravitational field and is in a nonaccelerated reference frame. One is such an observer if the laws of special relativity hold.

Strong Equivalence Principle (Uniform Field)

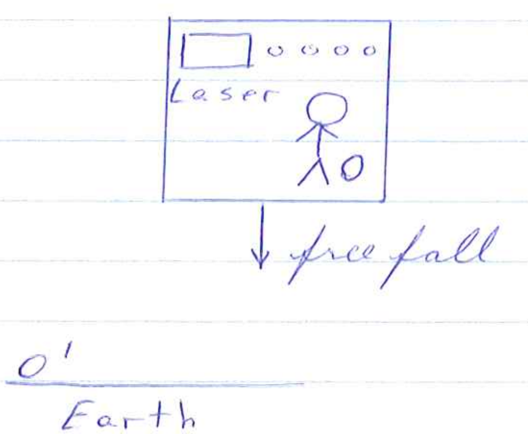
For an observer in a uniform gravitational field, all laws of nature (i.e. all known interactions) are the same as for an inertial observer.

Corollary

An observer in a closed elevator cannot determine that he/she is in free fall in a uniform gravitational field.

Implication

Gravity bends light. To see this consider observer O with a laser in an elevator falling in a uniform gravitational field.



O sees light go straight while O' who feels gravity sees it bend.

Nonuniform Gravitational Field

In reality, an observer in an elevator experiences a so called tidal force. i.e. Earth pulls somewhat more on his toes than on his head. However, one can always define a so called locally inertial coordinate system in the limit as observer shrinks to 0.

Strong Equivalence Principle

At every space-time point in an arbitrary gravitational field it is possible to choose a locally inertial coordinate system such that in a small region about the point in question, the laws of nature are the same as in an unaccelerated coordinate system in the absence of gravitation.

Gravitational Forces

Consider a massive point particle moving under the influence of only a gravitational field. The equivalence principle states the particle is not accelerated as observed in a locally inertial coordinate system ξ^α.

$$\therefore \quad \frac{d^2 \xi^\alpha}{d\tau^2} = 0 \qquad (1)$$

where $d\tau$ is the proper time defined by

$$d\tau^2 = \frac{\eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta}{c^2}$$

Next, we consider coordinate system x^μ, i.e $\xi^\alpha = \xi^\alpha(x^\mu)$

$$(1) \Longrightarrow \quad 0 = \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{dx^\mu}{d\tau} \right)$$

$$= \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{d^2 x^\mu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

Multiplying by $\dfrac{\partial x^\lambda}{\partial \xi^\alpha}$ and using $\dfrac{\partial \xi^\alpha}{\partial x^\mu} \dfrac{\partial x^\lambda}{\partial \xi^\alpha} = \delta^\lambda_\mu$ we get:

$$\boxed{0 = \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}$$

where $\Gamma^\lambda_{\mu\nu} \equiv \dfrac{\partial x^\lambda}{\partial \xi^\alpha} \dfrac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$ is the Affine Connection.

The proper time interval squared becomes:

$$d\tau^2 = \frac{\eta_{\alpha\beta}}{c^2} \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu \frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu$$

$$= \frac{g_{\mu\nu}\, dx^\mu dx^\nu}{c^2}$$

where $g_{\mu\nu} \equiv \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}$ is called the metric tensor.

Relation Between $g_{\mu\nu}$ & $\Gamma^\lambda_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \qquad (1)$$

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial x^\mu}{\partial \xi^\sigma} \frac{\partial^2 \xi^\sigma}{\partial x^\alpha \partial x^\beta} \qquad (2)$$

Multiplying (2) by $\frac{\partial \xi^\lambda}{\partial x^\mu}$ gives:

$$\frac{\partial^2 \xi^\lambda}{\partial x^\alpha \partial x^\beta} = \frac{\partial \xi^\lambda}{\partial x^\mu} \Gamma^\mu_{\alpha\beta} \qquad (3)$$

Differentiating (1) with respect to x^σ gives:

$$\frac{\partial g_{\mu\nu}}{\partial x^\sigma} = \eta_{\alpha\beta} \frac{\partial^2 \xi^\alpha}{\partial x^\sigma \partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} + \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 \xi^\beta}{\partial x^\sigma \partial x^\nu}$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\sigma} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\rho} \Gamma^\rho_{\sigma\mu} \frac{\partial \xi^\beta}{\partial x^\nu} + \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\rho} \Gamma^\rho_{\sigma\nu}$$

using (3)

$$\frac{\partial g_{\mu\nu}}{\partial x^\sigma} = g_{\rho\nu} \Gamma^\rho_{\sigma\mu} + g_{\rho\mu} \Gamma^\rho_{\sigma\nu} \quad (4)$$

Permuting μ, ν & σ gives:

$$\frac{\partial g_{\nu\sigma}}{\partial x^\mu} = g_{\rho\sigma} \Gamma^\rho_{\mu\nu} + g_{\rho\nu} \Gamma^\rho_{\mu\sigma} \quad (5)$$

$$\frac{\partial g_{\sigma\mu}}{\partial x^\nu} = g_{\rho\mu} \Gamma^\rho_{\nu\sigma} + g_{\rho\sigma} \Gamma^\rho_{\nu\mu} \quad (6)$$

Next, consider (4) + (5) − (6).

$$\frac{\partial g_{\mu\nu}}{\partial x^\sigma} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\sigma\mu}}{\partial x^\nu} = g_{\rho\nu} \Gamma^\rho_{\sigma\mu} + g_{\rho\mu} \Gamma^\rho_{\sigma\nu}$$
$$+ g_{\rho\sigma} \Gamma^\rho_{\mu\nu} + g_{\rho\nu} \Gamma^\rho_{\mu\sigma}$$
$$- g_{\rho\mu} \Gamma^\rho_{\nu\sigma} - g_{\rho\sigma} \Gamma^\rho_{\nu\mu}$$

Using $g_{\mu\nu} = g_{\nu\mu}$ and $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, this simplifies to:

$$2 g_{\rho\nu} \Gamma^\rho_{\sigma\mu} = \frac{\partial g_{\mu\nu}}{\partial x^\sigma} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\sigma\mu}}{\partial x^\nu}$$

Defining $g^{\nu\sigma}$ as inverse of $g_{\nu\sigma}$ i.e. $g^{\nu\sigma} g_{\kappa\nu} = \delta^\sigma_\kappa$

we get:

$$\boxed{\Gamma^{\chi}_{\sigma\mu} = \frac{1}{2} g^{\nu\chi} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} \right\}}$$

Newtonian limit

Consider a particle moving slowly in a weak stationary gravitational field.

Eqn. of motion is:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\sigma\nu} \frac{dx^{\sigma}}{d\tau} \frac{dx^{\nu}}{d\tau} = 0.$$

By slow, we mean $\left|\frac{d\vec{x}}{d\tau}\right| << c \frac{dt}{d\tau}$.

$$\therefore \frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{00} c^2 \left(\frac{dt}{d\tau}\right)^2 = 0$$

Affine Connection $\Gamma^{\mu}_{00} = \frac{1}{2} g^{\nu\chi} \left\{ \frac{\partial g_{0\nu}}{\partial x^{0}} + \frac{\partial g_{\nu 0}}{\partial x^{0}} - \frac{\partial g_{00}}{\partial x^{\nu}} \right\}$

$= -\frac{1}{2} g^{\nu\mu} \frac{\partial g_{00}}{\partial x^{\nu}}$ for a stationary (time independent) gravitational field

For a weak gravitational field:

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$$

where $|h_{\mu\nu}| << 1$. Hence, to first order in $h_{\mu\nu}$ we get:

$$\Gamma^{\mu}_{00} \simeq -\frac{1}{2}\eta^{\mu\nu}\frac{\partial h_{00}}{\partial x^{\nu}}.$$

Eqn. of motion becomes:

$$\frac{d^2x^{\mu}}{d\tau^2} - \frac{c^2}{2}\eta^{\mu\nu}\frac{\partial h_{00}}{\partial x^{\nu}}\left(\frac{dt}{d\tau}\right)^2 = 0$$

$\mu = 0$: $$c\frac{d^2t}{d\tau^2} - \frac{c^2}{2}\eta^{0\nu}\frac{\partial h_{00}}{\partial x^{\nu}}\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\frac{d^2t}{d\tau^2} - \frac{c}{2}\underbrace{\frac{\partial h_{00}}{\partial x^{0}}}_{=0 \text{ for stationary field}}\left(\frac{dt}{d\tau}\right)^2 = 0.$$

$$\therefore \frac{d^2t}{d\tau^2} = 0$$

$$\frac{dt}{d\tau} = K \text{ - constant}$$

$\therefore t = \tau$ using $K = 1$ since $\tau \to t$ in limit of low speeds.

$\mu = i$ $$\frac{d^2 x^i}{d\tau^2} - \frac{c^2}{2} \eta^{i\nu} \frac{\partial h_{00}}{\partial x^\nu} \left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\frac{d^2 x^i}{dt^2} + \frac{c^2}{2} \frac{\partial h_{00}}{\partial x^i} = 0.$$

$$\frac{d^2 \vec{x}}{dt^2} = -\frac{c^2}{2} \nabla h_{00}$$

The Newtonian result is $\frac{d^2 \vec{x}}{dt^2} = -\nabla \Phi$ where $\Phi = -\frac{GM}{r}$ is the gravitational potential at a distance r from the center of a mass M.

$$\therefore h_{00} = \frac{2\Phi}{c^2} + \text{constant}.$$

In the limit as $r \to \infty$, $g_{\mu\nu} \to \eta_{\mu\nu}$ which implies integration constant vanishes.

$$\therefore h_{00} = \frac{2\Phi}{c^2}$$

$$\therefore \boxed{g_{00} = 1 + \frac{2\Phi}{c^2}}$$

Time Dilation

Consider a clock in a gravitational field. The equivalence principle tells us the clock is unaffected by the gravitational field if it is observed in a locally inertial coordinate system ξ^α. Observed time between ticks is:

$$\Delta t = \left(\eta_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta\right)^{1/2}$$

$$= \left(g_{\mu\nu}\, dx^\mu\, dx^\nu\right)^{1/2}$$

In an arbitrary coordinate system, time between ticks is dt.

$$\frac{\Delta t}{dt} = \left(g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}\right)^{1/2}$$

For a clock at rest $\dfrac{dx^{i}}{d\tau} = 0$.

$$\frac{\Delta t}{dt} = \left(g_{00}\right)^{1/2}$$

$$\Delta t = dt\left(g_{00}\right)^{1/2}$$

Consider a clock observed at two different points in a gravitational field:

$$\Delta t = dt_1\left(g_{00}(1)\right)^{1/2} = dt_2\left(g_{00}(2)\right)^{1/2}$$

In practice, one measures frequencies $\nu_2 = dt_2^{-1}$, $\nu_1 = dt_1^{-1}$.

$$\therefore \quad \frac{\nu_2}{\nu_1} = \left(\frac{g_{00}(2)}{g_{00}(1)}\right)^{1/2}$$

In the weak field limit $g_{00} = 1 + \frac{2\Phi}{c^2}$.

Exercise: Defining $\Delta\nu \equiv \nu_2 - \nu_1$ show that

$$\boxed{\frac{\Delta\nu}{\nu} = \frac{\Phi(2) - \Phi(1)}{c^2}}$$

For the sun

$$\frac{\Phi}{c^2} = \frac{-GM_s}{R_s c^2}$$

$$= -\frac{6.67\times10^{-11} \times 2\times10^{30}\,\text{kg}}{7\times10^{8}\,\text{m} \times 9\times10^{16}}$$

$$= -2\times10^{-6}$$

Exercise: Evaluate $\frac{\Phi}{c^2}$ due to sun at Earth's surface.

Hence, the frequency of light emitted by an atom at the sun's surface will be observed on the Earth to be shifted to the red by 2×10^{-6} as compared to light emitted by the same atom on Earth. This so called Red Shift is much larger for white dwarfs which have a mass $\sim M_s$ and radius $\sim (0.1 - 0.01) R_s$.

TABLE 5.3 TIME DILATION EXPERIMENTS

Experimenter(s)	Year	Method	$\Delta\nu_{ex}/\Delta\nu_{th}$
Adams, Moore[12]	1925, 1928	redshift of H lines on Sirius B	0.2 to 0.5
Popper[15]	1954	redshift of H lines on 40 Eridani B	1.2 ± 0.3
Pound and Rebka[10]	1960	redshift of γ-rays on earth	1.05 ± 0.10
Brault[13]	1962	redshift of Na lines on sun	1.0 ± 0.05
Pound and Snider[11]	1964	redshift of γ-rays on earth	1.00 ± 0.01
Greenstein et al.[15]	1971	redshift of H lines on Sirius B	1.07 ± 0.2
Snider[14]	1971	redshift of K lines on sun	1.01 ± 0.06
Hafele and Keating[9]	1972	time gain of cesium-beam clocks	0.9 ± 0.2

Tests of Red Shift on Earth

1) Pound-Rebka Expt.

They measured the frequency shift of an X ray travelling 22.6 meters upwards in the Earth's gravitational field.

2) Time Dilation of Cs Atomic Clock

Two Cs clocks were put on airplanes, one travelling East, the other West. The average altitude was 30,000 ft. The clocks were then compared after the flights to a clock that had not flown. The time gains agreed with the prediction provided one subtracted the special relativistic Doppler shift.

TABLE 5.2 TIME DILATION WITH CESIUM-BEAM CLOCKS

	Time gain, westward	**Time gain, eastward**
Observed mean	$(273 \pm 7) \times 10^{-9}$ sec	$(-59 \pm 10) \times 10^{-9}$ sec
Kinematic correction	96 ± 10	-184 ± 18
Remainder (observed dilation)	177 ± 12	125 ± 21
Predicted dilation	179 ± 18	144 ± 14

Tensor Analysis

Consider the coordinate transformation $x^\mu \to x'^\mu(x^\nu)$. Note that the Lorentz transformation $x'^\mu = a^\mu{}_\nu x^\nu$ are a special subclass with $a^\mu{}_\nu$ = constant.

Scalar

This is an object ϕ which under coordinate transformation $x^\mu \to x'^\mu$ transforms into

$$\phi' = \phi$$

Contravariant Vector

This is an object V^μ which under coordinate transformation $x^\mu \to x'^\mu$ transforms into

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$$

eg. $dx'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} dx^\nu \Rightarrow dx^\mu$ is a contravariant vector.

Covariant Vector

This is an object U_μ which under coordinate transformation $x^\mu \to x'^\mu$ transforms into

$$U'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} U_\nu$$

eg. If ϕ is scalar then $\dfrac{\partial \phi}{\partial x'^\mu} = \dfrac{\partial x^\nu}{\partial x'^\mu} \dfrac{\partial \phi}{\partial x^\nu} \Rightarrow \dfrac{\partial \phi}{\partial x^\mu}$ is covariant vector

Tensor

A tensor with upper indices $\mu, \nu, \ldots$ and lower indices $\kappa, \lambda \ldots$ transforms like a product of contravariant vectors $U^\mu W^\nu \ldots$ and covariant vectors $V_\kappa Y_\lambda \ldots$.

e.g. Under coordinate transformation $x^\mu \to x'^\mu$

$$T'^{\mu}{}_{\nu}{}^{\lambda} = \frac{\partial x'^\mu}{\partial x^\kappa} \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial x'^\lambda}{\partial x^\sigma} T^{\kappa}{}_{\rho}{}^{\sigma}$$

A tensor with all indices upstairs is contravariant.
" " below " covariant.
" " up & down " mixed.

Metric Tensor

$$g'_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x'^\mu} \frac{\partial \xi^\beta}{\partial x'^\nu}$$

$$= \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial \xi^\beta}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\nu}$$

$$= g_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}$$

$\therefore g_{\mu\nu}$ is a covariant tensor.

Exercise: Show $g^{\mu\nu}$ is a contravariant tensor.

Exercise: Show $\delta^\mu{}_\nu$ is a scalar.

Tensor Algebra

1) Linear Combination

$T^{\mu}{}_{\nu} = a\,A^{\mu}{}_{\nu} + b\,B^{\mu}{}_{\nu}$ is a tensor if $A^{\mu}{}_{\nu}$ + $B^{\mu}{}_{\nu}$ are.
a, b are scalars

2) Direct Product

$T^{\mu}{}_{\nu}{}^{\rho} = A^{\mu}{}_{\nu}\,B^{\rho}$ is a tensor if $A^{\mu}{}_{\nu}$ + B^{ρ} are.

3) Contraction

$T^{\mu\rho} = T^{\mu}{}_{\nu}{}^{\rho\nu}$ is a tensor if $T^{\mu}{}_{\nu}{}^{\rho\nu}$ is.

Exercise: Prove the above.

Transformation of the Affine Connection

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}}$$

$$= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right)$$

$$= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \left\{ \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}} + \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right\}$$

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} \qquad (1)$$

An alternative expression for the second term is found by differentiating

$$\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} = \delta^{\lambda}_{\nu}$$

with respect to x'^{μ}.

$$\frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} = 0$$

$$\therefore \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} = - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}}$$

$$\therefore (1) \Rightarrow \Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} \qquad (2)$$

Covariant Differentiation

Consider the contravariant vector V^μ.

$$\text{i.e.}\quad V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$$

Differentiating with respect to x'^λ gives:

$$\frac{\partial V'^\mu}{\partial x'^\lambda} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial V^\nu}{\partial x^\rho} + \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho}\frac{\partial x^\rho}{\partial x'^\lambda} V^\nu \qquad (3)$$

If $\frac{\partial V^\mu}{\partial x^\lambda}$ were a tensor we would only have the first term on the right. $\therefore$ the derivative of a tensor isn't a tensor.
We next shall construct a tensor using the derivative. Using (2) we get:

$$\Gamma'^\mu_{\lambda\chi} V'^\chi = \left\{ \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\chi}\Gamma^\nu_{\rho\sigma} - \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\sigma}{\partial x'^\chi}\right\}\frac{\partial x'^\chi}{\partial x^\eta} V^\eta$$

$$= \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\Gamma^\nu_{\rho\sigma} V^\sigma - \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\lambda} V^\sigma \qquad (4)$$

Adding (3) + (4) gives:

$$\frac{\partial V'^\mu}{\partial x'^\lambda} + \Gamma'^\mu_{\lambda\chi} V'^\chi = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\left(\frac{\partial V^\nu}{\partial x^\rho} + \Gamma^\nu_{\rho\sigma} V^\sigma\right)$$

Hence $\frac{DV^\mu}{Dx^\lambda} \equiv \frac{\partial V^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\chi} V^\chi$ is a tensor and is called the covariant derivative.

One can also show that $\frac{DV_\mu}{Dx^\nu} \equiv \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} V_\lambda$ is a tensor

as is:

$$\boxed{\frac{DT^{\mu\sigma}{}_\lambda}{Dx^\rho} = \frac{\partial T^{\mu\sigma}{}_\lambda}{\partial x^\rho} + \Gamma^\mu_{\rho\nu} T^{\nu\sigma}{}_\lambda + \Gamma^\sigma_{\rho\nu} T^{\mu\nu}{}_\lambda - \Gamma^\kappa_{\lambda\rho} T^{\mu\sigma}{}_\kappa}$$

Covariant Differentiation of Metric Tensor

$$\frac{Dg_{\mu\nu}}{Dx^\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \Gamma^\rho_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho_{\lambda\nu} g_{\rho\mu}$$

$$= \Gamma^\rho_{\lambda\mu} g_{\rho\nu} + \Gamma^\rho_{\lambda\nu} g_{\rho\mu} - \Gamma^\rho_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho_{\lambda\nu} g_{\rho\mu}$$

$= 0$ using relation between $g_{\mu\nu}$ & $\Gamma^\rho_{\lambda\mu}$ found earlier.

This result was expected since the derivative of the metric is zero in a locally inertial coordinate system and a zero tensor is also zero in other coordinate systems.

Riemann Curvature Tensor

Consider the tensor $\dfrac{D}{Dx^\mu}\left(\dfrac{DV^\alpha}{Dx^\nu}\right) - \dfrac{D}{Dx^\nu}\left(\dfrac{DV^\alpha}{Dx^\mu}\right)$.

$$\frac{D}{Dx^\mu}\left(\frac{DV^\alpha}{Dx^\nu}\right) = \frac{\partial}{\partial x^\mu}\left(\frac{DV^\alpha}{Dx^\nu}\right) + \Gamma^\alpha_{\mu\sigma}\frac{DV^\sigma}{Dx^\nu} - \Gamma^\sigma_{\mu\nu}\frac{DV^\alpha}{Dx^\sigma}$$

$$= \frac{\partial}{\partial x^\mu}\left(\frac{\partial V^\alpha}{\partial x^\nu} + \Gamma^\alpha_{\nu\tau}V^\tau\right)$$

$$+ \Gamma^\alpha_{\mu\sigma}\left(\frac{\partial V^\sigma}{\partial x^\nu} + \Gamma^\sigma_{\nu\tau}V^\tau\right)$$

$$- \Gamma^\sigma_{\mu\nu}\frac{DV^\alpha}{Dx^\sigma}$$

$$= \underbrace{\frac{\partial^2 V^\alpha}{\partial x^\mu \partial x^\nu}}_{\text{symmetric under } \mu \leftrightarrow \nu} + \frac{\partial \Gamma^\alpha_{\nu\tau}}{\partial x^\mu}V^\tau$$

$$+ \underbrace{\Gamma^\alpha_{\nu\tau}\frac{\partial V^\tau}{\partial x^\mu} + \Gamma^\alpha_{\mu\sigma}\frac{\partial V^\sigma}{\partial x^\nu}}_{\text{symmetric under } \mu \leftrightarrow \nu} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\nu\tau}V^\tau$$

$$\underbrace{- \Gamma^\sigma_{\mu\nu}\frac{DV^\alpha}{Dx^\sigma}}_{\text{symmetric under } \mu \leftrightarrow \nu.}$$

$$\therefore \frac{D}{Dx^\mu}\left(\frac{DV^\alpha}{Dx^\nu}\right) - \frac{D}{Dx^\nu}\left(\frac{DV^\alpha}{Dx^\mu}\right) = -R_{\mu\nu}{}^\alpha{}_\tau V^\tau$$

where $R_{\mu\nu}{}^\alpha{}_\tau \equiv -\dfrac{\partial \Gamma^\alpha_{\nu\tau}}{\partial x^\mu} + \dfrac{\partial \Gamma^\alpha_{\mu\tau}}{\partial x^\nu} - \Gamma^\alpha_{\mu\sigma}\Gamma^\alpha_{\nu\tau} + \Gamma^\alpha_{\nu\sigma}\Gamma^\alpha_{\mu\tau}$

is called the Riemann Curvature Tensor. One can lower the α index using the metric tensor.

i.e. $R_{\mu\nu\lambda\tau} = g_{\lambda\alpha} R_{\mu\nu}{}^{\alpha}{}_{\tau}$.

One can show that $R_{\mu\nu\lambda\tau}$ is the only tensor that can be constructed from the metric tensor and its first and second derivatives and is linear in the 2nd derivatives. The following expression can be derived.

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left\{ \frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right\}$$

$$+ g_{\eta\sigma}\left\{ \Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu} \right\}$$

Properties of Curvature Tensor

1) Symmetry $R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu}$

2) Antisymmetry $R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = R_{\mu\lambda\kappa\nu}$

3) Cyclicity $R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0$

At first glance $R_{\lambda\mu\nu\kappa}$ has $4^4 = 256$ components. However, one can use the above properties to show that $R_{\lambda\mu\nu\kappa}$ has only 20 independent components.

Ricci Tensor $R_{\nu\beta} \equiv R_{\mu\nu\alpha\beta}\, g^{\mu\alpha}$

Ricci Scalar $R \equiv g^{\nu\beta} R_{\nu\beta}$

Bianchi Identities

In a locally inertial coordinate system, the Affine connection is zero and the Riemann Curvature tensor is given by:

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left\{ \frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right\}$$

$$\frac{D R_{\lambda\mu\nu\kappa}}{\partial x^\eta} = \frac{1}{2}\frac{\partial}{\partial x^\eta}\left\{ \frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right\}$$

By permuting ν, κ & η cyclically, we obtain:

$$\boxed{\frac{D R_{\lambda\mu\nu\kappa}}{D x^\eta} + \frac{D R_{\lambda\mu\eta\nu}}{D x^\kappa} + \frac{D R_{\lambda\mu\kappa\eta}}{D x^\nu} = 0}$$

This is called the Bianchi Identity and holds in all reference frames since it is a tensor.

Multiplying the above identity by $g^{\lambda\nu}$ & using $\frac{D g^{\lambda\nu}}{D x^\eta} = 0$, we get:

$$\frac{D R_{\mu\kappa}}{D x^\eta} - \frac{D R_{\mu\eta}}{D x^\kappa} + \frac{D R^\nu{}_{\mu\kappa\eta}}{D x^\nu} = 0.$$

Multiplying by $g^{\mu\kappa}$ gives:

$$\frac{D R}{D x^\eta} - \frac{D R^\kappa{}_\eta}{D x^\kappa} - \frac{D R^\nu{}_\eta}{D x^\nu} = 0$$

$$\frac{D}{Dx^\mu}\left(\delta^\mu{}_\eta R - 2R^\mu{}_\eta\right) = 0$$

$$\frac{D}{Dx^\mu}\left(R^\mu{}_\eta - \frac{1}{2}\delta^\mu{}_\eta R\right) = 0$$

Multiplying by $g^{\nu\eta}$ gives:

$$\boxed{\frac{D}{Dx^\mu}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu} R\right) = 0}$$

Principle of General Invariance

All laws of physics must maintain the same form under a general coordinate transformation.

e.g. $A^\mu = B^\mu \xrightarrow{x^\mu \to x'^\mu} A'^\mu = B'^\mu$

has same form

$$A^\mu = B_\mu \xrightarrow{x^\mu \to x'^\mu} \frac{\partial x^\mu}{\partial x'^\alpha} A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\mu} B_\alpha$$

does not have the same form

Einstein Field Equations

In 1916, Einstein derived an equation for the metric tensor making the following assumptions.

1) Equation is invariant under a general coordinate transformation.

2) Equation reduces to Newtonian result for weak gravitational fields and low velocities.

3) Equation is a second order differential equation and is linear in second derivatives of the metric.

Newtonian limit

The gravitational potential Φ produced by a mass distribution having mass density ρ satisfies:

$$\nabla^2 \Phi = 4\pi G \rho$$

To generalize this result, we note that $T_{00} = \rho c^2$ is the energy density component of the energy momentum tensor. Also, for weak fields we found $g_{00} = 1 + \frac{2\Phi}{c^2}$.

$$\therefore \quad \nabla^2 g_{00} = \frac{8\pi G}{c^4} T_{00}$$

We therefore search for an equation having the form

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (1)$$

Properties of $G_{\mu\nu}$

1) $G_{\mu\nu}$ is a tensor.

2) $G_{\mu\nu}$ is linear in second derivatives of $g_{\mu\nu}$.

3) $G_{\mu\nu} = G_{\nu\mu}$ since $T_{\mu\nu}$ is symmetric.

4) $\frac{DG_{\mu\nu}}{Dx^{\mu}} = 0$ since $T_{\mu\nu}$ is conserved.

5) In the Newtonian limit $G_{00} = \nabla^2 g_{00}$.

The only tensor that contains the second derivative of the metric linearly is $R_{\mu\nu\alpha\beta}$. Hence, the most general form of $G_{\mu\nu}$ is:

$$G_{\mu\nu} = -\left(a R_{\mu\nu} + b g_{\mu\nu} R + \Lambda g_{\mu\nu} \right)$$

This satisfies properties 1, 2 & 3. Setting $\frac{D G_{\mu\nu}}{D x^{\mu}} = 0$ and using $\frac{D g_{\mu\nu}}{D x^{\mu}} = 0$, we get:

$$0 = -\left(a \frac{D R_{\mu\nu}}{D x^{\mu}} + b\, g_{\mu\nu} \frac{D R}{D x^{\mu}} \right)$$

But from the Bianchi Identity $\frac{D R_{\mu\nu}}{D x^{\mu}} = \frac{1}{2} g_{\mu\nu} \frac{D R}{D x^{\mu}}$.

$$\therefore\ b = -\frac{a}{2} \text{ and } G_{\mu\nu} = -\left(a R_{\mu\nu} - \frac{a}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right)$$

Equation (1) then becomes:

$$-\left(a R_{\mu\nu} - \frac{a}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right) = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (2)$$

We now multiply by $g^{\mu\nu}$.

$$-\left(a R_{\mu\nu} g^{\mu\nu} - \frac{a}{2} g_{\mu\nu} g^{\mu\nu} R + \Lambda g_{\mu\nu} g^{\mu\nu} \right) = \frac{8\pi G}{c^4} T_{\mu\nu} g^{\mu\nu}$$

Using $g^{\mu\nu} g_{\mu\nu} = 4$ and defining $T \equiv T_{\mu\nu} g^{\mu\nu}$ the above simplifies.

$$-\left(aR - 2aR + 4\Lambda \right) = \frac{8\pi G}{c^4} T$$

$$aR - 4\Lambda = \frac{8\pi G}{c^4} T$$

$$aR = \frac{8\pi G T}{c^4} + 4\Lambda$$

Substituting this result into (2) yields:

$$-\left(aR_{\mu\nu} - \frac{g_{\mu\nu}}{2}\left[\frac{8\pi G}{c^4} T + 4\Lambda \right] + \Lambda g_{\mu\nu} \right) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$-aR_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T \right) - \Lambda g_{\mu\nu} \quad (3)$$

When $\mu = \nu = 0$, this gives:

$$-aR_{00} = \frac{8\pi G}{c^4}\left(T_{00} - \tfrac{1}{2} g_{00} T \right) - \Lambda g_{00} \quad (4)$$

We shall now find a & Λ by comparing the nonrelativistic limit of this equation to the Newtonian result.

Nonrelativistic Limit of $T_{\mu\nu}$

For low speeds $T_{00} \gg T_{ij}, T_{i0}$

For small fields $g^{\mu\nu} \simeq \eta^{\mu\nu}$

$$\begin{aligned} \therefore T &= T_{\mu\nu} g^{\mu\nu} \\ &\simeq T_{00} - T_{ii} \\ &\simeq T_{00} \\ &\simeq \rho c^2 \end{aligned}$$

Weak Field Limit

Consider $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

Affine Connection
$$\Gamma^{\alpha}_{\nu\beta} = \frac{1}{2} g^{\alpha\tau}\left\{ \frac{\partial g_{\tau\beta}}{\partial x^{\nu}} + \frac{\partial g_{\tau\nu}}{\partial x^{\beta}} - \frac{\partial g_{\nu\beta}}{\partial x^{\tau}} \right\}$$

$$\simeq \frac{1}{2}\left(\eta^{\alpha\tau} + h^{\alpha\tau}\right)\left\{ \frac{\partial h_{\tau\beta}}{\partial x^{\nu}} + \frac{\partial h_{\tau\nu}}{\partial x^{\beta}} - \frac{\partial h_{\nu\beta}}{\partial x^{\tau}} \right\}$$

$$= \frac{1}{2}\left\{ \frac{\partial h^{\alpha}{}_{\beta}}{\partial x^{\nu}} + \frac{\partial h^{\alpha}{}_{\nu}}{\partial x^{\beta}} - \eta^{\alpha\tau}\frac{\partial h_{\nu\beta}}{\partial x^{\tau}} \right\} + O(h^2)$$

Riemann Tensor
$$R_{\mu\nu}{}^{\alpha}{}_{\beta} = -\frac{\partial \Gamma^{\alpha}_{\nu\beta}}{\partial x^{\mu}} + \frac{\partial \Gamma^{\alpha}_{\mu\beta}}{\partial x^{\nu}} - \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\alpha}_{\nu\beta} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\alpha}_{\mu\beta}$$

$$\simeq -\frac{1}{2}\frac{\partial}{\partial x^{\mu}}\left\{ \frac{\partial h^{\alpha}{}_{\beta}}{\partial x^{\nu}} + \frac{\partial h^{\alpha}{}_{\nu}}{\partial x^{\beta}} - \eta^{\alpha\tau}\frac{\partial h_{\nu\beta}}{\partial x^{\tau}} \right\}$$

$$+\frac{1}{2}\frac{\partial}{\partial x^{\nu}}\left\{ \frac{\partial h^{\alpha}{}_{\beta}}{\partial x^{\mu}} + \frac{\partial h^{\alpha}{}_{\mu}}{\partial x^{\beta}} - \eta^{\alpha\tau}\frac{\partial h_{\mu\beta}}{\partial x^{\tau}} \right\}$$
$$+ O(h^2)$$

$$= \frac{1}{2}\left\{ -\frac{\partial^2 h^{\alpha}{}_{\nu}}{\partial x^{\mu}\partial x^{\beta}} + \frac{\partial^2 h^{\alpha}{}_{\mu}}{\partial x^{\nu}\partial x^{\beta}} + \eta^{\alpha\tau}\left(\frac{\partial^2 h_{\nu\beta}}{\partial x^{\mu}\partial x^{\tau}} - \frac{\partial^2 h_{\mu\beta}}{\partial x^{\nu}\partial x^{\tau}}\right)\right\}$$

Ricci Tensor
$$R_{\nu\beta} = R_{\mu\nu}{}^{\mu}{}_{\beta}$$

$$\simeq \frac{1}{2}\left\{ -\frac{\partial^2 h^{\mu}{}_{\nu}}{\partial x^{\mu}\partial x^{\beta}} + \frac{\partial^2 h^{\mu}{}_{\mu}}{\partial x^{\nu}\partial x^{\beta}} + \eta^{\mu\tau}\left(\frac{\partial^2 h_{\nu\beta}}{\partial x^{\mu}\partial x^{\tau}} - \frac{\partial^2 h_{\mu\beta}}{\partial x^{\nu}\partial x^{\tau}}\right)\right\}$$

$$\therefore R_{00} \simeq \frac{1}{2}\left\{ -\frac{\partial^2 h^\mu{}_0}{\partial x^\mu \partial x^0} + \frac{\partial^2 h^\mu{}_\mu}{\partial x^0 \partial x^0} + \eta^{\mu\tau}\left(\frac{\partial^2 h_{00}}{\partial x^\mu \partial x^\tau} - \frac{\partial^2 h_{\mu 0}}{\partial x^0 \partial x^\tau}\right)\right\}$$

For Newtonian fields, time derivatives vanish.

$$\therefore R_{00} = \frac{1}{2}\eta^{\mu\tau}\frac{\partial^2 h_{00}}{\partial x^\mu \partial x^\tau}$$

$$= \frac{1}{2}\left[\frac{\partial^2 h_{00}}{\partial x^0 \partial x^0} - \nabla^2 h_{00}\right]$$

$$= -\frac{1}{2}\nabla^2 h_{00}$$

Equation (4) then becomes:

$$-a\left(-\frac{1}{2}\nabla^2 h_{00}\right) = \frac{8\pi G}{c^4}\left(T_{00} - \frac{1}{2}T_{00}\right) - \Lambda$$

$$\frac{a}{2}\nabla^2 h_{00} = \frac{4\pi G}{c^4} T_{00} - \Lambda$$

$$a\nabla^2 h_{00} = \frac{8\pi G}{c^4} T_{00} - 2\Lambda$$

$$a\nabla^2 g_{00} = \frac{8\pi G}{c^4} T_{00} - 2\Lambda$$

Comparing with the Newtonian result gives $a = 1$
$\Lambda = 0$.

$$\therefore (2) \Longrightarrow \boxed{R_{\mu\nu} - g_{\mu\nu}\frac{R}{2} = -\frac{8\pi G T_{\mu\nu}}{c^4}}$$

This set of 10 equations are called Einsteins Equations.

Significance of Λ.

In the absence of matter $T_{\mu\nu}=0$ and Einstein's equations (3) becomes:

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

When $\mu=\nu=0$ and for weak fields we get:

$$-\frac{1}{2}\nabla^2 h_{00} = \Lambda \eta_{00}$$

Next recall that $h_{00} = \frac{2\Phi}{c^2}$.

$$\therefore \nabla^2 \Phi = -\Lambda c^2$$

This is a Poisson equation where the mass density is:

$$\rho_{eff} = \frac{\Lambda c^2}{4\pi G}$$

Hence there is a positive or negative mass density depending on the sign of Λ associated with the vacuum.

Exercise: For spherical coordinates with $\Phi(r=0)=0$ + $\frac{\partial\Phi}{\partial r}(r=0)=0$ show $\Phi = -\frac{\Lambda c^2 r^2}{6}$.

Hence, the existence of a Λ term, known as the Cosmological Constant modifies the $1/r$ Newtonian potential. From observations of the universe,

$$|\Lambda c^2| < 10^{-35} \text{ sec}^{-2}$$

Schwarzschild Solution of Einstein's Equations

We consider the gravitational field produced outside a spherically symmetric mass distribution which is at rest. This is a good approximation of the field generated by the sun. The space time interval is:

$$ds^2 = A''(R)\, c^2 dt^2 - B''(R)\, dR^2 - C(R)\, dR\, dt$$

$$- D(R)\left(d\theta^2 + \sin^2\theta\, d\phi^2 \right)$$

We shall assume time reversal symmetry i.e. $ds^2(t) = ds^2(-t)$ which implies $C(R) = 0$. Also define $r \equiv \sqrt{D(R)}$.

$$\therefore\ ds^2 = A(r)\, c^2 dt^2 - B(r)\, dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

Defining $x^0 \equiv ct$, $x^1 \equiv r$, $x^2 \equiv \theta$, $x^3 \equiv \phi$ we obtain the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

The nonzero Affine Connection elements are: $\begin{pmatrix} A' \equiv \partial A/\partial r \\ B' \equiv \partial B/\partial r \end{pmatrix}$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{A'}{2A} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$$

$$\Gamma^1_{00} = \frac{A'}{2B} \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^1_{11} = \frac{B'}{2B} \qquad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^1_{22} = -\frac{r}{B} \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$$

$$\Gamma^1_{33} = -\frac{r\sin^2\theta}{B}$$

Riemann Curvature tensor components are:

$$R^0_{101} = \frac{-A''}{2A} + \frac{A'^2}{4A^2} + \frac{A'B'}{4AB} \qquad R^0_{202} = -\frac{rA'}{2BA}$$

$$R^0_{303} = \frac{-rA'}{2BA}\sin^2\theta \qquad R^1_{212} = \frac{rB'}{2B^2}$$

$$R^1_{313} = \frac{rB'}{2B^2}\sin^2\theta \qquad R^2_{323} = \left(1 - \frac{1}{B}\right)\sin^2\theta$$

The Ricci tensor is then given by:

$$R_{00} = \frac{1}{B}\left\{\frac{-A''}{2} + \frac{A'^2}{4A} + \frac{A'B'}{4B} - \frac{A'}{r}\right\}$$

$$R_{11} = \frac{A''}{2A} - \frac{A'^2}{4A^2} - \frac{A'B'}{4AB} - \frac{B'}{rB}$$

$$R_{22} = \frac{1}{B}\left\{1 + \frac{r}{2}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right\} - 1$$

$$R_{33} = \frac{\sin^2\theta}{B}\left\{1 + \frac{r}{2}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right\} - \sin^2\theta$$

Finally, the Ricci scalar is:

$$R = \frac{1}{B}\left\{-\frac{A''}{A} + \frac{A'^2}{2A^2} + \frac{A'B'}{2AB} + \frac{2}{r}\left(\frac{B'}{B} - \frac{A'}{A}\right) - \frac{2}{r^2}\right\} + \frac{2}{r^2}$$

Solution of Einstein's Equations

Outside the star, $T_{\mu\nu} = 0$ and Einstein's equations become:

$$R_{\mu\nu} - g_{\mu\nu}\frac{R}{2} = 0$$

We then find the following:

$$R_{00} - g_{00}\frac{R}{2} = 0 \Rightarrow \quad \frac{-1}{B}\left(\frac{B'}{Br} - \frac{1}{r^2}\right) - \frac{1}{r^2} = 0$$

$$R_{11} - g_{11}\frac{R}{2} = 0 \Rightarrow \quad \frac{1}{B}\left(\frac{A'}{Ar} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0$$

$$\left.\begin{array}{l} R_{22} - g_{22}\frac{R}{2} = 0 \\ \\ R_{33} - g_{33}\frac{R}{2} = 0 \end{array}\right\} \Rightarrow \quad \frac{A''}{2A} - \frac{A'^2}{4A} - \frac{A'B'}{4AB} + \frac{1}{2r}\left(\frac{A'}{A} - \frac{B'}{B}\right) = 0$$

Exercise: Show the above equations have solutions

$$B = \frac{1}{1 - C/r} \quad + \quad A = 1 - \frac{C}{r}$$

where C is an integration constant.

Hence the space time interval is:

$$ds^2 = \left(1 - \frac{C}{r}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{C}{r}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

For weak gravitational fields, we had previously found:

$$g_{00} = 1 + \frac{2\Phi}{c^2}$$

$$= 1 - \frac{2GM}{rc^2}$$

$\therefore C = \frac{2GM}{c^2}$ and the space time interval becomes

$$\boxed{ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}$$

which is called the Schwarzschild solution.

Experimental Tests of General Relativity

Einstein suggested the following tests of General Relativity.

1) Gravitational red shift of spectral lines
2) Precession of the perihelia of the orbits of the inner planets
3) Deflection of light by the sun

A fourth test measuring the time delay of radar echoes passing the sun has also been carried out.

Equation of Motion

The equation of motion is given by

$$\frac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} = 0 \qquad (1)$$

where p is a parameter describing the trajectory. For a particle having a nonzero rest mass, p is the proper time τ. Hence we have:

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \begin{cases} 1 & \text{for a massive particle} \\ 0 & \text{for a massless particle} \end{cases} \qquad (2)$$

We shall solve equations (1) + (2) using the components of $g_{\mu\nu}$, $\Gamma^\mu_{\alpha\beta}$ found for the Schwarzschild solution.

$$ds^2 = B(r)c^2dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (3)$$

where $A(r) = \left(1 - \frac{2GM}{rc^2}\right)^{-1}$ & $B(r) = 1 - \frac{2GM}{rc^2}$.

Equations (1) then become:

$$\ddot{r} + \frac{A'}{2A}\dot{r}^2 - \frac{r}{A}\dot{\theta}^2 - \frac{r\sin^2\theta}{A}\dot{\phi}^2 + \frac{B'}{2A}c^2\dot{t}^2 = 0 \qquad (4)$$

$$\ddot{\theta} + \frac{2\dot{\theta}\dot{r}}{r} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 \qquad (5)$$

$$\ddot{\phi} + \frac{2\dot{r}\dot{\phi}}{r} + 2\cot\theta\,\dot{\phi}\dot{\theta} = 0 \qquad (6)$$

$$\ddot{t} + \frac{B'}{B}\dot{t}\dot{r} = 0 \qquad (7)$$

where dot means differentiation with respect to p.
i.e. $\dot{r} \equiv \frac{dr}{dp}$ $\dot{t} \equiv \frac{dt}{dp}$ etc.

We shall consider a particle which initially travels in the $\theta = \frac{\pi}{2}$ plane. i.e. $\dot{\theta} = 0$ Equation (5) then gives $\ddot{\theta} = 0$ showing the particle remains in this plane.

Exercise: Show that (6) can be written as:

$$\frac{d}{dp}\left\{\ln(r^2\dot{\phi})\right\} = 0$$

$$\therefore r^2\dot{\phi} = J \quad \text{a constant} \qquad (8)$$

For a massive particle, J is the angular momentum per unit mass.

Similarly, one can show that (7) can be expressed as:

$$\frac{d}{dp}\left\{\ln(\dot{t}B)\right\} = 0$$

We shall define p such that: $\dot{t}\, B = 1$ (9)

Rather than integrating (4) we use (2).

$$B\, c^2 \dot{t}^2 - A\, \dot{r}^2 - r^2 \dot{\theta}^2 - r^2 \sin^2\theta\, \dot{\phi}^2 = \begin{cases} 1 & \text{massive part.} \\ 0 & \text{rest mass} = 0 \end{cases}$$

Substituting for $\dot{\phi}$, $\dot{t}$ and $\theta = \frac{\pi}{2}$, we obtain:

$$\boxed{\frac{c^2}{B} - A\, \dot{r}^2 - \frac{J^2}{r^2} = \begin{cases} 1 & \text{massive particle} \\ 0 & \text{massless} \;'' \end{cases}} \qquad (10)$$

<u>Exercise:</u> Show $\dot{r} = \frac{dr}{d\phi} \frac{J}{r^2}$.

$$\therefore (10) \Rightarrow \boxed{\frac{c^2}{B} - A \left(\frac{dr}{d\phi}\right)^2 \frac{J^2}{r^4} - \frac{J^2}{r^2} = \begin{cases} 1 & \text{massive part.} \\ 0 & \text{massless} \;'' \end{cases}} \qquad (11)$$

The Motion of Planets: Perihelion Precession

For a massive particle such as a planet, the equation of motion (11) is:

$$\frac{c^2}{B} - A\left(\frac{dr}{d\phi}\right)^2 \frac{J^2}{r^4} - \frac{J^2}{r^2} = 1$$

Defining $u \equiv \frac{1}{r}$ and using $\frac{du}{d\phi} = \frac{-1}{r^2}\frac{dr}{d\phi}$ one gets:

$$\frac{c^2}{B} - A\left(\frac{du}{d\phi}\right)^2 J^2 - J^2u^2 = 1$$

Substituting $A^{-1} = B = 1 - \frac{2GMu}{c^2}$ gives:

$$c^2 - J^2\left(\frac{du}{d\phi}\right)^2 - J^2u^2\left(1 - \frac{2GM\,u}{c^2}\right) = 1 - \frac{2GM\,u}{c^2}$$

Differentiating this equation with respect to ϕ yields:

$$\frac{d^2u}{d\phi^2} + u - \frac{GM}{c^2J^2} - 3\frac{GM}{c^2}u^2 = 0 \qquad (12)$$

Exercise: Show Newton's corresponding equation is

$$\frac{d^2u}{d\phi^2} + u - \frac{GM}{c^2J^2} = 0 \qquad (13)$$

and has solution $u = \frac{GM}{c^2J^2}\left[1 + \epsilon\cos(\phi - \phi_0)\right]$ (14)

ϵ is the eccentricity and the perihelion is said to be located at ϕ_0.

The significance of the Einstein correction to Newton's result is given by the ratio of the fourth to second terms in (12), which is $\sim \frac{GM}{rc^2}$. For the case of Mercury

$$\frac{GM}{rc^2} = \frac{6.67 \times 10^{-11} \times 2 \times 10^{30}\,\text{kg.}}{5.5 \times 10^{10}\,\text{m.} \times (3 \times 10^{8}\,\text{m/sec})^2}$$

$$\sim 3 \times 10^{-8}$$

This is small and allows us to evaluate the $\frac{3GM}{c^2}u^2$ term in (12) by substituting in for u as given by (14). ($A \equiv J$)

$$(12) \Rightarrow \frac{d^2u}{d\phi^2} + u - \frac{GM}{A^2c^2} - \frac{3}{A^4}\left(\frac{GM}{c^2}\right)^3\left[1 + \epsilon\cos(\phi - \phi_0)\right]^2 = 0$$

For nearly circular orbits we may ignore terms of $O(\epsilon^2)$. Also $\frac{3}{A^4}\left(\frac{GM}{c^2}\right)^3 << \frac{1}{A}\frac{GM}{c^2}$.

$$\therefore \frac{d^2u}{d\phi^2} + u - \frac{GM}{A^2c^2} - \frac{6\epsilon}{A^4}\left(\frac{GM}{c^2}\right)^3\cos(\phi - \phi_0) = 0$$

This has solution:

$$u = \frac{1}{A^2}\frac{GM}{c^2}\left[1 + \epsilon\cos(\phi - \phi_0)\right] + \frac{3\epsilon}{A^4}\left(\frac{GM}{c^2}\right)^3 \phi\sin(\phi - \phi_0)$$

$$\simeq \frac{1}{A^2}\frac{GM}{c^2}\left[1 + \epsilon\cos\left(\phi - \phi_0 - \frac{3}{A^2}\left(\frac{GM}{c^2}\right)^2\phi\right)\right]$$

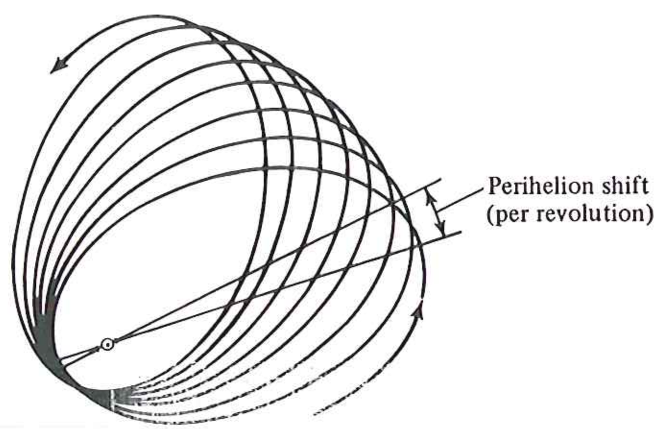
which is a precessing ellipse.

The argument of cosine changes by 2π when ϕ changes by:

$$\Delta\phi = 2\pi\left[1 - \frac{3}{A^2}\left(\frac{GM}{c^2}\right)^2\right]^{-1}$$

$$\simeq 2\pi\left[1 + \frac{3}{A^2}\left(\frac{GM}{c^2}\right)^2\right]$$

Note that $\Delta\phi > 2\pi$ meaning that each revolution, the perihelion advances by: $\frac{6\pi}{A^2}\left(\frac{GM}{c^2}\right)^2$.



For a nearly circular orbit of radius r, equation (14) gives: $\frac{1}{r} \simeq \frac{1}{A^2}\frac{GM}{c^2}$.

∴ the angular advance of the perihelion per revolution is $6\pi\frac{GM}{rc^2}$, which for Mercury is ~40" per century.

Taking into account the slight eccentricity of Mercury's orbit yields 43.03" per century.

Table 8.3. Comparison of Theoretical and Observed Centennial Precessions of Planetary Orbits.[6]

Planet	a (10^6 km)	e	$\frac{6\pi MG}{L}$	$\frac{\text{Revolutions}}{\text{Century}}$	$\Delta\varphi$ (seconds/century)	
					Gen. Rel.	Observed
Mercury (☿)	57.91	0.2056	0.1038″	415	43.03	43.11 ± 0.45
Venus (♀)	108.21	0.0068	0.058″	149	8.6	8.4 ± 4.8
Earth (⊕)	149.60	0.0167	0.038″	100	3.8	5.0 ± 1.2
Icarus	161.0	0.827	0.115″	89	10.3	9.8 ± 0.8

Deflection of Light By The Sun

For a photon, the equation of motion (11) is:

$$\frac{c^2}{B} - A\left(\frac{dr}{d\phi}\right)^2 \frac{J^2}{r^4} - \frac{J^2}{r^2} = 0$$

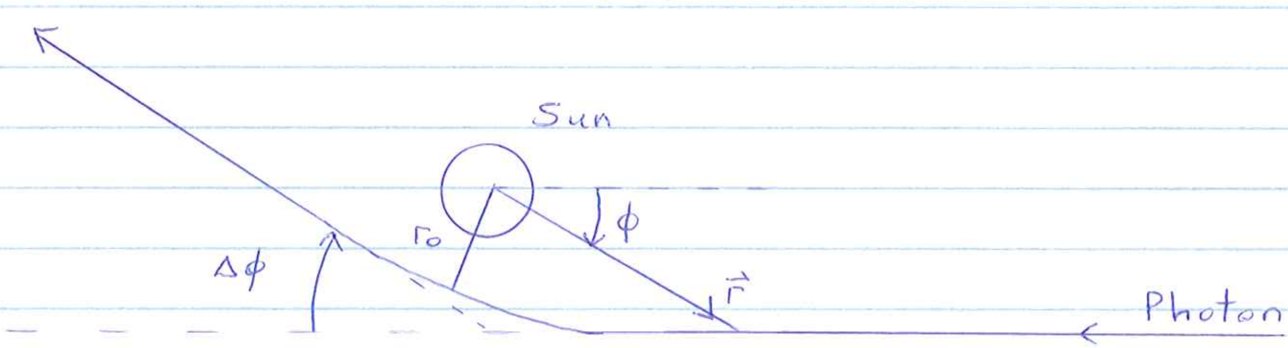
This can be rearranged to yield:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{A}\left(\frac{c^2}{J^2B} - \frac{1}{r^2}\right) \tag{15}$$

$$\frac{d\phi}{dr} = \pm \frac{A^{1/2}}{r^2\left(\frac{c^2}{J^2B} - \frac{1}{r^2}\right)^{1/2}}$$

Integrating for a light ray travelling from infinity to distance r from sun (and taking negative root) gives: an angular deviation:

$$\phi(r) - \phi(\infty) = -\int_{\infty}^{r} \frac{A^{1/2}\,dr}{r^2\left(\frac{c^2}{J^2B} - \frac{1}{r^2}\right)^{1/2}} \tag{16}$$



at closest approach to sun r_0, $\frac{dr}{d\phi} = 0$, and (15) yields

$$J = \frac{c r_0}{(B(r_0))^{1/2}} \tag{17}$$

The total change in ϕ as the photon comes from infinity, bends around the sun and goes back to infinity is $2[\phi(r_0)-\phi(\infty)]$. The deflection of the orbit from a straight line is:

$$\Delta\phi = 2\left[\phi(r_0)-\phi(\infty)\right]-\pi$$

where:

$$\phi(r)-\phi(\infty) = \int_r^\infty A^{1/2}(r)\left[\frac{r^2 B(r_0)}{r_0^2 B(r)}-1\right]^{1/2}\frac{dr}{r}$$

Expanding the integrand in terms of $\frac{GM}{rc^2}$ yields:

$$\phi(r)-\phi(\infty) = \operatorname{Arcsin}\left(\frac{r_0}{r}\right)+\frac{GM}{r_0c^2}\left\{2-\sqrt{1-\left(\frac{r_0}{r}\right)^2}-\sqrt{\frac{r-r_0}{r+r_0}}\right\} + O\left(\left(\frac{GM}{rc^2}\right)^2\right)$$

$$\therefore \boxed{\Delta\phi = \frac{4GM}{r_0c^2}}$$

For a photon grazing the sun $M = 2\times10^{30}$ kg.
$r_0 = 6.95\times10^8$ m.

$\Rightarrow \Delta\phi = 1.75''$

TABLE 3.2 EXPERIMENTAL RESULTS ON THE DEFLECTION OF LIGHT[2]

Observatory	Eclipse	Site	θ (sec)
Greenwich	May 29, 1919	Sobral	1.98 ± 0.16
Greenwich	" "	Principe	1.61 ± 0.40
Adelaide-Greenwich	Sept. 21. 1922	Australia	1.77 ± 0.40
Victoria	" "	Australia	1.42 to 2.16
Lick	" "	Australia	1.72 ± 0.15
Lick	" "	Australia	1.82 ± 0.20
Potsdam	May 9, 1929	Sumatra	2.24 ± 0.10
Sternberg	June 19, 1936	U.S.S.R.	2.73 ± 0.31
Sendai	" "	Japan	1.28 to 2.13
Yerkes	May 20, 1947	Brazil	2.01 ± 0.27
Yerkes	Feb. 25, 1952	Sudan	1.70 ± 0.10
U. of Texas[3]	June 30, 1973	Mauritania	1.58 ± 0.16[a]

[a] According to a preliminary data analysis.

TABLE 3.3 EXPERIMENTAL RESULTS ON THE DEFLECTION OF RADIO WAVES

Radio Telescope	Wavelength (cm)	Baseline (km)	θ (sec)
Owens Valley[4]	3.1	1.07	1.77 ± 0.20
Goldstone[5]	12.5	21.56	$1.82 \begin{cases} +0.26 \\ -0.17 \end{cases}$
National RAO[6]	11.1 and 3.7	~ 2	1.64 ± 0.10
Mullard RAO[7]	11.6 and 6.0	~ 1	1.87 ± 0.30
Cambridge[8]	6.0	4.57	1.82 ± 0.14
Westerbork[9]	6.0	1.44	1.68 ± 0.09
Haystack and National RAO[10]	3.7	845	1.73 ± 0.05
National RAO[11]	11.1 and 3.7	35.6	1.76 ± 0.02
Westerbork[12]	21.2 and 6.0	~ 1	1.82 ± 0.06

Radar Echo Delay

For a photon, the equation of motion (10) is:

$$\frac{c^2}{B} - A\dot{r}^2 - \frac{J^2}{r^2} = 0$$

Exercise: Using (9), show that $\dot{r} = \frac{1}{B}\frac{dr}{dt}$.

$$\therefore \frac{c^2}{B} - \frac{A}{B^2}\left(\frac{dr}{dt}\right)^2 - \frac{J^2}{r^2} = 0 \qquad (18)$$

At closest approach $r = r_0$, $\frac{dr}{dt} = 0$ and (18) yields:

$$J^2 = \frac{c^2 r_0^2}{B(r_0)} \qquad (19)$$

Substituting J^2 into (18) and solving for $\frac{dr}{dt}$ gives:

$$\frac{dr}{dt} = c\left(\frac{B(r)}{A(r)}\right)^{1/2}\left(1 - \frac{r_0^2 B(r)}{r^2 B(r_0)}\right)^{1/2}$$

Or

$$\frac{dt}{dr} = \frac{1}{c}\left(\frac{A(r)}{B(r)}\right)^{1/2}\left(1 - \frac{r_0^2 B(r)}{r^2 B(r_0)}\right)^{-1/2}$$

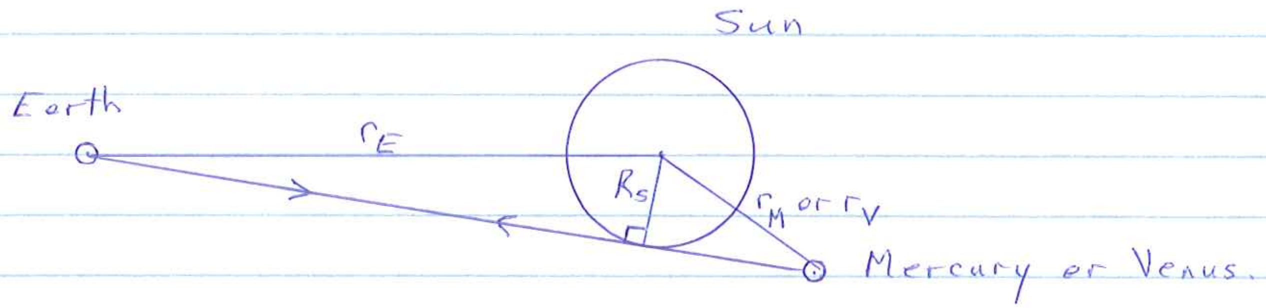
Hence time for light to travel from r_0 to r or r to r_0 is:

$$\Delta t(r, r_0) = \frac{1}{c}\int_{r_0}^{r}\left(\frac{A(r)/B(r)}{1 - \frac{r_0^2 B(r)}{r^2 B(r_0)}}\right)^{1/2} dr$$

Expanding the integrand in terms of $\frac{GM}{rc^2}$ yields:

$$\Delta t(r, r_0) = \frac{1}{c}\left\{ \sqrt{r^2 - r_0^2} + \frac{2GM}{c^2} \ln\left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + \frac{GM}{c^2} \left(\frac{r - r_0}{r + r_0} \right)^{1/2} \right\} \tag{20}$$

Experimental Setup



The maximum effect of the general relativistic terms in (20) occur when light from the Earth grazes the sun and bounces off Mercury or Venus. The general relativistic correction is then given by:

$$\Delta t_{max} = 2\left\{ \Delta t(r_E, R_S) + \Delta t(r_M, R_S) \underbrace{- \frac{1}{c}\sqrt{r_E^2 - R_S^2} - \frac{1}{c}\sqrt{r_M^2 - R_S^2}}_{\text{Newtonian Result}} \right\}$$

light bounces back to Earth

$$\Delta t_{max} \simeq \frac{4GM}{c^2}\left\{ 1 + \ln\left(\frac{4 r_E r_M}{R_S^2} \right) \right\}$$

For Mercury $\Delta t_{max} \simeq 240\ \mu sec$ which compares to a time of about 20 minutes for light to travel from Earth to Mercury and back.

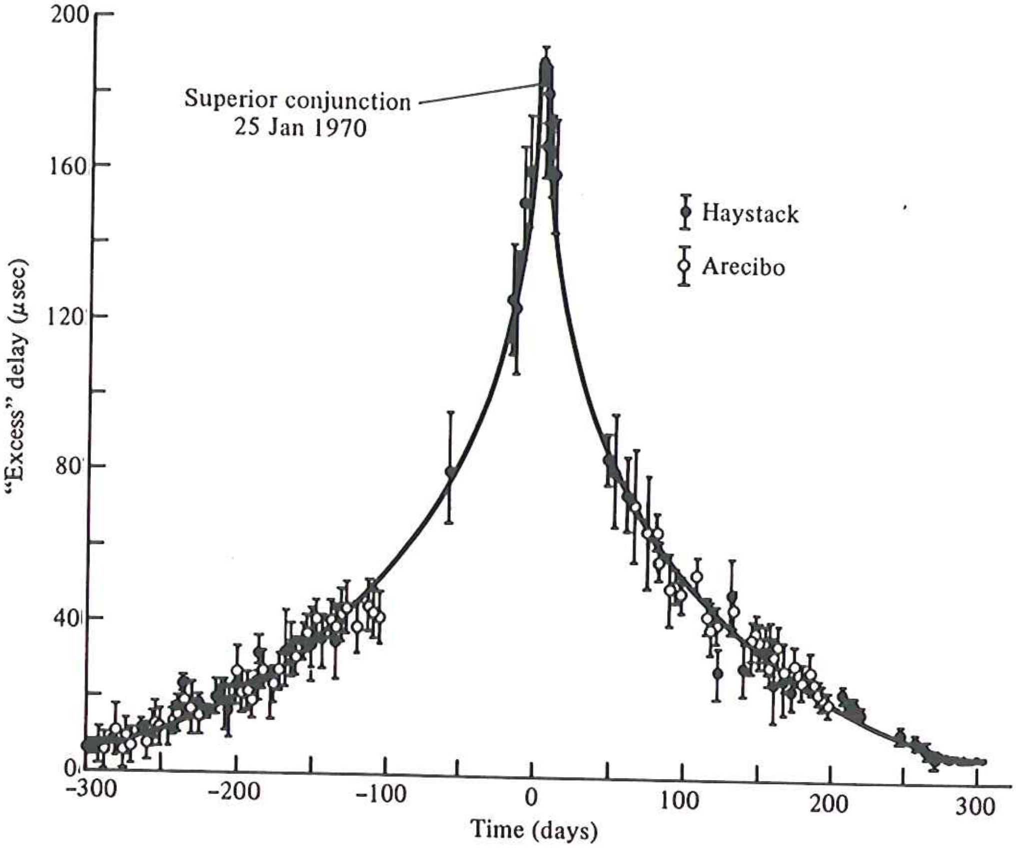


Fig. 3.7 *Earth-Venus time-delay measurements. The solid curve gives the theoretical prediction.* (*From Shapiro et al.,* Phys. Rev. Lett. *26, 1132 (1971)*.)

Gravitational Radiation

We shall now show how Einstein's equations

$$R_{\mu\nu} = -\frac{8\pi G}{c^4} S_{\mu\nu} \qquad (1)$$

where $S_{\mu\nu} \equiv T_{\mu\nu} - g_{\mu\nu}\frac{T^\lambda_\lambda}{2}$ generates gravitational waves.

In the weak field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the first term of the Ricci tensor is:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left\{ \Box^2 h_{\mu\nu} - \frac{\partial^2 h^\lambda{}_\nu}{\partial x^\lambda \partial x^\mu} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\lambda \partial x^\nu} + \frac{\partial^2 h^\lambda{}_\lambda}{\partial x^\mu \partial x^\nu} \right\} \qquad (2)$$

where $\Box^2 \equiv \partial^\lambda \partial_\lambda$.

Gauge Invariance

Exercise: Show that $h_{\mu\nu}$ and $h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \epsilon_\mu}{\partial x^\nu} - \frac{\partial \epsilon_\nu}{\partial x^\mu}$ (3)

yield the same result for $R^{(1)}_{\mu\nu}$.

This permits one to specify a so called gauge constraint on $h_{\mu\nu}$. We use the harmonic coordinate system gauge given by:

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0 \qquad (4)$$

Exercise: Show that the weak field limit of (4) is

$$\frac{\partial}{\partial x^\mu} h^\mu{}_\nu = \frac{1}{2}\frac{\partial h^\mu{}_\mu}{\partial x^\nu} \qquad (5)$$

Exercise: Show if (4) is not satisfied by $h_{\mu\nu}$, that it holds for $h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \epsilon_\mu}{\partial x^\nu} - \frac{\partial \epsilon_\nu}{\partial x^\mu}$

where ϵ_ν satisfies $\Box^2 \epsilon_\nu = \frac{\partial h^\mu{}_\nu}{\partial x^\mu} - \frac{1}{2}\frac{\partial h^\mu{}_\mu}{\partial x^\nu}$.

Using (5) + (2), the Einstein equations (1) become

$$\boxed{\Box^2 h_{\mu\nu} = -\frac{16\pi G}{c^4} S_{\mu\nu}.} \qquad (6)$$

This is a wave equation where $S_{\mu\nu}$ acts as a source of gravitational waves travelling at the speed of light.

Plane Waves.

In vacuum, the wave equation (6) reduces to

$$\Box^2 h_{\mu\nu} = 0 \qquad (7)$$

which has the plane wave solution

$$h_{\mu\nu} = e_{\mu\nu}\, e^{ik_\lambda x^\lambda} + e^*_{\mu\nu}\, e^{-ik_\lambda x^\lambda}. \qquad (8)$$

where $e_{\mu\nu}$ is called the polarization tensor and k^λ is the wave vector. Substitution of (8) into (7) yields

$$k_\mu k^\mu = 0 \qquad (9)$$

and of (8) into (5) gives: $k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e^\mu{}_\mu$ (10)

Properties of Polarization Tensor $e_{\mu\nu}$

$e_{\mu\nu}$ is a 4×4 matrix having 16 components. However $h_{\mu\nu}$ is symmetric implying $e_{\mu\nu} = e_{\nu\mu}$ which leaves only 10 independent components. The four gauge equations (10) further reduce this number to 6 independent components.

The gauge condition doesn't uniquely specify the field. Consider

$$\epsilon_\mu(x) = i\epsilon^\mu e^{ik_\lambda x^\lambda} - i\epsilon^{\mu *} e^{-ik_\lambda x^\lambda}$$

$$(3) \Rightarrow \quad h'_{\mu\nu} = e'_{\mu\nu} e^{ik_\lambda x^\lambda} + e'^{*}_{\mu\nu} e^{-ik_\lambda x^\lambda}$$

$$\text{where} \quad e'_{\mu\nu} = e_{\mu\nu} + k_\mu \epsilon_\nu + k_\nu \epsilon_\mu \tag{11}$$

The four ϵ_ν can be chosen to further reduce the number of independent polarization tensor components from 6 to 2.

Example

Consider a plane wave travelling in the $+\hat{z}$ direction. i.e. $k_\mu = \frac{\omega}{c}(1, 0, 0, -1)$ The gauge equation (10) yields:

$$e_{31} + e_{01} = e_{32} + e_{02} = 0$$

$$e_{33} + e_{03} = -e_{03} - e_{00} = \frac{1}{2}(e_{11} + e_{22} + e_{33} - e_{00})$$

$$\text{OR} \quad e_{01} = -e_{31} \quad e_{02} = -e_{32} \quad e_{03} = \frac{-1}{2}(e_{33} + e_{00}) \quad e_{22} = -e_{11}$$

Hence, only 6 independent $e_{\mu\nu}$ remain.

We next transform $h_{\mu\nu} \rightarrow h'_{\mu\nu}$. Equation (11) gives:

$$e'_{11} = e_{11} \qquad e'_{12} = e_{12}$$
$$e'_{13} = e_{13} - k\epsilon_1 \qquad e'_{23} = e_{23} - k\epsilon_2$$
$$e'_{33} = e_{33} - 2k\epsilon_3 \qquad e'_{00} = e_{00} + 2k\epsilon_0$$

The ϵ_μ are now chosen such that $e'_{13} = e'_{23} = e'_{33} = e'_{00} = 0$.
$\therefore$ e_{11} + e_{12} are only independent components of $e_{\mu\nu}$, which is:

$$e_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{11} & e_{12} & 0 \\ 0 & e_{12} & -e_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A gravitational wave has 2 linearly independent polarization states described by:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad + \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Energy and Momentum of Plane Waves

Einstein's equations are:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

where $T_{\mu\nu}$ is the energy momentum tensor of matter. Using the weak field approximation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and Einstein's equations become

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)} = -\frac{8\pi G}{c^4}\left[T_{\mu\nu} + t_{\mu\nu}\right]$$

where the superscript (1) denotes that only terms linear in $h_{\mu\nu}$ and its derivatives appear, and

$$t_{\mu\nu} \equiv \frac{c^4}{8\pi G}\left\{ R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} - R^{(1)}_{\mu\nu} + \frac{R^{(1)}}{2}\eta_{\mu\nu}\right\}$$

is the energy momentum tensor of the gravitational field.

Exercise: Show that $R^{(1)}_{\mu\nu} = 0$ for a plane wave.

$$\therefore t_{\mu\nu} = \frac{c^4}{8\pi G}\left\{ R^{(2)}_{\mu\nu} - \frac{R^{(2)}}{2}\eta_{\mu\nu}\right\}$$

Exercise: For a plane wave show that (a lot of work!)

$$\langle t_{\mu\nu}\rangle = \frac{c^4}{16\pi G} k_\mu k_\nu \left\{ e^{\lambda\rho *} e_{\lambda\rho} - \frac{1}{2}|e^\lambda_\lambda|^2\right\}$$

time average
over many wave periods

Exercise: For the plane wave considered earlier propagating in the $\hat{z}$ direction, show that

$$\langle t_{\mu\nu} \rangle = \frac{c^4}{8\pi G} k_\mu k_\nu \left\{ |e_{11}|^2 + |e_{12}|^2 \right\}$$

Generation of Gravitational Waves

The gravitational wave equation is:

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4} S_{\mu\nu}$$

where $S_{\mu\nu} = T_{\mu\nu} - \eta_{\mu\nu}\frac{T^{\alpha}{}_{\alpha}}{2}$ The solution is given by:

$$h_{\mu\nu}(\vec{x},t) = \frac{4G}{c^4}\int d^3x' \frac{S_{\mu\nu}(\vec{x}', t - |\vec{x}-\vec{x}'|/c)}{|\vec{x}-\vec{x}'|}$$

Note that $h_{\mu\nu} \propto r^{-1}$ where $r \equiv |\vec{x}-\vec{x}'|$. Hence energy leaving a sphere of radius r, $h^2_{\mu\nu} 4\pi r^2$ is independent of r as expected.

We first consider a single Fourier frequency component:

i.e. $T_{\mu\nu}(\vec{x},t) = T_{\mu\nu}(\vec{x},\omega)e^{i\omega t} + c.c$

where c.c. denotes complex conjugate.

$$\therefore S_{\mu\nu}(\vec{x}', t - |\vec{x}-\vec{x}'|/c) = S_{\mu\nu}(\vec{x}',\omega)e^{i\omega(t-|\vec{x}-\vec{x}'|/c)} + c.c.$$

where $S_{\mu\nu}(\vec{x},\omega) = T_{\mu\nu}(\vec{x},\omega) - \frac{1}{2}\eta_{\mu\nu}T^{\alpha}{}_{\alpha}(\vec{x},\omega)$

$$\therefore h_{\mu\nu}(\vec{x},t) = \frac{4G}{c^4}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\left(S_{\mu\nu}(\vec{x}',\omega)e^{i\omega(t-|\vec{x}-\vec{x}'|/c)} + c.c.\right)$$

We shall consider gravitational fields far away from the source in the so called radiation zone. i.e. $|\vec{x}| >> |\vec{x}'|$.

Exercise: If $|\vec{x}| >> |\vec{x}'|$, show $|\vec{x} - \vec{x}'| = |\vec{x}| - \hat{x}\,\vec{x}'$

$$\therefore\ h_{\mu\nu}(\vec{x},t) = e_{\mu\nu}(\vec{x},\omega)\, e^{ik_\lambda x^\lambda} + c.c.$$

where $k^\mu \equiv \left(\frac{\omega}{c}, \vec{k}\right)$ and

$$e_{\mu\nu}(\vec{x},\omega) \equiv \frac{4G}{c^4}\frac{1}{|\vec{x}|}\int d^3x'\, S_{\mu\nu}(\vec{x}',\omega)\, e^{i\vec{k}\cdot\vec{x}'}$$

The polarization tensor can be rewritten as

$$e_{\mu\nu}(\vec{x},\omega) = \frac{4G}{c^4}\frac{1}{|\vec{x}|}\left\{ T_{\mu\nu}(\vec{k},\omega) - \frac{1}{2}\eta_{\mu\nu}\, T^\lambda{}_\lambda(\vec{k},\omega)\right\}$$

where $T_{\mu\nu}(\vec{k},\omega) \equiv \int d^3x'\, T_{\mu\nu}(\vec{x}',\omega)\, e^{i\vec{k}\cdot\vec{x}'}$

We now make the dipole approximation i.e. $\vec{k}\cdot\vec{x}' << 1$
This means the source radius is much smaller than the wavelength.

$$\therefore\ T_{\mu\nu}(\vec{k},\omega) \simeq \int d^3x'\, T_{\mu\nu}(\vec{x}',\omega)$$

Substituting $e_{\mu\nu}(\vec{x}, \omega)$ into $\langle t_{\mu\nu} \rangle$, we find the time averaged gravitational field energy density

$$\langle t_{00} \rangle = \frac{G\omega^2}{\pi c^6 |\vec{x}|^2} \left\{ |T_{\mu\nu}(\vec{k}, \omega)|^2 - \frac{1}{2} |T^\lambda{}_\lambda(\vec{k}, \omega)|^2 \right\}$$

The power emitted into unit solid angle is

$$\frac{dP}{d\Omega} = \langle t_{00} \rangle \; c \; |\vec{x}|^2$$

$$\frac{dP}{d\Omega} = \frac{G\omega^2}{\pi c^5} \left\{ |T_{\mu\nu}(\vec{k}, \omega)|^2 - \frac{1}{2} |T^\lambda{}_\lambda(\vec{k}, \omega)|^2 \right\}$$

Exercise: Show the conservation law $\frac{\partial T_{\mu\nu}(\vec{x}, t)}{\partial x^\mu} = 0$ yields $k^\mu T_{\mu\nu}(\vec{k}, \omega) = 0$ which in turn gives:

$$T_{0i}(\vec{k}, \omega) = -\hat{k}^j \, T_{ji}(\vec{k}, \omega) \qquad \hat{k} \equiv \frac{\vec{k}}{|\vec{k}|}$$

$$T_{00}(\vec{k}, \omega) = \hat{k}^i \hat{k}^j \, T_{ji}(\vec{k}, \omega)$$

After much boring algebra one then finds

$$\frac{dP}{d\Omega} = \frac{G\omega^2}{\pi c^5} \Lambda_{ijlm} \, T^*_{ij}(\vec{k}, \omega) \, T_{lm}(\vec{k}, \omega)$$

where
$$\Lambda_{ijlm} \equiv \delta_{il}\delta_{jm} - 2\delta_{il}\hat{k}_j\hat{k}_m + \frac{1}{2}\hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m - \frac{1}{2}\delta_{ij}\delta_{lm} + \frac{1}{2}\delta_{ij}\hat{k}_l\hat{k}_m + \frac{1}{2}\hat{k}_i\hat{k}_j\delta_{lm}$$

$$\therefore -\omega^2 D_{ij}(\omega) = \int dt\, d^3x\, e^{i\omega t}\, x_i x_j \frac{\partial^2 T_{k\ell}(\vec{x},t)}{\partial x_k \partial x_\ell}$$

Integrating by parts twice on the right side and using boundary conditions to set other terms to zero gives:

$$-\omega^2 D_{ij}(\omega) = 2 \int dt\, d^3x\, e^{i\omega t}\, T_{ij}(\vec{x},t)$$

$$\therefore T_{ij}(\vec{k},\omega) = -\frac{\omega^2}{2} D_{ij}(\omega)$$

Hence, the power emitted into unit solid angle is:

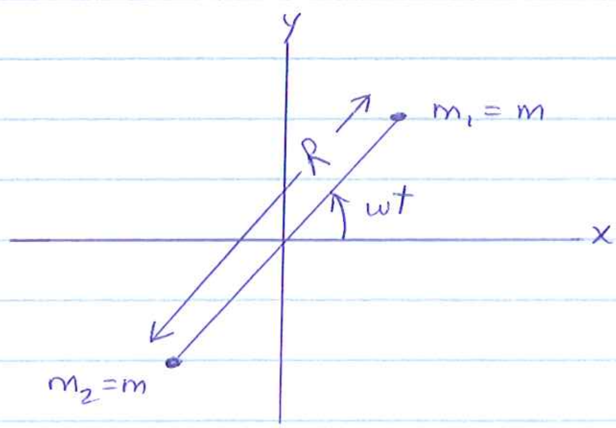
$$\frac{dP}{d\Omega} = \frac{G\omega^6}{4\pi c^5} \Lambda_{ij\ell m} D^*_{ij}(\omega) D_{\ell m}(\omega)$$

Integrating over the solid angle (lots of algebra) yields the total power emitted.

$$\boxed{P = \frac{2G\omega^6}{5c^5} \left\{ D^*_{ij}(\omega) D_{ij}(\omega) - \frac{1}{3} |D_{ii}(\omega)|^2 \right\}}$$

Example of a Radiating System
A Binary Star

Consider two stars of equal mass m orbiting about their combined center of mass. As the stars emit gravitational radiation, they slowly spiral inward.



$$\vec{x}_1 = \frac{R}{2}(\cos\omega t, \sin\omega t, 0)$$

$$\vec{x}_2 = -\frac{R}{2}(\cos\omega t, \sin\omega t, 0)$$

Exercise: Show $\omega^2 = \frac{2Gm}{R^3}$.

Quadrupole Moments

$$D_{ij} = \sum_{a=1,2} m\, x_i^{(a)} x_j^{(a)}$$

$$D_{11} = 2m\left(\frac{R}{2}\right)^2 \cos^2\omega t = \frac{mR^2}{4} + \frac{mR^2}{8}\left(e^{2i\omega t} + e^{-2i\omega t}\right)$$

$$D_{22} = 2m\left(\frac{R}{2}\right)^2 \sin^2\omega t = \frac{mR^2}{4} - \frac{mR^2}{8}\left(e^{2i\omega t} - e^{-2i\omega t}\right)$$

$$D_{12} = D_{21} = m\left(\frac{R}{2}\right)^2 \cos\omega t \sin\omega t = -i\,\frac{mR^2}{8}\left(e^{2i\omega t} - e^{-2i\omega t}\right)$$

$$D_{i3} = D_{3i} = 0$$

Radiation is only generated by the time dependent part of D_{ij}. The radiation frequency is at $\bar{\omega} = 2\omega$ which makes sense since the binary star appears unchanged every half period. The total power emitted at frequency $\bar{\omega}$ is:

$$P = \frac{2G\bar{\omega}^6}{5c^5}\left\{ D^*_{ij}(\bar{\omega})D_{ij}(\bar{\omega}) - \frac{1}{3}|D_{ii}(\bar{\omega})|^2 \right\}$$

Now $D_{ij}(t) = D_{ij}(\bar{\omega})e^{i\bar{\omega}t} + c.c.$

$$\therefore D_{11}(\bar{\omega}) = \frac{mR^2}{8}$$

$$D_{22}(\bar{\omega}) = -\frac{mR^2}{8}$$

$$D_{12}(\bar{\omega}) = D_{21}(\bar{\omega}) = -i\frac{mR^2}{8}$$

The total power emitted then becomes:

$$P = \frac{64}{5}\frac{G^4m^5}{c^5R^5}.$$

Rate of Inward Spiraling

Orbit Period $T = \frac{2\pi}{\omega} = \frac{2\pi R^{3/2}}{\sqrt{2Gm}}$

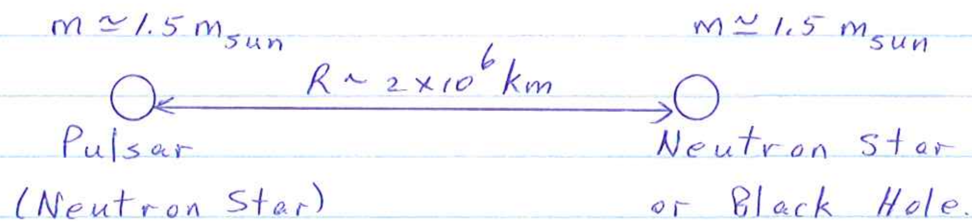
Exercise: Show $\frac{\dot{T}}{T} = \frac{3}{2}\frac{\dot{R}}{R} = -\frac{192}{5}\frac{G^3m^3}{R^4c^5}.$

For an orbit having eccentricity e the preceding equation is modified as follows:

$$\frac{\dot{T}}{T} = -\frac{192}{5}\,\frac{G^3 m^3}{R^4 c^5}\, f(e)$$

where $f(e) = \dfrac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}}$

Binary Pulsar



A pulsar acts like a lighthouse sending a beam of light towards the Earth every period of the binary. The above binary has been observed having an orbital eccentricity $e = 0.617$. and a period of about 8 hrs.

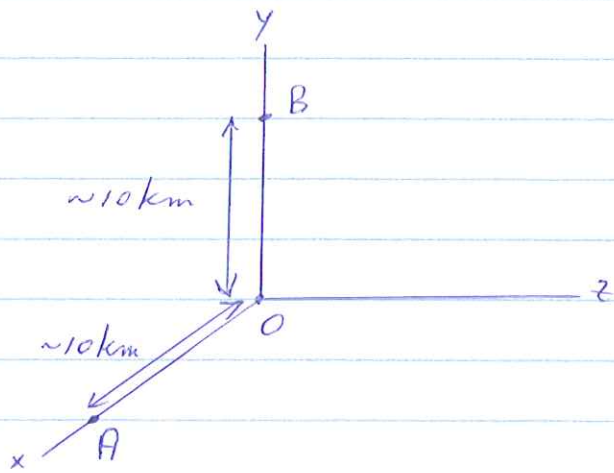
$$\therefore \dot{T} = -\frac{192}{5}\,\frac{(6.67\times10^{-11})^3\,(3\times10^{30})^3}{(2\times10^9)^4\,(3\times10^8)^5}\, f(0.617)\,(8\times3600)$$

$$\simeq -2.40\times10^{-12}$$

The observed rate is $\dot{T} = -(2.30 \pm 0.22)\times10^{-12}$. Hence in one year the pulsar period decreases. 7×10^{-5} sec. At present the period $T = 27906.98161(3)$ sec. This measurement is continuing and will produce a better result for $\dot{T}$ the longer the pulsar is observed. However, it already provides powerful although indirect evidence for the existence of gravitational radiation.

Detection of a Gravitational Wave

Work is presently underway to construct a laser interferometer having two arms as shown below.



Suppose the interferometer encounters a gravitational wave propagating in the $\hat{z}$ direction. The wave has field:

$$h_{11} = -h_{22} = e_{11}\, e^{i(\omega t - kz)} + e_{11}^{*}\, e^{-i(\omega t - kz)}$$

$$\text{Distance } AO = \int_0^L \sqrt{g_{xx}}\, dx$$

$$\simeq \int_0^L (\eta_{11} + h_{11})^{1/2}\, dx$$

$$\simeq \int_0^L \left(\eta_{11} + \frac{h_{11}}{2}\right) dx$$

$$AO = L + \frac{e_{11}}{2} L \cos\omega t$$

Similarly $BO = \int_0^L \sqrt{g_{yy}}\, dy = L - \frac{e_{11}}{2} L \cos\omega t$

Hence, the gravitational wave changes the length AO relative to BO. Plans are to construct two interferometers to help eliminate noise caused by vibrations such as earthquakes. A major difficulty is that the frequency and power of gravitational waves emitted by various astrophysical sources can be only crudely estimated.

TABLE 4.1 FREQUENCY BANDS FOR GRAVITATIONAL WAVES

Designation	Frequency	Typical Sources
Extremely low frequency	10^{-7}/sec to 10^{-4}/sec	cosmological? explosions in quasars and galactic nuclei binaries
Very low frequency	10^{-4}/sec to 10^{-1}/sec	short-period binaries huge black holes ($\sim 10^5$ to $10^8\ M_\odot$)
Low frequency	10^{-1}/sec to 10^{2}/sec	pulsars
Medium frequency	10^{2}/sec to 10^{5}/sec	black holes (1–$10^3\ M_\odot$) collapse of stars supernovae, birth of neutron stars
High frequency	10^{5}/sec to 10^{8}/sec	man-made?
Very high frequency	10^{8}/sec to 10^{11}/sec	black-body cosmological?

TABLE 4.3 ASTROPHYSICAL SOURCES OF GRAVITATIONAL RADIATION[a]

Source	Spectrum	Energy received[b]	Comments
Binary star system (of the AM CVn type)	discrete, $\nu = 2 \times 10^{-3}$/sec	10^{-9} erg/cm² sec	not detectable
Collapse of neutron binary system	glissando, $\nu \sim 200$/sec increasing to $\nu \sim 2 \times 10^3$/sec	10^{11} erg/cm²	detectable
Pulsating neutron star	discrete, $\nu = 10^3 - 10^4$/sec	10^9 erg/cm²	detectable with tuned detector
Rotating neutron star (with rigid deformation)	discrete, $\nu = 3 \times 10^2$/sec	10^{-1} erg/cm² sec	not detectable
Rapidly rotating neutron star (with rotation-induced deformation)	discrete, $\nu = 1.5 \times 10^3$/sec (slight drift to higher frequency)	10^9 erg/cm²	detectable with tuned detector
Neutron star falling into black hole (10 $M_\odot$)	continuous, peaked near $\nu \sim 10^4$/sec	10^{10} erg/cm²	detectable
Gravitational collapse of star (10 $M_\odot$) to form a black hole	continuous, peaked near $\nu \sim 10^3$/sec	10^{13} erg/cm²	detectable

[a] Based, in part, on the summary given by Rees, Ruffini, and Wheeler.[4]
[b] The energy is given in terms of the flux (erg/cm² sec) for sources that radiate continuously or in terms of the integrated flux (erg/cm²) for sources that radiate only for a short time ($\lesssim$ 1 sec).

It has been assumed that the distance to the source has the standard value of 100 light years; if the distance is r, the energy must be multiplied by $(100/r)^2$.

Stellar Evolution

In a star such as the sun, the gravitational attraction of the star on itself is countered by the thermal gas pressure produced by burning hydrogen to produce helium. When the supply of hydrogen is exhausted, the star implodes and becomes one of the following: White Dwarf, Neutron Star or Black Hole. During the implosion, the outer layers may be ejected in an enormous explosion called a supernovae. A supernovae in 1087 in the Crab Nebula produced sufficient light to rival the moon in brightness for several weeks.

White Dwarf

This is a star that may be thought of as a degenerate Fermi gas of electrons. From statistical mechanics, one can show that a gas of N electrons enclosed in volume V has Fermi momentum

$$p_F = \left(\frac{3\pi^2\hbar^2 N}{V}\right)^{1/3} \qquad (1)$$

The Fermi energy is given by:

$$\epsilon_F = \begin{cases} \dfrac{\hbar^2 p_F^2}{2m_e} & \text{nonrelativistic case} \\ p_F c & \text{relativistic case} \end{cases} \qquad (2)$$

Using $-\frac{GM^2}{R}$ as the gravitational binding energy, we find the following expressions for the total energy.

$$E = \begin{cases} \dfrac{N^{5/3}(9\pi\hbar^3/4)^{2/3}}{2m_e R^2} - \dfrac{GM^2}{R} & \text{(nonrelativistic)} \\[2ex] \dfrac{N^{4/3}(9\pi\hbar^3/4)^{1/3}}{R} - \dfrac{GM^2}{R} & \text{(relativistic)} \end{cases} \qquad (3)$$

<u>Exercise:</u> Show the total energy only has a minimum for the nonrelativistic case given by

$$R = \frac{N^{5/3}(9\pi\hbar^3/4)^{2/3}}{GM^2 m_e} \qquad (4)$$

The critical mass at which the equilibrium disappears may be estimated by finding when the electron gas turns relativistic. Setting $p_F = m_e c$ in (1), we find:

$$R = \frac{(9\pi\hbar^3 N/4)^{1/3}}{m_e c} \qquad (5)$$

Equating (4) + (5) yields:

$$M_{crit}^2 = \frac{cN^{4/3}(9\pi\hbar^3/4)^{1/3}}{G}$$

For helium nuclei, $N \simeq M/2m_p$ which gives:

$$M_{crit} = \sqrt{\frac{9\pi}{4}}\left(\frac{\hbar c}{G m_p^2}\right)^{3/2} m_p$$

$$M_{crit} = \sqrt{\frac{9\pi}{4}} \left(\frac{1\times10^{-27} \times 3\times10^{10}}{6.67\times10^{-8}\,(1.67\times10^{-24})^2} \right)^{3/2} 1.67\times10^{-24}$$

$$= 9\times10^{33}\ \text{gm}.$$

$$\sim M_{sun}$$

A more realistic model of nuclear matter yields $M_{crit} = 1.44\, M_{sun}$ which is called the Chandrasekhar Limit. Stars having masses exceeding M_{crit} can't be White Dwarfs.

Neutron Stars

At high densities, the electrons combine with protons producing neutrons. The electron degeneracy is then no longer able to counter the gravitational attraction. The star then implodes creating a neutron star where a degenerate neutron gas counters the gravitational attraction. Neutron stars are very compact having a diameter of about 20km and a mass $\sim 2M_{sun}$. Due to their small size, they can spin up to 10^3 rev/sec. They sometimes emit light like a lighthouse beacon and are called pulsars.

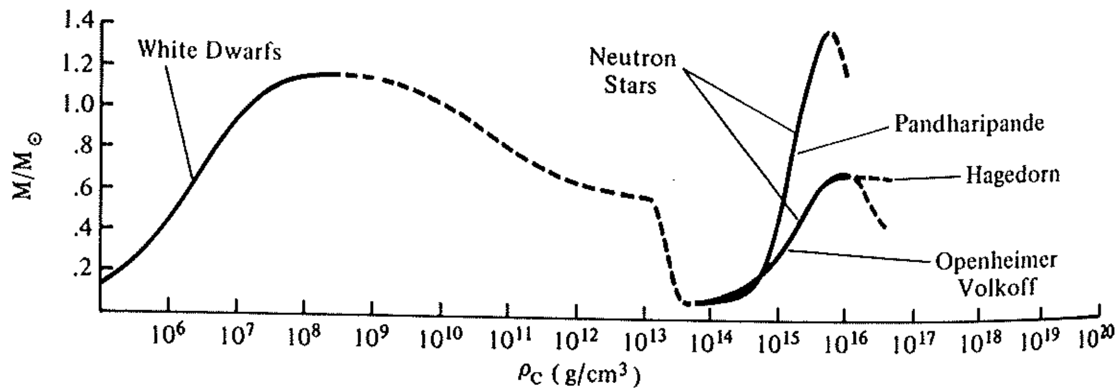


Fig. 9.20 *Total mass as a function of central density for cold stars. Different curves result from different assumptions about the equation of state. (Courtesy of Prof. R. Ruffini, Institute for Advanced Study.)*

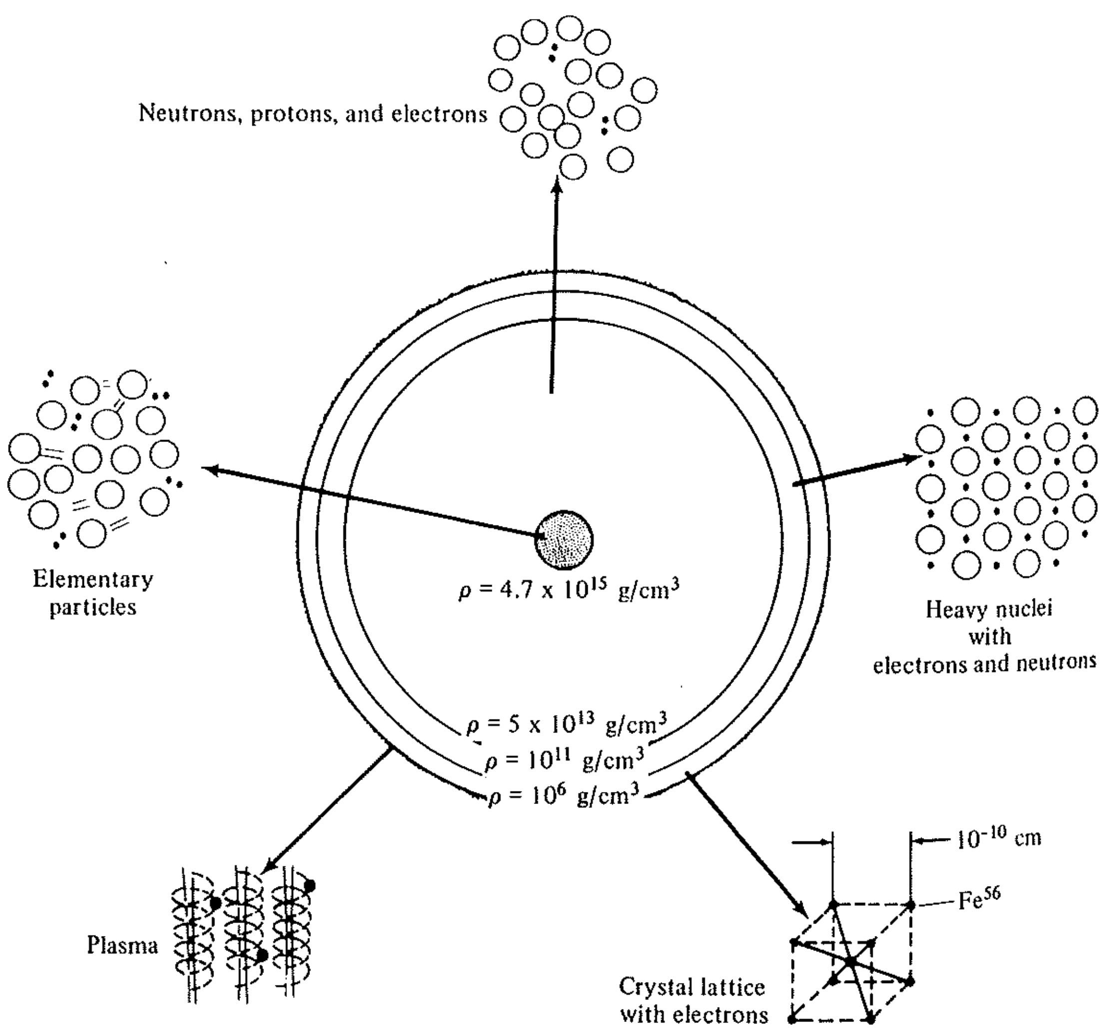


Fig. 9.21 *Structure of a neutron star. (After Ruffini,* in *DeWitt and DeWitt, eds.* Black Holes)

Black Holes

Neutron stars can only exist below a critical mass of about $4M_{sun}$. If the star has a larger mass, it continues to contract indefinitely to a point called a Black Hole.

Exercise: Equating $\frac{mv^2}{2}$ to $\frac{GmM}{r}$, show that the maximum escape velocity $v=c$ occurs at $r = \frac{2GM}{c^2}$.

Hence, we expect that not even light can escape from the region $r < \frac{2GM}{c^2}$. This result is somewhat naive since the Newtonian expression for kinetic energy was used.

The Schwarzschild space-time distance is:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Note that ds^2 becomes singular at radius

$$r_s \equiv \frac{2GM}{c^2}$$

which is called the Schwarzschild radius. This result fortuitously agrees with the Newtonian estimate.

Exercise: If $M = M_{sun}$, show $r_s \simeq 3\text{km}$.

A clock placed at rest at $r = r_s$ shows a proper time

$$d\tau = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} dt \rightarrow 0 \quad \text{as } r \rightarrow r_s$$

Hence, an astronaut who falls into a Black Hole, will appear to be frozen in time to an outside observer.

Detection of Black Holes

This is difficult since Black Holes emit no light. However, a Black Hole that is part of a binary star can suck material from its partner star. This material is accelerated to high speeds and emits X-rays.

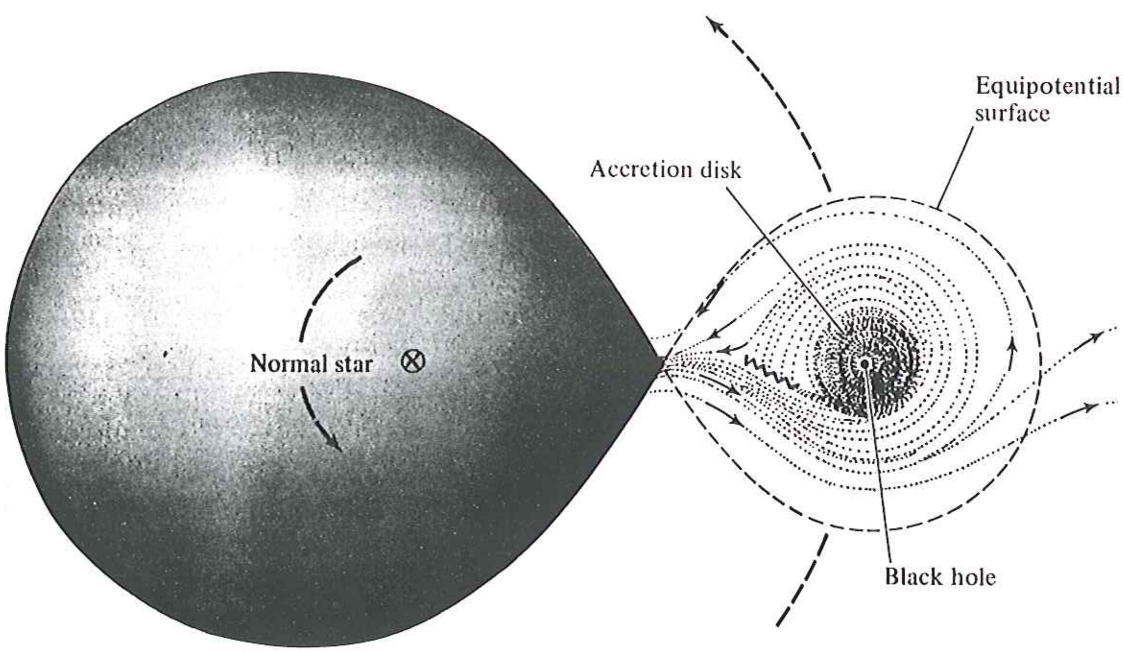


Fig. 9.25 *Primary star, accretion disc, and black hole. (After Eardley and Press,* Annual Review of Astronomy and Astrophysics, *1975.)*

Possible Black Holes are given in the table below.

TABLE 9.2 FLICKERING X-RAY SOURCES

X-ray source	Time scale of bursts	Orbital period (days)	Primary star	Mass of primary star	M_X
Cygnus X-1[a]	~ 0.001 sec	5.6	HDE 226868	$30 M_\odot$	$\geq 6\ M_\odot$
SMC X-1	~ 1 min	3.89	Sk 160	20	~ 6
Vela X-1	~ 0.1 sec	8.9	HD 77581	~ 15	~ 1.5

[a] Cygnus X-1 is not an eclipsing binary. The orbital period is determined from the velocity curve of the primary star (obtained from Doppler shifts).

Recently, it has been thought that Black Holes as massive as $10^6\ M_{sun}$ exist at the center of galaxies.

Cosmology

The stars visible to us with the naked eye are not uniformly distributed. Our star is part of a disk shaped galaxy called the Milky Way which contains about 10^{11} stars and has a size of about 3×10^4 parsecs. (1 parsec = 3.26 light years)

Galaxies are also not uniformly distributed but make up clusters. Our cluster is called the Local Group and has about 20 galaxies and a size of $\sim 10^6$ pc.

Clusters appear to be randomly distributed. Hence, on a scale of $\sim 10^7$ pc, the universe appears homogeneous. It is therefore impossible to distinguish one position in the universe from another. This is called the Cosmological Principle.

Determining Cosmic Distances

1) Geometric Methods

The distance to nearby stars can be determined using triangulation. i.e. by noting the different position as measured by two separated observers. One can determine distances up to ~30 pc using the diameter of the Earth's orbit as the baseline.

2) Main Sequence Stars

Distances of up to $\sim 10^5$ pc can be determined using the so called Hertzsprung-Russell diagram. This is a relation between a star's temperature and the amount of light $\mathcal{L}$ it emits.

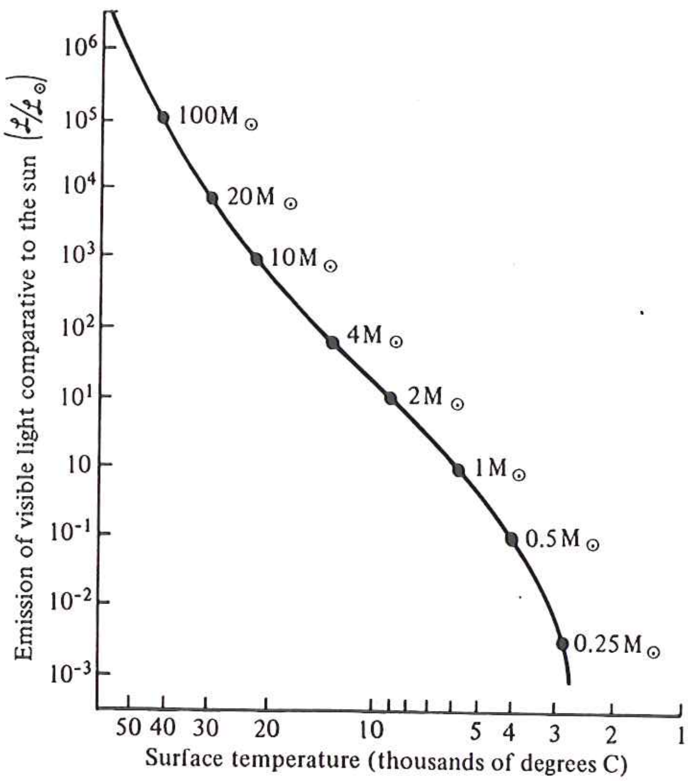


Fig. 10.3 *Hertzsprung-Russell diagram for main-sequence stars. (After Hoyle,* Frontiers of Astronomy.)

The star's temperature is found from its spectra which in turn enables $\mathcal{L}$ to be determined. The distance to the star d is then found by measuring the energy flux S observed on Earth and using:

$$S = \frac{\mathcal{L}}{4\pi d^2} \qquad (1)$$

3) Cepheid Stars

Distances of $\sim 10^6$ pc can be found using Cepheid stars. These stars have a large luminosity $\sim 10^{3-4} \mathcal{L}_{sun}$ which varies in time. The oscillation period is in the range of 2-40 days. An empirical relation between the luminosity and the period has been observed.

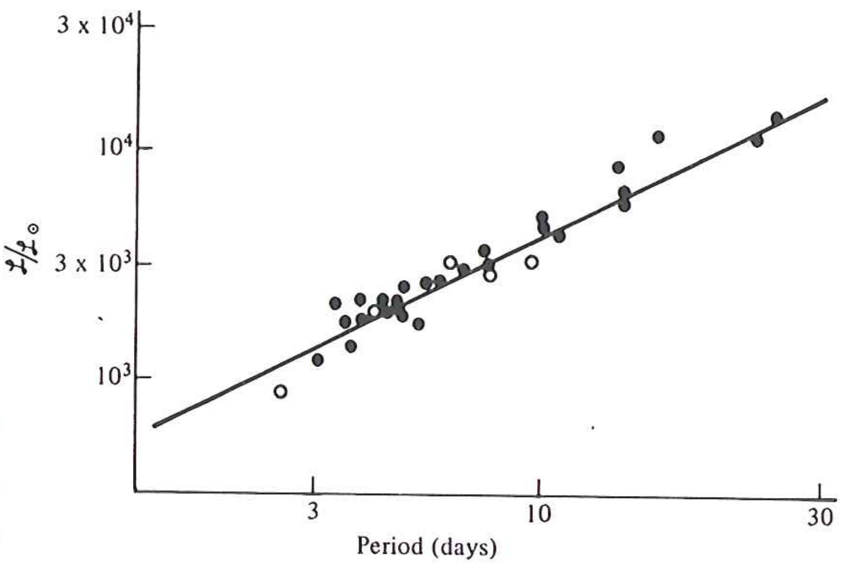


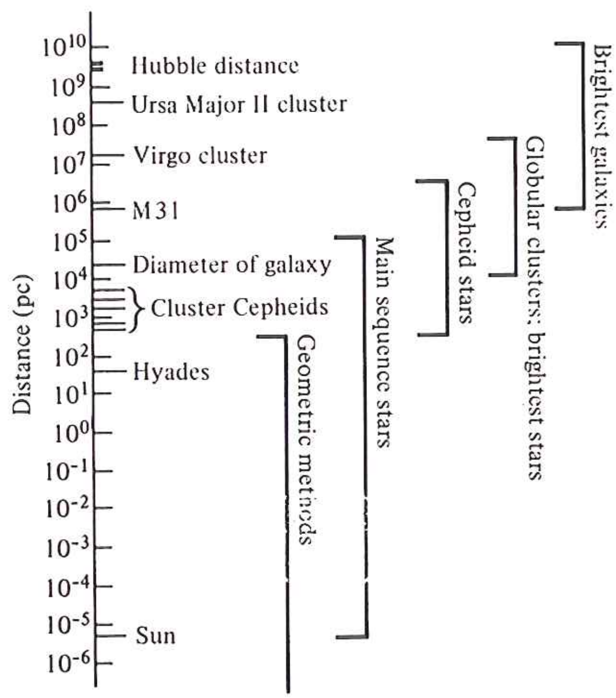
Fig. 10.4 *The period-luminosity relation for Cepheid stars. (After Kraft,* Ap. J. ***134****, 616 (1961).)*

The distance to the star is then found by first measuring the luminosity period, next finding the luminosity using the above diagram and finally measuring the star's light flux on Earth and using equation (1).

4) Larger Distances

Distances greater than 10^6 pc are estimated using the brightest star in a galaxy or the brightest galaxy in a cluster. An educated guess of the luminosity emitted by this object is made and its energy flux on Earth is measured. The distance is then determined albeit with substantial uncertainty.

Distance Summary

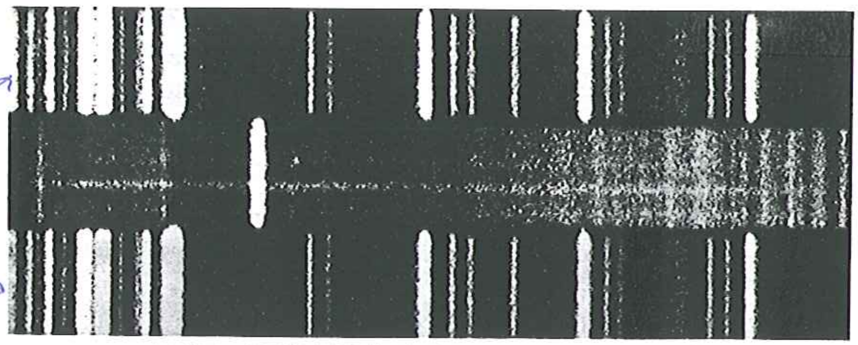


The cosmic distance ladder. The brackets indicate the range for which a particular method may be used. Each method depends on all the other methods that are to the left of it in the diagram. (After Weinberg, Gravitation and Cosmology.)

Cosmological Redshift

Astronomers have observed the spectral lines produced by distant stars to be shifted to the red, indicating the universe is expanding. One can think of our solar system as being a point on the surface of a balloon that is being blown up.

spectrum on Earth



red shifted spectrum from Bootes

Figure 14.11 The radio galaxy 3C295 in Boötes. The spectrum of this galaxy, shown below, reveals a red shift $z = 0.46$, the largest yet observed for any galaxy. This photograph and spectrograph were taken with the 200-in. telescope at Mt. Palomar. (Courtesy Mt. Wilson and Mt. Palomar observatories.)

The red shift is defined as:

$$z = \frac{\lambda - \lambda_o}{\lambda_o}$$

where $\begin{cases} \lambda_o \text{ is wavelength emitted by atom on Earth} \\ \lambda \quad \text{"} \qquad\qquad\qquad \text{"} \quad \text{star} \end{cases}$

Exercise: Show $z = \frac{v}{c}$ where v is speed of star and c is speed of light.

The red shift has been found to be proportional to the distance l of the receding star.

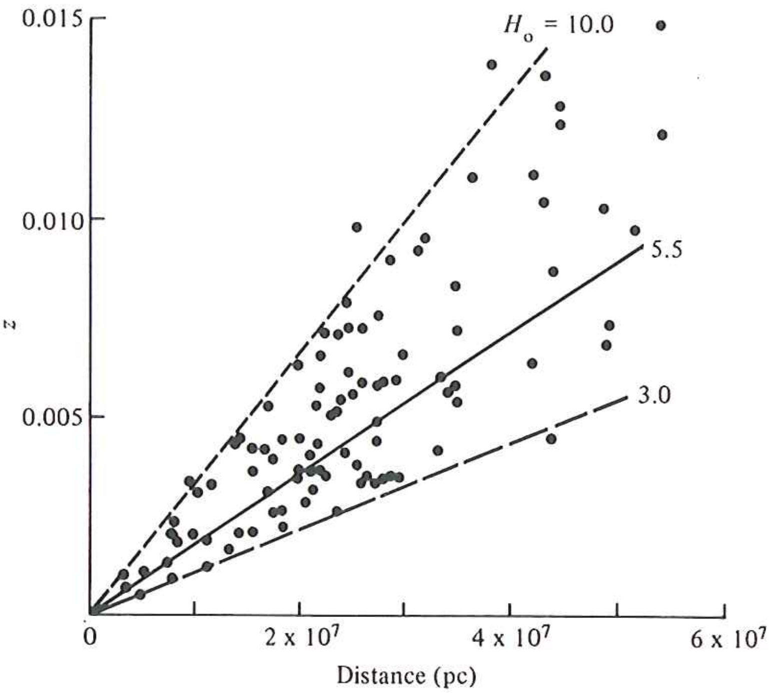


Fig. 10.9 *Observed redshift of galaxies as a function of distance. The straight line corresponds to H_0 = 5.5 cm sec^{-1} pc^{-1}. (After Sandage and Tammann,* Ap. J. ***196****, 313 (1975).)*

Hence, the velocity with which a star recedes from us is also proportional to l.

$$\therefore \; v = H_o l$$

where $H_o = \left(5.5 \, {}^{+6}_{-2.5}\right)$ cm/sec/pc is called Hubble's constant.

Age of Universe

The age of the universe can be estimated to be $H_0^{-1} = 1.8$ billion years assuming the expansion rate has remained constant. Another age estimate is found using radioactive decay rates. Astrophysics predicts that when the elements were synthesized inside stars that

$$\frac{[Th^{232}]}{[U^{238}]} = 1.9 \pm 0.3$$

The decay rate of Th^{232} is 1.55×10^{-10} /yr.
" " U^{238} " 4.95×10^{-11} /yr.

Exercise: Using the observed ratio $\frac{[Th^{232}]}{[U^{238}]} = 3.9 \pm 0.5$

show the age of these elements is 6.8×10^{9} yrs.

Other radioactive elements yield lifetimes as high as 15 billion years.

Cosmic Black Body Radiation

An expanding universe implies that at earlier times, the universe was much denser and hotter. Residual radiation should remain and has indeed been observed corresponding to a Black Body at a temperature of 2.7 K.

Exercise: What wavelength does kT correspond to?

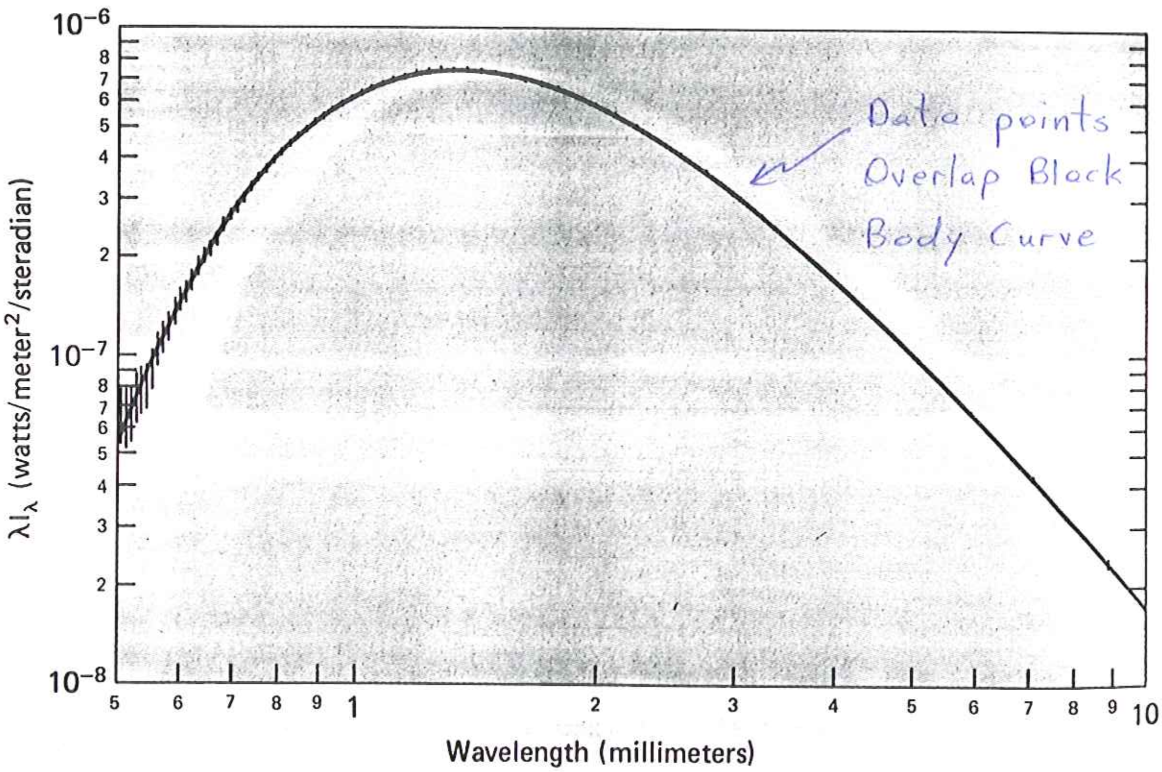


Figure 33–16 The first spectrum based on COBE observations drew an ovation from astronomers when it was first shown at an American Astronomical Society meeting. It represents a proof at astonishingly high accuracy that the cosmic background radiation is a black body. The data points and their error bars (shown in black) are fit precisely by a black body curve (color) for 2.735 ± .06°.

Mass Density of Universe

Matter exists in a variety of forms in the universe as shown in the following table.

TABLE 10.1 THE MEAN MASS DENSITY OF THE UNIVERSE

Form of Matter	Mean Mass Density (g/cm^3)[a]
Galaxies	2×10^{-31}
Galactic halos, and intergalactic stars, dwarf galaxies, or black holes (in clusters)	$\lesssim 2 \times 10^{-30}$
Electromagnetic radiation	
blackbody (2.7 °K)	4×10^{-34}
radio waves	10^{-40}
starlight (optical)	10^{-35}
X-rays	10^{-37}
Cosmic rays	10^{-35}
Intergalactic hydrogen (atomic)	$< 4 \times 10^{-36}$ g/cm^3 (2×10^{-30})[b]
Intergalactic hydrogen (molecular)	$< 5 \times 10^{-34}$
Intergalactic plasma ($\sim 10^6$ °K)	$\leq 10^{-29}$
Total protons, neutrons, and other baryons	$\leq 6 \times 10^{-31}$[c]

[a] Densities in this table are based on the value $H_0 = 5.5$ cm sec^{-1} pc^{-1} for Hubble's constant.

[b] The second number does not assume a cosmological redshift for quasars.

[c] From theory of deuterium production.

The density of matter is estimated to be $\lesssim 6 \times 10^{-31}$ gm/cm^3. The actual density may be much higher due to a variety of objects that are as yet not visible. These include:

- Black Holes
- Massive Neutrinos
- Brown Dwarfs (i.e. Jupiter like planets)
- Weak Interacting Massive Particles (WIMP's) i.e. New Physics.

It can be shown that if the density exceeds the critical value given by:

$$\rho_{crit} = \frac{3}{8\pi G} H_0^2$$
$$= 5.7 \times 10^{-30}\ gm/cm^3 \quad \left(\text{using } H_0 = 5.5\ \frac{cm/sec}{pc}\right)$$

that in the future the universe will stop expanding and begin to contract. Hence the Big Bang may become the Big Crunch.

Standard Model of Cosmology

The universe was created in a gigantic explosion called the Big Bang. The hot dense fireball cooled and became less dense as it expanded. As it cooled, particles "froze out".

The Hadron Era ($0 < t < 10^{-4}$ sec, $T > 10^{12}$ K)

Universe contains particles and antiparticles which annihilate to produce photons and vice versa.

Exercise: Show minimum temperature for

$$2\gamma \rightarrow p + \bar{p}$$

where photon energy is kT is $T \sim 10^{13}$ K.

When the temperature fell below this minimum, the baryons and antibaryons annihilated leaving the small excess of baryons that presently make up the matter in the universe.

The Lepton Era ($10^{-4} < t < 10$ sec, $10^{12} > T > 4\times10^{9}$ K)

Universe contains photons, electrons, neutrinos and their antiparticles as well as baryons left from the hadron era. This era ends when the photon energy for the process $2\gamma \rightarrow e^- + e^+$ is insufficient.

The Radiation Era ($10 < t < 10^{13}$ sec, $4 \times 10^9 > T > 10^4$ K)

Universe consists of mainly photons and neutrinos as well as smaller numbers of electrons, protons and neutrons. This era ends when the photons have insufficient energy to ionize hydrogen and helium, i.e. atoms form

The Matter Era ($t > 10^{13}$ sec, $T < 10^{3-4}$ K)

This is the present era where matter and radiation are no longer in equilibrium.

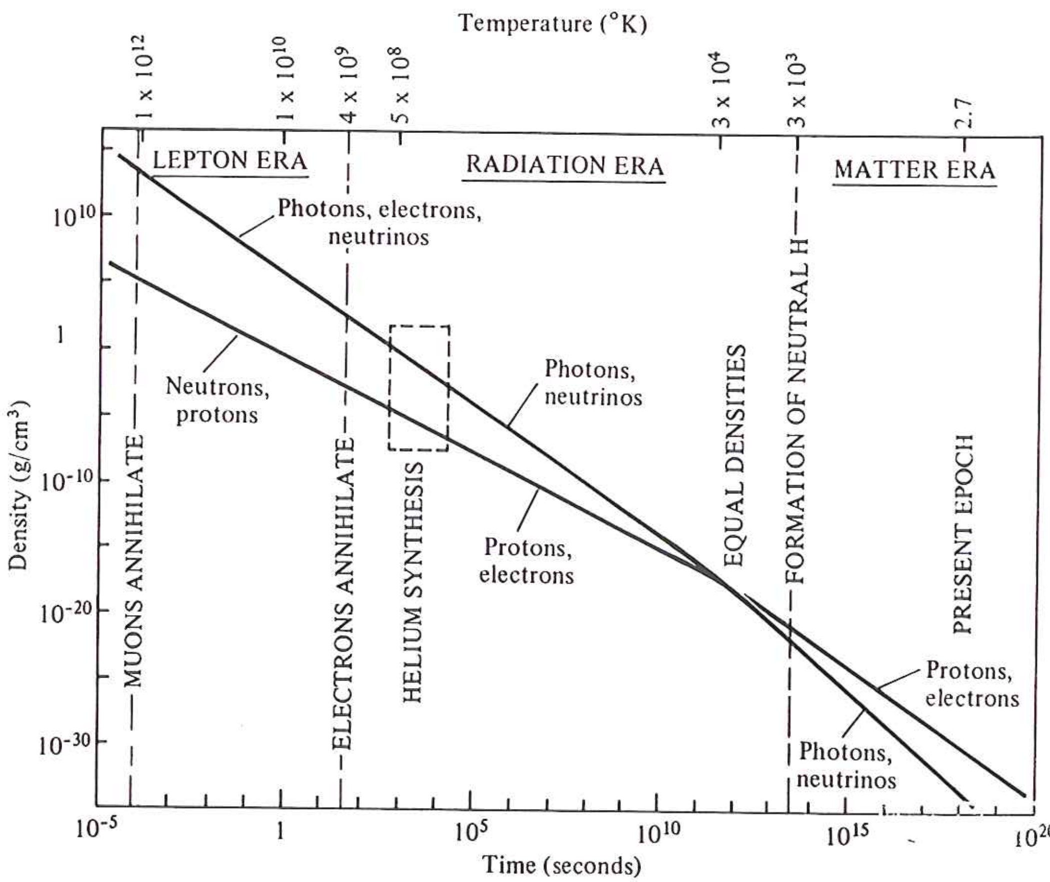


Fig. 10.17 *Eras in the evolution of the universe.* (*After Harrison,* Annual Review of Astronomy and Astrophysics, *1973*.)